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Multiplicatively closed bases for $C(A)$

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Abstract. A canonical basis for the centralizer $C(M)$ of a $n \times n$ complex matrix M is obtained from the one $C(A)$ constructed for a Jordan canonical form A of M. These bases are closed under nonzero products and we obtain simple multiplication tables for their products. Two matrices have **the same centralizer invariants** iff their Jordan canonical forms have the same block structure. For square matrices M and N we prove that $C(M)$ and $C(N)$ are isomorphic as algebras over C iff they have the same size and the same centralizer invariants. We produce a similar canonical basis for the centralizer algebra of a real matrix over the real numbers which has analogous multiplicative properties in addition to conjugation properties. We also give some additional consequences of these results.

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1 Introduction

For any $n \times n$ matrix M over the complex field $\mathbb{C}$, as in [4], one can use its Jordan canonical form to find a basis $\mathfrak{B}$ for its centralizer algebra $C(M) = \{ B : B \text{ is } n\times n \text{ matrix over } \mathbb{C} \text{ and } MB = BM \}$ as a vector space over $\mathbb{C}$. We show that this basis $\mathfrak{B}$ as described in [4] has the property that if $B_1, B_2 \in \mathfrak{B}$ and $B_1B_2 \neq 0$, then $B_1B_2 \in \mathfrak{B}$. We introduce a notation for the elements of $\mathfrak{B}$ in order to study their multiplicative properties and develop formulas for their products. $\mathfrak{B}$ can also be considered a canonical multiplicative basis for $C(M)$ since its products are easily computed and $C(M)$ is determined up to isomorphism as an algebra by its multiplication table. We are able to characterize exactly when the nilpotent elements of $C(M)$ form a subalgebra of $C(M)$. Using properties of these canonical bases we are able to show our main result that any two centralizer algebras $C(M)$ and $C(N)$ which are isomorphic as algebras over $\mathbb{C}$ have Jordan canonical forms which have the same block structure. One can also study over any field $\mathbb{F}$ containing the entries of M, $C_{\mathbb{F}}(M) = \{ B : B \text{ is } n \times n$

matrix over $\mathbb{F}$ and $MB = BM\}$; we prove for this situation that there is a canonical basis for $C_{\mathbb{F}}(M)$ with the same multiplication properties as the above canonical basis $\mathfrak{B}$ iff all of the eigenvalues of M belong to $\mathbb{F}$. It is interesting to observe that whenever a power series $\sum_{n=0}^{\infty} a_n M^n$ converges that it must converge to an element in the center $ZC(M)$ of $C(M)$ which is by [4] equal to a polynomial $p(x)$ evaluated at M. We give a necessary and sufficient condition for when such a power series converges which as in [3] gives a method to calculate $p(x)$. For any polynomial $p(x)$ we give a necessary and sufficient condition for when $C(M) = C(p(M))$. Moreover, we show that $C(M) \cong C(N)$ iff there is a polynomial $p(x)$ such that N is similar to $p(M)$. We show how to use the minimal polynomial $m(x)$ of M to produce k linearly independent elements in $ZC(M)$ when M has k distinct eigenvalues. We close the second section with a result which for a selfadjoint matrix M characterizes when a polynomial $p(M)$ of M is also selfadjoint.

In the third section we use our work for $C(N)$ over $\mathbb{C}$ of the second section to find a kind of canonical basis for $C_{\mathbb{R}}(N)$ over the real number field $\mathbb{R}$ where N is any $n \times n$ matrix with entries from $\mathbb{R}$. When N has complex eigenvalues we know that this basis can not be closed under multiplication by our previous work. However, in this case we obtain the simplest kind of violation of this closure property in that any nonzero product of basis elements is ± 1 times another basis element. These bases for $C_{\mathbb{R}}(N)$ have pleasant geometric and conjugation properties. Again by [4] the center $ZC_{\mathbb{R}}(N)$ is precisely the set of all matrices of the form $p(N)$ with $p(x) \in \mathbb{R}[x]$. Moreover, we show how to calculate powers of N using the real and imaginary parts of powers of its eigenvalues in terms of these bases and similarity maps.

2 Canonical bases for the centralizer of a matrix

As in [4] we work mainly with a Jordan canonical form A of M where $A = SMS^{-1}$ since results about $C(A)$ are easily translated in to the corresponding results for $C(M)$ by means of the automorphism σ of the algebra of $n \times n$ matrices given by $\sigma(X) = S^{-1}XS$. So as in [2] we assume that we have a Jordan canonical form A of M in the usual block form

$$A = \begin{bmatrix} A_1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & A_2 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & A_k \end{bmatrix}$$

with square matrices $A_1, \ldots, A_k$ where k is the number of distinct eigenvalues of M. We denote by λ_i for $i = 1, \ldots, k$ these distinct eigenvalues and assume that each direct summand A_i for $i = 1, \ldots, k$ is the block in A associated with eigenvalue λ_i for $i = 1, \ldots, k$ of size $d_i \times d_i$ where d_i is the multiplicity of the root λ_i in the characteristic polynomial of M for $i = 1, , , k$. By Lemma 1.3.4 of [4] it is sufficient to consider the $C(A_i)$ for $i = 1, \ldots, k$ since $C(A)$ is isomorphic to the direct sum of the $C(A_i)$ for $i = 1, \ldots, k$. So we restrict our attention to determining $C(A_i)$ for a single block A_i in the Jordan matrix A associated with λ_i. So as in [2], we assume that this block

$$A_i = \begin{bmatrix} J_1^{(i)} & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & J_2^{(i)} & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & J_{m_i}^{(i)} \end{bmatrix},$$

where each $J_s^{(i)}$ is an elementary Jordan matrix for $s = 1, \ldots, m_i$ and where each elementary Jordan matrix has the following form

$$J_s^{(i)} = \begin{bmatrix} \lambda_i & 0 & \cdot & \cdot & \cdot & 0 \\ 1 & \lambda_i & 0 & \cdot & \cdot & \cdot \\ 0 & 1 & \lambda_i & & & \cdot \\ \cdot & \cdot & 1 & \cdot & & \cdot \\ 0 & \cdot & & \cdot & \lambda_i & 0 \\ 0 & 0 & \cdot & 0 & 1 & \lambda_i \end{bmatrix}$$

which for $t(s) > 1$ is a lower triangular $t(s) \times t(s)$ matrix with all entries 0 except for the diagonal entries equal to λ_i and the $(j+1, j)$-entry which is 1 for $j = 1, \ldots, t(s) - 1$ and for $t(s) = 1$, $J_s^{(i)} = [\lambda_i]$, for $s = 1, \ldots, m_i$. We also require $t(s+1) \leq t(s)$ for $s = 1, \ldots, m_i - 1$.

Now for notational convenience we drop the subscript i and consider $C(A)$ for A just one block of a Jordan canonical form associated with a fixed eigenvalue λ. So as in [2] we consider

$$A = \begin{bmatrix} J_1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & J_2 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & J_m \end{bmatrix}$$

where J_s is an elementary Jordan matrix of size $t(s) \times t(s)$ for $s = 1, \dots, m$ and $t(s+1) \le t(s)$ for $s = 1, \dots, m-1$. Let us set $d = t(1) + \dots + t(m)$ so that A is a $d \times d$ matrix.

1 Remark. One could use a less specific form as in [4] but since all these Jordan forms of a matrix are cogredient (similar to each other using a permutation matrix) in the study of their centralizer algebra over $\mathbb{C}$ it makes no difference.

By [4] a $\mu \times \nu$ matrix B is **triangularly striped** iff

$$\text{(1) for } \mu = \nu,\ B = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ a_2 & a_1 & \cdots & 0 \\ a_3 & a_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ a_\mu & a_{\mu-1} & \cdots & a_1 \end{bmatrix},$$

$$\text{(2) for } \mu < \nu,\ B = \begin{bmatrix} a_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ a_2 & a_1 & \cdots & 0 & 0 & \cdots & 0 \\ a_3 & a_2 & \cdots & & & & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_\mu & a_{\mu-1} & \cdots & a_1 & 0 & \cdots & 0 \end{bmatrix}, \text{ and}$$

$$\text{(3) for } \mu > \nu,\ B = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \\ a_1 & 0 & \cdots & 0 \\ a_2 & a_1 & \cdots & 0 \\ a_3 & a_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ a_\nu & a_{\nu-1} & & a_1 \end{bmatrix}.$$

We observe that the nonzero entries in a triangularly striped matrix always start in first column and then that entry is repeated diagonally until one reaches the last row. For any triangular striped matrix with a_1 nonzero has rank r equal to the minimum of $\{\mu, \nu\}$. So one can find the obvious basis for the space of $\mu \times \nu$ triangular striped matrices of size equal to the $\min\{\mu, \nu\}$ by letting the entry a_i be 1 and a_j with $j \ne i$ equal to 0 for each $i \le \min\{\mu, \nu\}$ (since clearly they are linearly independent, span, and number exactly the minimum of $\{\mu, \nu\}$).

By Theorem 1.3.6 and Corollary 1.3.9 of [4] we have an explicit representation of the elements in $C(A)$ for A as described above:

2 Theorem. *If A is a single block of a Jordan canonical form associated with a single eigenvalue λ, then $C(A)$ is the set of all $d \times d$ matrices of the form*

$$B = \begin{bmatrix} B_{1,1} & B_{1,2} & \cdots & B_{1,m} \\ B_{2,1} & B_{2,2} & \cdots & B_{2,m} \\ \cdots & \cdots & \cdots & \cdots \\ B_{m,1} & B_{m,2} & \cdots & B_{m,m} \end{bmatrix},$$

where $B_{i,j}$ is an arbitrary $t(i) \times t(j)$ triangularly striped matrix and $C(A)$ has dimension equal to $t(1) + 3t(2) + \cdots + (2m-1)t(m)$.

We wish to introduce a notation for a basis $\mathfrak{B}$ obtained from the result above in [4] in order to study its multiplicative properties. By our hypothesis, we know that for $1 \le i, j \le m$ that the block $B_{i,j}$ in B above is a $t(i) \times t(j)$ triangularly striped matrix. For $u = \min\{\, t(i), t(j) \,\}$ and $1 \le q \le u$, there is a unique choice of a diagonal with q 1's in the block $B_{i,j}$ in B and all other entries in B equal to zero. This $d \times d$ matrix is denoted by $\gamma_{i,j}(q)$. So $\mathfrak{B} = \{\, \gamma_{i,j}(q) : \text{with } 1 \le i, j \le m$ and $1 \le q \le u = \min\{\, t(i), t(j) \,\}$ is a basis for $C(A)$.

There is the following alternative description of these basis elements for $C(A)$. Let $D(1), D(2), \ldots, D(m)$ denote the partition of the set $\{\, 1, \ldots, d \,\}$ into consecutive blocks of size $t(s)$ for $s = 1, \ldots, m$, respectively, i.e., $D(1) = \{\, 1, \ldots, t(1) \,\}$ and $D(s) = \{\, t(1) + \cdots + t(s-1) + 1, \ldots, t(1) + \cdots + t(s) \,\}$ for $1 < s \le m$. For $1 \le s \le m$, we define $f(s)$ and $l(s)$ to be the smallest and largest elements of $D(s)$, respectively, so that $D(s) = \{\, f(s), \ldots, l(s) \,\}$ for $s = 1, \ldots, m$. For any $1 \le i, j \le d$, $E_{i,j}$ is the $d \times d$ matrix with a 1 in its i,j-entry and zeroes everyplace else; recall that these are just the standard basis elements for $d \times d$ matrices and that $E_{i,j}E_{k,l} = E_{i,l}$ if $j = k$ and 0 if $j \ne k$. More analytically for $1 \le i < j \le m$, we obtain $\gamma_{i,j}(t(j)) = \sum_{q=0}^{t(j)-1} E_{l(i)+q+1-t(j), f(j)+q}$ where $l(i)$ denotes the largest element of $D(i)$ and $f(j)$ the smallest element of $D(j)$ and, in general, for $1 \le r < t(j)$ we define $\gamma_{i,j}(r) = \sum_{q=0}^{r-1} E_{l(i)+q+1-r, f(j)+q}$. Also for $1 \le j < i \le m$, $\gamma_{i,j}(t(i)) = \sum_{p=0}^{t(i)-1} E_{l(i)+p+1-t(i), f(j)+p}$ and, in general, for $1 \le r < t(i)$, $\gamma_{i,j}(r) = \sum_{p=0}^{r-1} E_{l(i)+p+1-r, f(j)+p}$. For $i = j$ and $1 \le r \le t(i)$, we have that $\gamma_{i,i}(r)$ lies in the diagonal block $B_{i,i}$ of B and it is defined for $1 \le i \le m$ and for $1 \le r \le t(i)$ by, $\gamma_{i,i}(r) = \sum_{q=0}^{r-1} E_{l(i)+q+1-r, f(i)+q}$. These basis elements $\gamma_{i,i}(r)$ from the diagonal blocks $B_{i,i}$ with $1 \le i \le m$ and $1 \le r \le t(i)$ are called **the primary basis elements for** $C(A)$ while $\gamma_{i,j}(r)$ with $1 \le i \ne j \le m$ and $1 \le r \le \min\{\, t(i), t(j) \,\}$ are called **the secondary basis elements for** $C(A)$.

For a particular matrix A, the elements of $\mathfrak{B}$ have a geometric relationship to each other. We illustrate this with an example.

Let

$$A = \begin{bmatrix} \lambda & 0 & 0 & 0 & 0 & 0 \\ 1 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 1 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 1 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{bmatrix}.$$

The primary basis vectors of $C(A)$ are the following

$$P(A) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

While the secondary basis vectors are the following

$$S(A) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

The whole basis $\mathfrak{B}$ is represented by

$$\Gamma(A) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

Note that one may view the secondary basis vectors as being obtained by sliding the diagonal blocks below the first diagonal block straight upward so that its last row coincides with the last row of each diagonal block above it to form the striped triangular blocks above them and also one slides the diagonal blocks below the first diagonal block directly to the left so that its first column coincides with the first column of the each diagonal block above it to obtain the striped triangular blocks to its left (this is completely general).

3 Proposition. *If A is a single block of a Jordan canonical form for the eigenvalue λ, then $\mathfrak{B} = \{\, \gamma_{i,j}(r) : 1 \leq i, j \leq m$ and $1 \leq r \leq \min\{\, t(i), t(j) \,\}$ is a basis for $C(A)$ as a vector space. If A is an arbitrary Jordan canonical form with $\lambda_1, \ldots, \lambda_k$ distinct eigenvectors, then one obtains a basis for $C(A)$ by forming the direct sum of the bases $\mathfrak{B}_i$ found for $C(A_i)$ as above for $i = 1, \ldots, k$ where A_i is the block on the diagonal of A containing all occurrences of the eigenvalue λ_i in A for $i = 1, \ldots, k$ and preserving the same block structures within A.*

For a matrix A in Jordan canonical form we note that in this section whenever we refer to a basis for any $C(A)$ we mean the one we just produced above. We will call this basis $\mathfrak{B}$ **a canonical basis for** $C(A)$ as an algebra. We examine in more detail products of these basis elements. We again assume A is a single block from a Jordan form. We give some special names and notation to certain primary basis elements; namely, $\alpha_s = \gamma_{s,s}(t(s))$ is the basis vector along the main diagonal in the block $B_{s,s}$ for $s = 1, \ldots, m$, it is a nonzero idempotent, for any other basis vector of form $\gamma_{s,j}(r)$, $\gamma_{s,s}(t(s))\gamma_{s,j}(r) = \gamma_{s,j}(r)$, and for any other basis vector of the form $\gamma_{i,s}(r)$, $\gamma_{i,s}(r)\gamma_{s,s}(t(s)) = \gamma_{i,s}(r)$. Now for $1 \leq s \leq m$ such that $t(s) > 1$, we define $\beta_s = \gamma_{s,s}(t(s) - 1)$, it is a nonzero nilpotent element with degree of nilpotency exactly $t(s)$ if $t(s) > 1$ and its nonzero powers give all the other nilpotent basis elements from the block $B_{s,s}$, i.e., $\beta_s^j = \gamma_{s,s}(t(s) - j)$ if $j < t(s)$; sometimes we may still write β_s when $t(s) = 1$ but in this case β_s does not exist. We determine next what the products of basis elements when at least one is a secondary basis element. It is easy to see that $\gamma_{i,j}(r)\gamma_{k,l}(q)$ is 0 if $j \neq k$. In particular, if secondary basis vectors lie in the same rows or in the same columns then their product is 0. Before going on we show that any nonzero product of elements of $\mathfrak{B}$ is an element of $\mathfrak{B}$. It is evident that $C(A)$ is not only a vector space but they are algebras, i.e., closed under products. In general for an arbitrary basis for $C(A)$ the most one knows is that any nonzero product of its elements is a linear combination of these basis elements. But the basis $\mathfrak{B}$ from [4] as described above also satisfy the following special closure property:

4 Lemma. *Let A be any Jordan canonical form with canonical basis $\mathfrak{B}$ for $C(A)$. If $b, b' \in \mathfrak{B}$ and $bb' \neq 0$, then $bb' \in \mathfrak{B}$ and*

$$\operatorname{rank}(bb') \leq \min\{\, \operatorname{rank}(b), \operatorname{rank}(b') \,\}.$$

Proof. It suffices to show this for $C(A)$ where A is a single block of a Jordan canonical matrix containing all occurrences of an eigenvalue value λ. Suppose we have that the product of two basis elements b and b' from $\mathfrak{B}$ whose product is nonzero. So we have that $b = E_{i,j+1} + \cdots + E_{i+u,j+u}$ and $b' = E_{k,l} + \cdots + E_{k+v,l+v}$. There must be one of the $j+1, \ldots, j+u$ equal to one of the numbers $k, \ldots, k+v$ since the product is not zero and the nonzero coefficient of the $E_{i',l'}$ must be 1. This says there is a set $D(y)$ in our partition of $\{ 1, \ldots, d \}$ such that $\{ j+1, \ldots, j+u \} \subseteq D(y)$ and $\{ k, \ldots, k+v \} \subseteq D(y)$. There is an x such that $\{ i, i{+}1, \ldots, i{+}u \} \subseteq D(x)$ and a z such that $\{ l, \ldots, l{+}v \} \subseteq D(z)$. The product lies in the same rows as b and the same columns as b' and, consequently, it must be a linear combination of basis vectors lying in this x, z block (in $B_{x,z}$). But any nonzero component $E_{i',l'}$ of bb' establishes just what "diagonal" it lies on in this block and it can't have nonzero entries from two different diagonals of the same block because of the consecutive nature of the indices in b and b'. Once one understands this pattern the statement about ranks is apparent. $\boxed{QED}$

It is convenient to have a multiplication table for computing these products. The next result produces this table for a single block A associated with λ and illustrates the usefulness of our notation. This table is easily expandable to $C(A)$ for any Jordan canonical form. In what follows we make the convention that for $1 \le i, j \le m$ that when we write $\gamma_{i,j}(r)$ we assume that $r \le \min\{ t(i), t(j) \}$and for such an r with $0 < r$, $\gamma_{i,j}(r)$ has rank r. If $q \le 0$ appears in an expression $\gamma_{i,j}(q)$, then $\gamma_{i,j}(q) = 0$ the zero matrix. We note that sometimes we have two different notations for the same basis elements namely for $1 \le i \le m$, $\alpha_i = \gamma_{i,i}(t(i))$ and $\beta_i = \beta_i(t(i)-1) = \gamma_{i,i}(t(i)-1)$ if $1 < t(i)$.

5 Proposition. *Assuming the notation introduced above, the following hold:*

(1) If $1 \le i < j \le m$ and $1 \le r \le t(j)$, then $\alpha_i\alpha_i = \alpha_i$, $\alpha_i\alpha_j = \alpha_j\alpha_i = 0$, $\alpha_i\gamma_{i,j}(r) = \gamma_{i,j}(r)$, $\gamma_{i,j}(r)\alpha_j = \gamma_{i,j}(r)$, $\beta_i(t(i)-1)\gamma_{i,j}(r) = \gamma_{i,j}(r-1)$, $\gamma_{i,j}(r)\beta_j(t(j)-1) = \gamma_{i,j}(r-1)$, $\alpha_j\gamma_{j,i}(r) = \gamma_{j,i}(r)$, $\gamma_{j,i}(r)\alpha_i = \gamma_{j,i}(r)$, $\beta_j(t(j)-1)\gamma_{j,i}(r) = \gamma_{j,i}(r-1)$, and $\gamma_{j,i}(r)\beta_i(t(i)-1) = \gamma_{j,i}(r-1)$;

(2) If $1 \le i, j, k \le m$, then $\gamma_{i,j}(r)\gamma_{j,k}(s) = \gamma_{i,k}(r+s-t(j))$;

(3) If $1 \le h, i, j, k, \le m$ and $i \ne j$, then $\gamma_{h,i}(r)\gamma_{j,k}(s) = 0$; and

(4) If $1 \le i, j, k \le m$ and $\gamma_{i,j}(r)\gamma_{j,k}(s) \ne 0$, then $r+s-t(j) \le \min\{ r, s \}$.

Proof. We will show a part of (2) where we assume that $1 \le i < j < k \le m$ since the others are similar. Let

$$\gamma_{i,j}(r)\gamma_{j,k}(s) = \left(\sum_{q=0}^{r-1} E_{l(i)+q+1-r, f(j)+q} \right) \left(\sum_{p=0}^{s-1} E_{l(j)+p+1-s, f(k)+p} \right).$$

For this product to be nonzero we must have $f(j)+r-1 = l(j)+p+1-s$ which solving for p one obtains $p = r+s-t(j)-1$ since $l(j)-f(j) = t(j)-1$. Thus, $\gamma_{i,j}(r)\gamma_{j,k}(s) = \gamma_{i,k}(r+s-t(j))$. $\boxed{QED}$

One can alternatively describe the rows and columns that these basis vectors lie in as follows: Given any basis vector $\gamma_{i,j}(r)$ in the canonical basis $\mathfrak{B}$ of $C(A)$ with $1 \leq i, j \leq m$ and $1 \leq r \leq \min\{\, t(i), t(j)\,\}$, $\gamma_{i,j}(r)$, then there are unique primary idempotent basis elements α_i and α_j such that $\alpha_i \gamma_{i,j}(r) = \gamma_{i,j}(r)$ and $\gamma_{i,j}(r)\alpha_j = \gamma_{i,j}(r)$ (we say that $\gamma_{i,j}(r)$ **lies in the rows of** α_i and **lies in the columns of** α_j), i.e., referring to matrix in Theorem 2 $\gamma_{i,j}(r)$ lies in the block $B_{i,j}$. Now there may be certain *minimal idempotent primary basis vectors* which are simply those basis vectors α_s where $t(s) = 1$; these can be characterized also by the following property that it is a basis vector b such that $bb = b$ and for all $c \in C(A_i)$, $bc = cb$ implies that $c = rb$ for some scalar r. It is interesting to note that if there are exactly j minimal idempotent basis vectors that they determine in our construction a block of j^2 associated basis elements forming a subalgebra isomorphic to the algebra of all $j \times j$ matrices (so one does not totally entirely avoid full matrix rings this way). We note in the special case when A is the $n \times n$ zero matrix, then the basis $\mathfrak{B}$ obtained this way for $C(A)$ is $\{\, E_{i,j} : 1 \leq i, j \leq n\,\}$ the usual standard basis for $n \times n$ matrices and its familiar multiplicative properties.

One can observe that any secondary $\gamma_{i,j}(r)$ basis vector with $i \neq j$ is not in the center $ZC(A)$ of $C(A)$ since there are an idempotent primary basis vector α_i and α_j such that $\alpha_i\gamma = \gamma$ and $\gamma\alpha_i = 0$ or $\gamma\alpha_j = \gamma$ and $\alpha_j\gamma = 0$. Thus, if there are no secondary basis vectors, then all the basis vectors found for $C(A)$ will be primary basis vectors all which commute with each other. So in this case one would have that $C(A)$ is commutative. By [2] or [4] this happens for $C(A)$ iff the algebraic multiplicity of λ_i in the minimal polynomial of M is equal to the algebraic multiplicity of λ_i in the characteristic polynomial of M. In this case we have the associated A_i consists of just a single elementary Jordan block and the dimension of $C(A_i)$ is exactly $d_i = t(1)$ with basis $\{\, \alpha_1(t(1)), \beta_1(t(1) - 1), \ldots, \beta_1(1)\,\}$ and $C(A_i) = \{\, p(A_i) : p(x) \in C[x]\,\}$. If this were to happen for each eigenvalue, i.e., the minimal polynomial for M equals the characteristic polynomial for M, then the dimension of $C(M) = \sum_{i=1}^{k} d_i = n$ and so $C(M) = \{\, p(M) : p(x) \in \mathbb{C}[x]\,\}$ and it is commutative. This gives an alternative viewpoint to these known results [2]:

Next we list some properties concerning the rank of elements of $C(A_i)$.

6 Proposition. *Let $\mathfrak{B}_i$ be the above basis for $C(A_i)$. If $S \subseteq \mathfrak{B}_i$ and all elements of S lie in the same block $D(s)$ of rows (columns) of A_i, then the rank of $\sum_{b_j \in S} k_j b_j$ is equal to* $\max\{\, \mathrm{rank}(b_j) : k_j \neq 0\,\}$. *If $b, b' \in \mathfrak{B}_i$, then* $\mathrm{rank}(bb')$ *is less than or equal to minimum of* $\mathrm{rank}(b)$ *and* $\mathrm{rank}(b')$. *If $S \subseteq \mathfrak{B}_i$ and* $\mathrm{rank}(\sum_{b_j \in S} k_j b_j) = 1$, *then for all the b_j's with $k_j \neq 0$ have* $\mathrm{rank}(b_j)$*=1 and they all lie in the same row of A_i or they all lie in the same column of A_i.*

Now we can give precise information about the rank of elements in $C(A_i)$.

7 Proposition. *Any $C(A_i)$ has an element of every possible rank r with $0 \leq r \leq d_i$. In $C(A)$ or $C(M)$ there are elements of any possible rank $0 \leq r \leq n$.*

Proof. The first part comes from an inspection of the various possibilities for a Jordan block A_i. Because $C(A)$ is a direct sum of the $C(A_i)$'s and the rank of the sum of these elements is simply the sum of their ranks the result here follows from the first part. Observe that $C(M)$ is isomorphic to $C(A)$. $\boxed{QED}$

In looking at the bases produced above for $C(A_i)$ it is possible that there are secondary basis vectors whose product is an idempotent basis vector. In fact this will happen exactly when there are $1 \leq k < j \leq m_i$ with $t(k) = t(j)$ since then $\gamma_{k,j}(t(j))\gamma_{j,k}(t(j)) = \alpha_k(t(k))$ and $\gamma_{j,k}(t(j))\gamma_{k,j}(t(j)) = \alpha_j(t(j))$. However, if this does not happen then we have an even stronger result about the nilpotent elements.

8 Proposition. *Let A be a single block in a Jordan canonical form associated with eigenvalue λ and $C(A)$ be as above. The set of nilpotent elements in $C(A)$ form a subalgebra of $C(A)$ iff for all $1 \leq k < j \leq m$, $t(j) < t(k)$.*

Proof. By the above computation this is a necessary condition. Suppose that for all $1 \leq k < j \leq m$, $t(j) < t(k)$. In the canonical basis $\mathfrak{B}$ for $C(A)$, consider c a linear combination of nilpotent elements of $\mathfrak{B}$. For two nilpotent basis elements one can verify that $\gamma_{i,j}(r)\gamma_{j,k}(s) = \gamma_{i,k}(\min\{r, s\})$ iff $r = t(j)$ and $j > i$ or $s = t(j)$ and $j > k$. Suppose to the contrary that for all $q > 0$ that $c^q \neq 0$. Let r be the maximum rank such that for some i, j, $\gamma_{i,j}(r)$ appears with nonzero coefficient in n^q for infinitely many q. We may choose u large enough so that for $q \geq u$ that n^q is a linear combination of basis elements of the form $\gamma_{i,j}(s)$ with $s \leq r$. Consider $n^q n^t$ with $q \geq u$ and $t \geq u$. By hypothesis and Proposition 5 part (4), there must be an element of the form $\gamma_{i,j}(r)$ appearing with nonzero coefficient in the representation of $n^q n^t$. But then there is an element $\gamma_{i,k}(r)$ with nonzero coefficient in n^q and an element $\gamma_{k,j}(r)$ with nonzero coefficient in n^t such that $\gamma_{i,k}(r)\gamma_{k,j}(r) = \gamma_{i,j}(r)$. But by remark above one must have that $t(k) = r$, $k > i$, and $k > j$. But then observe that any element of the form $\gamma_{i,j}(r)$ with nonzero coefficient in $n^q n^t$ has the property that $t(i) > r$ and $t(j) > r$. Consequently, in $(n^q n^t)(n^q n^t)$ there can be no element of the form $\gamma_{i,j}(r)$ with nonzero coefficient and so they all have the form $\gamma_{i,j}(s)$ with $s < r$. This is a contradiction. $\boxed{QED}$

9 Definition. Two $n \times n$ matrices M and N have **the same centralizer invariants** iff they have Jordan canonical forms which have the same block structure. Two Jordan canonical forms have the same block structure means that there is a bijection of their spectra so that applying this bijection to the diagonal entries of one of these Jordan forms and leaving the other entries not on the diagonal unchanged results in the other Jordan canonical form.

In view of our construction we have the following result:

10 Proposition. *If M and N are $n \times n$ matrices with the same centralizer invariants, then $C(M)$ is isomorphic as an algebra to $C(N)$ over $\mathbb{C}$.*

Proof. One may assume that M and N have Jordan canonical forms A and B, respectively, with the same block structure. Referring to the bijection σ of the eigenvalues of A onto the eigenvalues of B which applied to the diagonal entries in A yields B. One just maps the canonical basis $\gamma_{i,j}(r)$ for the block associated with λ in A onto the corresponding $\gamma'_{i,j}(r)$ in the block of B associated with $\sigma(\lambda)$. The only thing one need further to check is that this is yields an algebra isomorphism which also preserve matrix products. But this is immediate because of the Proposition 3 and Proposition 5. $\boxed{QED}$

Now we formulate our main result:

11 Theorem. *Let M and N be two square matrices over $\mathbb{C}$. $C(M)$ is isomorphic to $C(N)$ as algebras over $\mathbb{C}$ iff they have the same centralizer invariants.*

We need first to find a basis for the center $ZC(A) = \{ Z \in C(A) :$ for all $X \in C(A) XZ = ZX \}$ of $C(A)$ for a Jordan canonical form A. But this is easy to do in view of our canonical basis for $C(A)$ and the basis for $ZC(A)$ will also be canonical in the sense that nonzero products of its basis elements are again basis elements.

12 Lemma. *If A is a single block in a Jordan canonical form associated with eigenvalue λ, then a basis for the center $ZC(A)$ is given by $(\alpha_1 + \cdots + \alpha_m)$, $(\beta_1 + \cdots + \beta_m)$, $(\beta_1^2 + \cdots + \beta_m^2)$, ..., $(\beta_1^{r-1} + \cdots + \beta_{n_i}^{r-1})$ where $r = d(1)$ the multiplicity of λ in the minimal polynomial of A (if $d(1) = 1$, there are no nilpotent β's), its dimension is $d(1)$, and $(\beta_1 + \cdots + \beta_m)^j = (\beta_1^j + \cdots + \beta_m^j)$ for $j = 1, \ldots, r-1$. If A is a Jordan canonical form with more than one block, then one does the above for each block and takes their direct sum to obtain a basis for the center $ZC(A)$ of $C(A)$.*

For A a single block, we refer to $(\alpha_1 + \cdots + \alpha_m)$ as **the idempotent basis element of** $ZC(A)$ and to $(\beta_1 + \cdots + \beta_m)$, $(\beta_1^2 + \cdots + \beta_m^2)$, ..., $(\beta_1^{r-1} + \cdots + \beta_m^{r-1})$ as **the nilpotent basis elements of** $ZC(A)$. By Theorem 1.3.7 [4] $ZC(A)$ is precisely $\{ p(A) : p(x) \in \mathbb{C}[x] \}$ where $\mathbb{C}[x]$ is the algebra of polynomials in x with complex coefficients. In particular, each element in the above basis for $ZC(A)$ is a polynomial in A.

Now we complete the proof of Theorem 11:

13 Theorem. *If M and N are two square matrices of perhaps different sizes and $C(M) \cong C(N)$ as algebras, then M and N have the same size and they have the same centralizer invariants.*

Proof. We introduce some notational conventions. First we may assume that M and N are their Jordan canonical forms A and B, respectively, and $C(A) \cong C(B)$ via an isomorphism f. We use the canonical basis elements of $C(A)$ and of $C(B)$ to describe important elements in each. First for each block corresponding to eigenvalue λ_i of A we denote now by α_i the idempotent basis element and β_i the nilpotent generator of the nilpotent elements in $ZC(A_i)$ so that each element in $ZC(A_i)$ has a basis representation of the form $a\alpha_i + \sum_{j=1}^{d(i)-1} b_j\beta_i^j$ where λ_i has multiplicity $d(i)$ in the minimal polynomial of A for $i = 1, \dots, k$. We point out that $\sum_{i=1}^{k} \alpha_i$ is the identity matrix of the same size as $A's$ and $\alpha_i\alpha_j = 0$ for $i \neq j$. We use the notation ψ_t, δ_t to denote the analogous elements in $ZC(B_t)$ for $t = 1, \dots, j$ where $s(t)$ is the algebraic multiplicity of the eigenvalue λ'_t in the minimal polynomial for B. Since f is an algebra isomorphism it must map $ZC(A)$ onto $ZC(B)$. So now consider $f(\alpha_1) = \sum_{t=1}^{j} c_t\psi_t + \sum_{t=1}^{j}\sum_{i=1}^{s(t)-1} d_i\delta_t^i$. Now since $\alpha_1\alpha_1 = \alpha_1$, one has $f(\alpha_1)f(\alpha_1) = f(\alpha_1)$. Writing $f(\alpha_1)$ as $\sum_{k=1}^{j} c_k\psi_k + \delta$, one obtains $\sum_{t=1}^{j} c_t\psi_t + \delta = \sum_{t=1}^{j} c_t^2\psi_t + 2(\sum_{t=1}^{j} c_t\psi_t)\delta + \delta^2$. Obviously, we need that $c_t^2 = c_t$ for $t = 1, \dots, j$ and at least one is 1. One also obtains that $\delta = 0$ by considering for each δ_t its minimal power occurring with nonzero coefficient in δ. Thus, $f(\alpha_1) = \sum_{i \in S} \psi_i$ where $\varnothing \neq S \subseteq \{1, \dots, j\}$ and also for different $\alpha_i's$ their S sets are disjoint. This shows that $k \leq j$. The same argument for f^{-1} yields $j \leq k$. Thus, $k = j$ and f is a bijection of $\{\alpha_1, \dots, \alpha_k\}$ onto $\{\psi_1, \dots, \psi_k\}$. By renumbering we may assume that $f(\alpha_i) = \psi_i$ for $i = 1, \dots, k$.

Now we wish to examine how f maps the elements γ in $C(A)$ such that $\gamma\alpha_1 = \alpha_1\gamma = \gamma$ which are precisely the elements of $C(A_1)$. Clearly, it must map these elements 1-1 and onto the elements δ in $C(B)$ such that $\delta\psi_1 = \psi_1\delta$ which are the elements of $C(B_1)$. First of all it must map β_1 in $ZC(A_1)$ to a nilpotent element in $ZC(B_1)$ of the same degree. Consequently the degree of nilpotency of δ_1 is greater than or equal to the degree of nilpotency of β_1. Using f^{-1} we can conclude that their degrees of nilpotency must be the same. Doing this for each block one obtains that $d(i) = s(i)$ for $i = 1, \dots, k$. Thus the minimal polynomials of A and B have the same degree and algebraic multiplicities for their roots. By the multiplicative properties of the basis elements in the canonical basis for $C(A)$ it is clear that the γ above are all the elements in $C(A_1)$ and their images under f are precisely all elements in $C(B_1)$.

Next we examine how f maps the elements in $C(A_1)$ onto the elements of $C(B_1)$. We introduce a new notation at this point (actually this notation for the primary basis elements of $C(A_1)$ is the same as we introduced prior to Proposition 5) which abuses the above notation since we use the same letters. So now we define $\alpha_1, \dots, \alpha_m$ to be the primary idempotent basis vectors for $C(A_1)$ and β_i be the primary nilpotent basis vector associated with α_i of maximum nilpotency degree $t(i)$ if there are any for $i = 1, \dots, m$ (for $t(i) = 1$ there is no nonzero nilpotent element β such that $\beta\alpha_i = \alpha_i\beta = \beta$). We let $\psi_1, \dots, \psi_j$ be the primary idempotent basis vectors for the $C(B_1)$ together with the primary nilpotent basis vector δ_t associated with ψ_t of maximum nilpotency degree $s(t)$ if there are any for $t = 1, \dots, j$. From above we know $f(a_1 + \cdots + \alpha_m) = \psi_1 + \cdots + \psi_j$. We know that $\alpha_i\alpha_q = 0$ for $1 \leq i, q \leq m$ and $i \neq q$. Now we consider the linear operators determined by $A_1 = \lambda(\alpha_1 + \cdots + \alpha_m) + (\beta_1 + \cdots + \beta_m)$ and $\alpha_1, \dots, \alpha_m$ restricted to the vector space V of all v such that $(\alpha_1 + \cdots + \alpha_m)v = v$ which has dimension $(t(1) + \cdots + t(m))$. We know that the maximum number of such subspaces of V invariant under A_1 is exactly m and this is reflected by the fact that $\alpha_1, \dots \alpha_m$ are idempotents that commute with A_1 in view of Theorem 6.7.10 of [2]. Also By Theorem 6.7.10 of [2] since the idempotents $f(\alpha_1), \dots, f(\alpha_m)$ commute with $B_1 = \lambda'(\psi_1 + \cdots + \psi_j) + (\delta_1 + \cdots + \delta_j)$ (after all B_1 is in center of $C(B_1)$) and, consequently, they determine m subspaces of W invariant under B_1 where W equals the set of all vectors w such that $(\psi_1 + \cdots + \psi_j)w = w$. But the exact number of subspaces of W which are invariant under B_1 is j and so $j \geq m$. By the same argument using f^{-1}we obtain that $j = m$. Thus, the number of blocks in $C(A_1)$ and $C(B_1)$ are equal. By relabeling we may assume that $f(\alpha_i) = \psi_i$ for $i = 1, \dots, m$. From above we know that $t(1) = s(1)$. Now considering α_2 one can argue if $t(2) > 1$ that since there is a nonzero nilpotent element β_2 in $C(A_1)$ such that $\beta_2\alpha_2 = \alpha_2\beta_2 = \beta_2$ and its degree of nilpotency is $t(2)$, then $f(\beta_2)$ must be a corresponding element of same nilpotency degree associated with ψ_2. Thus, $t(2) \leq s(2)$, and considering the map f^{-1} we obtain that $t(2) = s(2)$. A similar argument works if $t(i)$ or $s(i)$ is 1. Thus, we obtain that $t(i) = s(i)$ for $i = 1, \dots, k$. This shows that the block $C(A_1)$ and the block $C(B_1)$ have the same size and that $C(A_1)$ and $C(B_1)$ have the same centralizer invariant. Thus, $C(A)$ and $C(B)$ have the same centralizer invariants. $\boxed{QED}$

3 Applications of the centralizer invariants

Let M be any $n \times n$ matrix with distinct eigenvalues $\lambda_1, \dots \lambda_k$ and let $m(x)$ be its minimal polynomial. For each $i = 1, \dots, k$ we define $\widehat{m_i}(x)$ equal to the quotient polynomial obtained by dividing $m(x)$ by $(x - \lambda_i)$. Clearly, $\widehat{m_i}(M) \neq 0$ for $i = 1, \dots, k$. Since $m(M) = (M - \lambda_i)\widehat{m_i}(M) = 0$, the nonzero columns of $\widehat{m_i}(M)$ are eigenvectors for λ_i. For $1 \leq i, j \leq k$ and $i \neq j$, $\widehat{m_i}(M)\widehat{m_j}(M) = 0$. The next result demonstrates how these elements are linearly independent elements in $ZC(M)$:

14 Proposition. *For $i = 1, \dots, k$, $\widehat{m_i}(M)$ are linearly independent elements of $ZC(M)$.*

Proof. Suppose that $c_1\widehat{m_1}(M) + \cdots + c_k\widehat{m_k}(M) = 0$ with not all $c_1, \dots, c_k$

zero. But then $c_1\widehat{m_1}(x) + \cdots + c_k\widehat{m_k}(x)$ is a polynomial of degree less than the minimal polynomial whose value at M is 0. Thus, $c_1\widehat{m_1}(x) + \cdots + c_k\widehat{m_k}(x) = 0$. Fix $i = 1, \dots, k$, let $j(i)$ be the algebraic multiplicity of λ_i in $m(x)$. So one has that $(x - \lambda_i)^{j(i)} \mid c_i\widehat{m_i}(x)$ but then $(x - \lambda_i) \mid c_i$; so $c_i = 0$. $\boxed{QED}$

We next consider a necessary and sufficient condition that a power series of the form $\sum_{n=0}^{\infty} a_n M^n$ with $a_n \in \mathbb{C}$ converges. This question was considered in [3] where a sufficient condition was demonstrated. Clearly, it converges iff $\sum_{n=0}^{\infty} a_n A^n$ converges where A is a Jordan canonical form of M; the latter converge iff for each of the blocks A_i of A associated with eigenvalue λ_i for $i = 1, \dots, k$, $\sum_{n=0}^{\infty} a_n A_i^n$ converges. We note that it is apparent that when this converges that it converges to an element in the center of $C(A)$ which is a polynomial in A and since we have a basis for these centers one can express this limit as a linear combination of those basis elements.

We prove the next result for each block A_i in the Jordan canonical form associated with eigenvalue λ_i for $i = 1, \dots, k$ and this result carries over for the full Jordan canonical form of A of M and to M. Following our conventions above we drop the subscript i and consider such a single block A for eigenvalue λ where A is comprised of m elementary Jordan matrices for λ and $t(1) \geq \cdots \geq t(m)$ and $t(1)$ is the algebraic multiplicity of λ in the minimal polynomial of M. In terms of the canonical primary basis elements of $C(A)$ we can write $A = \lambda(\alpha_1 + \cdots + \alpha_m) + \beta$ where $\beta = \beta_1 + \cdots + \beta_m$ and some of these β_i need not exist when $t(i) = 1$ where β has nilpotency degree equal to $t(1)$.

15 Theorem. *Let A be a single block for eigenvalue λ and $r = t(1) - 1$. The power series $\sum_{n=0}^{\infty} a_n A^n$ converges iff for $j = 0, \dots, r$ the ordinary series $\sum_{n=j}^{\infty} \binom{n}{j} a_n \lambda^{n-j}$ all converge. Indeed, when these do all converge we have that*

$$\sum_{n=0}^{\infty} a_n A^n = \left(\sum_{n=0}^{\infty} a_n \lambda^n\right)(\alpha_1 + \cdots + \alpha_m) + \sum_{j=1}^{r}\left(\sum_{n=j}^{\infty}\binom{n}{j} a_n \lambda^{n-j}\right)\beta^j.$$

Proof. Let $n \geq r$, then $A^n = \lambda^n(\alpha_1 + \cdots + \alpha_m) + \binom{n}{1}\lambda^{n-1}\beta^1 + \cdots + \binom{n}{r}\lambda^{n-r}\beta^r$ where $\beta^j = (\beta_1^j + \cdots + \beta_m^j)$ for $j = 1, \dots, r$. Now since the $(\alpha_1 + \cdots + \alpha_m)$ and β^j for $j = 1, \dots, r$ are linearly independent we have that for $p \geq r$, $\sum_{n=0}^{p} a_n A^n = (\sum_{n=0}^{p} a_n \lambda^n)(\alpha_1 + \cdots + \alpha_m) + \sum_{j=1}^{r}(\sum_{n=j}^{m}\binom{n}{j} a_n \lambda^{n-j})\beta^j$ has a limit iff the coefficients of $(\alpha_1 + \cdots + \alpha_m)$ and β^j for $j = 1, \dots, r$ all have limits. $\boxed{QED}$

As in [3] and for A a block in a Jordan form associated with an eigenvalue λ, it is useful to observe for a polynomial $p(x)$ one can obtain $p(A)$ as follows which in turn can be used to motivate the definition of $f(A)$ for arbitrary functions $f(x)$ with enough derivatives at λ.

16 Corollary. *If $p(x)$ is any polynomial, then $p(A) = p(\lambda)(\alpha_1 + \cdots + \alpha_m) + \sum_{j=1}^{r} \frac{p^{(j)}(\lambda)}{j!}\beta^j$.*

Let $\mathbb{F}$ be any field and M be a $n \times n$ square matrix with entries from $\mathbb{F}$. We define $C_{\mathbb{F}}(M) = \{\, B : B$ is an an $n \times n$ matrix with entries from $\mathbb{F}$ with $B \in C(M)\,\}$. Let us say that a matrix M with all entries in a field $\mathbb{F}$ has **a canonical basis for** $C_{\mathbb{F}}(M)$ **over** $\mathbb{F}$ iff there is a vector space basis for $C_{\mathbb{F}}(M)$ over $\mathbb{F}$ which has all the multiplicative properties as in our results above in Proposition 5 and such that for distinct elements $a_1, \ldots, a_k$ of $\mathbb{F}$ one has that $M = a_1\alpha_1 + \beta_1 + \cdots + a_k\alpha_k + \beta_k$ where for $i = 1, \ldots, k$ α_i are the canonical basis elements for the idempotent elements in $ZC_{\mathbb{F}}(M)$ and β_i and its nonzero powers forms a basis for the nilpotent elements δ in $ZC_{\mathbb{F}}(M)$ such that $\alpha_i\delta = \delta\alpha_i = \beta_i$ where β_i has nilpotency degree $d(i)$ for $i = 1, \ldots, k$ if $d(i) > 1$. As observed in [4] for any field $\mathbb{F}$ containing the entries of M, one can write down a system of homogeneous linear equations over $\mathbb{F}$ whose solution space over $\mathbb{F}$ is precisely all matrices in $C_{\mathbb{F}}(M)$ and, consequently, its dimension is always the same for all such fields $\mathbb{F}$. We now have the following:

17 Proposition. *A square matrix M has a canonical basis over $\mathbb{F}$ iff all the eigenvalues of M belong to $\mathbb{F}$.*

Proof. Clearly if $\mathbb{F}$ has all of the eigenvalues of M in it one can carry out the reduction to its Jordan canonical form using an invertible matrix S over $\mathbb{F}$ and then proceed as above. Suppose that $M = a_1\alpha_1 + \beta_1 + \cdots + a_k\alpha_k + \beta_k$ over such a canonical basis for $C_{\mathbb{F}}(M)$ over $\mathbb{F}$. Let $m(x)$ be the minimal polynomial for M. One can compute $m(M) = 0 = m(a_1)\alpha_1 + \sum_{i=1}^{d(1)-1} \frac{m^{(i)}(a_1)}{i!}\beta_1^i + \cdots m(a_k)\alpha_k + \sum_{i=1}^{d(k)-1} \frac{m^{(i)}(a_k)}{i!}\beta_k^i$. Since all the listed elements are in the basis and linearly independent one has that $a_1, \ldots, a_k$ are k distinct roots of $m(x)$ and furthermore the multiplicity of a_i in the minimal polynomial is at least $d(i)$ or 1 if there is no β_i for that i. $\boxed{QED}$

Now it is fairly easy to understand for any polynomial $p(x)$ how $C(M)$ and $C(p(M))$ are related. Clearly, $C(M)$ is always a subalgebra of $C(p(M))$. We say that an eigenvalue λ of M is **linked** iff in its Jordan canonical form there is at least one 1 below a main diagonal entry containing λ.

18 Proposition. *If M has distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ and $p(x)$ is a polynomial such that $p(\lambda_i)$ are all distinct for $i = 1, \ldots, k$ and $p'(\lambda_i) \neq 0$ for any linked eigenvalue λ_i with $i = 1, \ldots, k$, then $C(M) = C(p(M))$. If M has distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ and $p(x)$ is a polynomial such that $p(\lambda_i)$ are not all distinct for $i = 1, \ldots, k$ or $p'(\lambda_i) = 0$ for some linked eigenvalue λ_i with $i = 1, \ldots, k$, then $C(M) \subset C(p(M))$.*

Proof. We can assume that A is a Jordan canonical form of M. One has that $p(A)$ has the same centralizer invariants as does A iff $p(\lambda_i)$ are all distinct for $i = 1, \ldots, k$ and $p'(\lambda_i) \neq 0$ for any linked eigenvalue λ_i with $i = 1, \ldots, k$. Consequently, if this condition is satisfied, by Theorem 11 we have that $C(A) \cong C(p(A))$. But since $C(A) \subseteq C(p(A))$ and since they have the same dimension, $C(A) = C(p(A))$. On the other hand if the hypothesis about $p(x)$ is not satisfied, then $C(M) \subset C(p(M))$. $\boxed{QED}$

Next we have the following:

19 Proposition. *If A and B are matrices such that for polynomials $p_1(A) = B$ and $p_2(B) = A$, then $C(A) = C(B)$ and so A and B have the same centralizer invariants.*

Proof. Since $C(A) \subseteq C(B)$ and $C(B) \subseteq C(A)$, the result follows. $\boxed{QED}$

20 Corollary. *If A is invertible, then $C(A) = C(A^{-1})$ and so A and A^{-1} have the same centralizer invariants.*

Proof. This follows trivially from 'only if' part of Theorem 11 since $XA = AX$ iff $A^{-1}X = XA^{-1}$. $\boxed{QED}$

Incidentally, this shows that if $\prod_{i=1}^{k}(\lambda - \lambda_i)^{n_i}$ and $\prod_{i=1}^{k}(\lambda - \lambda_i)^{m_i}$ are the characteristic polynomial and minimal polynomial for A, respectively, then $\prod_{i=1}^{k}(\lambda - \frac{1}{\lambda_i})^{n_i}$ and $\prod_{i=1}^{k}(\lambda - \frac{1}{\lambda_i})^{m_i}$ are the characteristic and minimal polynomials for A^{-1}.

In a canonical basis for $C(A)$ it is easy to see that using the canonical basis for $Z(CA)$ one has that $A^{-1} = \frac{1}{\lambda_1}\alpha_1 + \sum_{i=1}^{d(1)-1} \frac{f^{(i)}(\lambda_1)}{i!}\beta_1^i + \cdots + \frac{1}{\lambda_k}\alpha_k + \sum_{i=1}^{d(k)-1} \frac{f^{(i)}(a_k)}{i!}\beta_k^i$ where $f(x) = \frac{1}{x}$. In view of this we can now do the following since it is always possible for distinct λ_i for $i = 1, \ldots, k$ and distinct λ_i' for $i = 1, \ldots, k$ to find a polynomial $p(x)$ of degree at most k such that $p(\lambda_i) = \lambda_i$ for $i = 1, \ldots, k$ and $p'(\lambda_i) \neq 0$ for $i = 1, \ldots, k$.

21 Proposition. *If $C(A) \cong C(B)$, then there is a polynomial $p(x)$ such that B is similar to $p(A)$.*

Proof. By Theorem 11 we know that A and B have the same centralizer invariants. Consequently, let λ_i and λ_i' denote their distinct eigenvalues for $i = 1, \ldots, k$, respectively, for which their Jordan forms have the same centralizer invariants (so the Jordan blocks for λ_i and λ_i' have exactly the same shape). Choosing a polynomial $p(x)$ as above one has that $p(A)$ has a Jordan form exactly the same as the Jordan form for B. Consequently, B is similar to $p(A)$. $\boxed{QED}$

22 Corollary. *If M is a selfadjoint matrix and $B \in ZC(M)$, then for some polynomial $p(x) \in \mathbb{C}[x]$, $p(M) = B$ and B is selfadjoint iff $p(\lambda) \in \mathbb{R}$ for all $\lambda \in spec(A)$. If M is symmetric, then so is every element of $ZC(M)$.*

PROOF. Since M is selfadjoint, then all eigenvalues of A are real and M is diagonalizable. Let $p(M) = B$. Suppose that $SMS^{-1} = J$ a Jordan form of A. So $p(J) = Sp(A)S^{-1} = SBS^{-1}$. So now if B is selfadjoint all of its eigenvalues are real and so are $p(\lambda)$ for $\lambda \in spec(A)$. Suppose now that $p(\lambda)$ for $\lambda \in spec(A)$ are all real, consequently we can find a polynomial $p'(x) \in \mathbb{R}[x]$ so that $p'(\lambda) = p(\lambda)$ for all $\lambda \in spec(A)$. So $Sp'(A)S^{-1} = p'(J) = p(J)$; thus, $B = p'(A)$ is selfadjoint. The second assertion is obvious since $p(M)$ is symmetric for any $p(x) \in \mathbb{C}[x]$. $\boxed{QED}$

4 The real centralizer algebra of a real matrix

In this section we assume that A is always an $n \times n$ matrix with real number entries. We want to find a basis for the its centralizer $C(A)$ over the scalar field of real numbers so that any real matrix in $C(A)$ is a linear combination of these basis elements using coefficients that are all real numbers. We show how to use the results in the previous section to obtain this basis together with its multiplicative properties. In view of Proposition 17, it follows that this basis can not be closed under nonzero products of its elements if A has complex nonreal eigenvalues. However this defect is not too serious since we can obtain a basis such that any nonzero product of basis elements is either a basis element or the negative of a basis element. In this way our previous work lifts to this setting. Moreover our basis for the centralizer of $C(A)$ over the reals has pleasant "conjugation properties".

By the same considerations as in the previous section, it is sufficient to consider now the blocks in its Jordan form associated with its eigenvalues $\lambda_1, \ldots, \lambda_k$. If a λ_i is a real number, one can use the same basis for that block as done in the previous section. If λ_i is a complex number with nonzero imaginary part, then we treat its block together with the block associated with its complex conjugate $\overline{\lambda_i}$ simultaneously. These two blocks must have the same size $m \times m$ and have the same centralizer invariants, so they are the same $m \times m$ matrix except for the elements on their main diagonal where one has λ_i and the other $\overline{\lambda_i}$. We demonstrate how to handle these two blocks now.

It is useful to consider an example which is representative of the general situation. Consider the following 10×10 matrix obtained from two such blocks.

$$A = \begin{bmatrix} \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \overline{\lambda} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \overline{\lambda} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \overline{\lambda} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \overline{\lambda} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \overline{\lambda} \end{bmatrix}.$$

We have the following basis elements for $C(A)$ as from the previous section given by

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

Consider labeling the diagonal elements of A by λ_1, λ_2, λ_3, λ_4, λ_5, $\overline{\lambda_1}$, $\overline{\lambda_2}$, $\overline{\lambda_3}$, $\overline{\lambda_4}$, $\overline{\lambda_5}$. Consider the permutation σ of $\{1, \ldots, 10\}$ which moves these diagonal elements to the following successive positions down the main diagonal λ_1, $\overline{\lambda_1}$, λ_2, $\overline{\lambda_2}$, λ_3, $\overline{\lambda_3}$, λ_4, $\overline{\lambda_4}$, λ_5, $\overline{\lambda_5}$ so $\sigma = (6, 2, 3, 5, 9, 8)(4, 7)$. First we apply a permutation of the rows and columns of A using a permutation matrix P given by the permutation σ applied to the identity matrices rows and P^T which comes from σ applied to the columns of I, i.e., to obtain $B = PAP^T$ so that entry $a_{i,j}$

of A is moved to position $\sigma(i), \sigma(j)$ in B. So in our example we have that

$$B = \begin{bmatrix} \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \overline{\lambda} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \overline{\lambda} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \overline{\lambda} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \overline{\lambda} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \overline{\lambda} \end{bmatrix}.$$

Next we use a special similarity mapping of B given by

$$C = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{21}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}.$$

This matrix C is an involution, i.e., $CC = I$.

$$CBC = \begin{bmatrix} \alpha & \beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\beta & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & \alpha & \beta & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -\beta & \alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \alpha & \beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\beta & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\beta & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \alpha & \beta \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\beta & \alpha \end{bmatrix},$$

if $\lambda = \alpha + \beta i$.

This matrix CBC is the real form of the Jordan matrix of A, denote it by $J_r(A) = CP(J(A)P^TC)$ as in [1]. Now we will obtain our basis for $J_r(A)$ by applying the same similarity map to our basis for $A = J(A)$ encoded in the above matrix M above. However, as you will see, the resulting basis elements have both complex entries and so will need to be decomposed into their real part and their imaginary part. We use the same notation for the basis encoded in M as in the previous section for the block for the eigenvalue λ, namely $\gamma_{i,j}(r)$ but the corresponding basis element for the block for the eigenvalue $\overline{\lambda}$ we use $\overline{\gamma}_{i,j}(r)$.

Consider now $\gamma_{1,1}(3) = E_{1,1} + E_{2,2} + E_{3,3}$ so now $P\gamma_{1,1}(3)P^T = E_{1,1} + E_{3,3} + E_{5,5}$. So next

$$C(E_{1,1} + E_{3,3} + E_{5,5})C = \begin{bmatrix} \frac{1}{2} & \frac{-i}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{i}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{-i}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{-i}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{i}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now carrying out the same computation for $\overline{\gamma}_{1,1}(3) = E_{6,6} + E_{7,7} + E_{8,8}$. First $P\overline{\gamma}_{1,1}(3)P^T = E_{2,2} + E_{4,4} + E_{6,6}$. Next we obtain

$$C(E_{2,2} + E_{4,4} + E_{6,6})C = \begin{bmatrix} \frac{1}{2} & \frac{i}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-i}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{i}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-i}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{i}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-i}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let us denote $CPQP^TC$ by $f(Q)$. So now we would like to select the real part of $f(\gamma_{1,1}(3))$ as a basis element for $C_r(A)$ but it is the same as the real part of $f(\overline{\gamma}_{1,1}(3))$ and since we prefer entries in our basis elements to be units we finally decided to define for $C_5(A)$ the basis element $\gamma^*_{1,1}(6) = f(\gamma_{1,1}(3)) + f(\overline{\gamma}_{1,1}(3))$.

But now looking at the imaginary parts of $f(\gamma_{1,1}(3))$ and $f(\overline{\gamma}_{1,1}(3))$, we define for $C_r(A)$ another basis element denoted by $\overline{\gamma}^*_{1,1}(6) = \frac{1}{i}(f(\overline{\gamma}_{1,1}(3)) - f(\gamma_{1,1}(3)))$. Now one can verify the following multiplicative properties of $\gamma^*_{1,1}(6)\gamma^*_{1,1}(6) = \gamma^*_{1,1}(6)$, $\overline{\gamma}^*_{1,1}(6)\overline{\gamma}^*_{1,1}(6) = -\gamma^*_{1,1}(6)$, and $\gamma^*_{1,1}(6)\overline{\gamma}^*_{1,1}(6) = \overline{\gamma}^*_{1,1}(6)\gamma^*_{1,1}(6) = \overline{\gamma}^*_{1,1}(6)$.

Next we consider $f(\gamma_{1,1}(2))$ and $f(\overline{\gamma}_{1,1}(2))$.

$$f(\gamma_{1,1}(2)) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{-i}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{i}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{-i}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$f(\overline{\gamma}_{1,1}(2)) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{i}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-i}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{i}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-i}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

So now we define two more corresponding basis elements for $C_r(A)$, namely $\gamma^*_{1,1}(4) = f(\gamma_{1,1}(2)) + f(\overline{\gamma}_{1,1}(2))$ and $\overline{\gamma}^*_{1,1}(4) = \frac{1}{i}(f(\overline{\gamma}_{1,1}(2)) - f(\gamma_{1,1}(2)))$.

One can check that the above situation is quite general. One defines in $C_r(A)$ in addition to the above four basis elements two more for each $\gamma_{i,j}(r)$ for the block for λ. So $\gamma^*_{1,1}(2) = f(\gamma_{1,1}(1)) + f(\overline{\gamma}_{1,1}(1))$, $\overline{\gamma}^*_{1,1}(2) = \frac{1}{i}(f(\overline{\gamma}_{1,1}(1)) - f(\gamma_{1,1}(1)))$, $\gamma^*_{2,2}(4) = f(\gamma_{1,1}(2)) + f(\overline{\gamma}_{1,1}(2))$, $\overline{\gamma}^*_{2,2}(4) = \frac{1}{i}(f(\overline{\gamma}_{2,2}(2)) - f(\gamma_{2,2}(2)))$, $\gamma^*_{2,2}(2) = f(\gamma_{2,2}(1)) + f(\overline{\gamma}_{2,2}(1))$, $\overline{\gamma}^*_{2,2}(2) = \frac{1}{i}(f(\overline{\gamma}_{2,2}(1)) - f(\gamma_{2,2}(1)))$, $\gamma^*_{1,2}(4) = f(\gamma_{1,2}(2)) + f(\overline{\gamma}_{1,2}(2))$, $\overline{\gamma}^*_{1,2}(4) = \frac{1}{i}(f(\overline{\gamma}_{1,2}(2)) - f(\gamma_{1,2}(2)))$, $\gamma^*_{1,2}(2) = f(\gamma_{1,2}(1)) + f(\overline{\gamma}_{1,2}(1))$, $\overline{\gamma}^*_{1,2}(2) = \frac{1}{i}(f(\overline{\gamma}_{1,2}(1)) - f(\gamma_{1,2}(1)))$, $\gamma^*_{2,1}(4) = f(\gamma_{2,1}(2)) + f(\overline{\gamma}_{2,1}(2))$, $\overline{\gamma}^*_{2,1}(4) = \frac{1}{i}(f(\overline{\gamma}_{2,1}(2)) - f(\gamma_{2,1}(2)))$, $\gamma^*_{2,1}(2) = f(\gamma_{2,1}(1)) + f(\overline{\gamma}_{2,1}(1))$, and $\overline{\gamma}^*_{2,1}(2) = \frac{1}{i}(f(\overline{\gamma}_{2,1}(1)) - f(\gamma_{2,1}(1)))$ which total 18 in number which is exactly the same size as the basis for the two blocks associated with λ and $\overline{\lambda}$. Geometrically, we can represent these basis elements as follows. First the following matrix represents all matrices in $C_r(CBC)$ by assigning to the scalars arbitrary real numbers and by setting one scalar equal to 1 and all the others equal to 0 one obtains exactly the basis listed above with $\gamma^*_{i,j}$ following the "diagonals" down while $\overline{\gamma}^*_{i,j}$ follow "diagonals" going up,e.g., $a = 1$ and others 0 gives $\gamma^*_{1,1}(6)$ and $b = 1$ and the others 0 gives $\overline{\gamma}^*_{1,1}(6)$:

$$M_r = \begin{bmatrix} a & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -b & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ c & d & a & b & 0 & 0 & r & s & 0 & 0 \\ -d & c & -b & a & 0 & 0 & -s & r & 0 & 0 \\ e & f & c & d & a & b & t & u & r & s \\ -f & e & -d & c & -b & a & -u & t & -s & r \\ v & w & 0 & 0 & 0 & 0 & g & h & 0 & 0 \\ -w & v & 0 & 0 & 0 & 0 & -h & g & 0 & 0 \\ x & y & v & w & 0 & 0 & p & q & g & h \\ -y & x & -w & v & 0 & 0 & -q & p & -h & g \end{bmatrix}.$$

Now we do the above for each block in the Jordan canonical form of the initial real matrix A to obtain a basis for $C_{\mathbb{R}}(A)$ over the reals consisting entirely of real matrices. For real eigenvalues this basis is the same as that obtained in section one and for a complex eigenvalue and its conjugate we follow the strategy just outlined in the above example. The total number of such basis elements is of course exactly the same as for $C(A)$ in the first section. The multiplication table for these new basis elements is given by the next result of course the product between ones from different blocks is always zero except for the ones involved with λ and $\overline{\lambda}$.

23 Definition. Let λ and $\overline{\lambda}$ be two eigenvalues of A in its Jordan canonical form J over the complexes. One assumes that these blocks Q associated with λ and $\overline{\lambda}$ are adjacent to each other as in our example. Let $J_r(Q) = CPQP^TC$ where P is the permutation matrix associated with rearranging the eigenvalues λ and $\overline{\lambda}$ alternating along the main diagonal in exactly the same manner as done above in our example and C is the "involution" matrix on these blocks of the correct size analogous to our C above. We define $\gamma^*_{i,j}(2r) = f(\gamma_{i,j}(r)) + f(\overline{\gamma}_{i,j}(r))$ and $\overline{\gamma}^*_{i,j}(2r) = \frac{1}{i}(f(\overline{\gamma}_{i,j}(r)) - f(\gamma_{i,j}(r)))$ where $f(X) = CPXP^TC$ and $\gamma_{i,j}$ and $\overline{\gamma}_{i,j}$ are the basis elements of the separate blocks of Q as in the second section. We continue the convention that $\gamma^*_{i,j}(2r) = 0$ ($\overline{\gamma}^*_{i,j}(2r) = 0$) if $r \leq 0$:

24 Theorem. *Let Q be two blocks in the Jordan canonical matrix associated with the eigenvalues λ and $\overline{\lambda}$ of a matrix with real entries in standard position, P is the permutation matrix for Q illustrated above, and C is the involution matrix as above of the appropriate size for this J. We obtain a basis* $\{\, \gamma^*_{i,j}(2r) :$

$1 \le i,j \le m$ *and* $1 \le r \le \min\{\, t(i), t(j)\,\}\} \cup \{\, \overline{\gamma}^*_{i,j}(2r) : 1 \le i,j \le m$ *and* $1 \le r \le \min\{\, t(i), t(j)\,\}\,\}$ *for* $J_r(CPQP^TC)$ *over the real numbers where* $t(j)$ *is the function used in the previous section giving the size of the elementary Jordan blocks for* λ*. Moreover, these basis elements have products given as follows:*

(1) $\gamma^*_{i,j}(2r)\gamma^*_{j,k}(2s) = \gamma^*_{i,k}(2(r+s-t(j)))$,

(2) $\overline{\gamma}^*_{i,j}(2r)\overline{\gamma}^*_{j,k}(2s) = -\gamma^*_{i,k}(2(r+s-t(j)))$,

(3) $\gamma^*_{i,j}(2r)\overline{\gamma}^*_{j,k}(2s) = \overline{\gamma}^*_{i,k}(2(r+s-t(j)))$,

(4) $\overline{\gamma}^*_{i,j}(2r)\gamma^*_{j,k}(2s) = \overline{\gamma}^*_{i,k}(2(r+s-t(j)))$, and

(5) $\gamma^*_{i,h}(2r)\gamma^*_{j,k}(2s) = 0$, $\gamma^*_{i,h}(2r)\overline{\gamma}^*_{j,k}(2s) = 0$, $\overline{\gamma}^*_{i,h}(2r)\overline{\gamma}^*_{j,k}(2s) = 0$, and

(6) $\overline{\gamma}^*_{i,h}(2r)\gamma^*_{j,k}(2s) = 0$ when $h \ne j$.

PROOF. We check a few of these but they all follow from the multiplication tables from Proposition 5 $\gamma^*_{i,j}(2r)\gamma^*_{j,k}(2s) = (f(\gamma_{i,j}(r)) + f(\overline{\gamma}_{i,j}(r)))(f(\gamma_{j,k}(s)) + f(\overline{\gamma}_{j,k}(s))) = f(\gamma_{i,k}(r+s-t(j)) + \overline{\gamma}_{i,k}(r+s-t(j))) = \gamma^*_{i,k}(2(r+s-t(j)))$. $\gamma^*_{i,j}(2r)\overline{\gamma}^*_{j,k}(2s) = (f(\gamma_{i,j}(r)) + f(\overline{\gamma}_{i,j}(r)))\,\frac{1}{i}(f(\overline{\gamma}_{j,k}(s)) - f(\gamma_{j,k}(s))) = \frac{1}{i}(f(\overline{\gamma}_{i,k}(r+s-t(j))) - f(\gamma_{i,k}(r+s-t(j))))) = \overline{\gamma}^*_{i,k}(2(r+s-t(j)))$ $\overline{\gamma}^*_{i,j}(2r)\overline{\gamma}^*_{j,k}(2s) = \frac{1}{i}(f(\overline{\gamma}_{i,j}(r)) - f(\gamma_{i,j}(r)))\,\frac{1}{i}(f(\overline{\gamma}_{j,k}(s)) - f(\gamma_{j,k}(s))) = -(f(\overline{\gamma}_{i,k}(r+s-t(j))) + f(\gamma_{i,k}(r+s-t(j))))) = -\gamma^*_{i,k}(2(r+s-t(j)))$. The rest is similar. $\boxed{QED}$

One obtains an easy characterization of the center $ZC_r(CPJP^TC)$ and by [4] it consists of all polynomials $p(CPQP^TC)$ where $p(x)$ is a polynomial with real coefficients.

25 Corollary. *A basis of dimension* $2t(1)$ *for the center of* $C_r(CPQP^TC)$ *as above is given by* $\alpha = \sum_{i=1}^m \gamma^*_{i,i}(2(t(i))), \overline{\alpha} = \sum_{i=1}^m \overline{\gamma}^*_{i,i}(2(t(i)))$*, together with* $1 \le j < t(1), \beta_j = \sum_{i=1}^m \gamma^*_{i,i}(2j)$ *and* $\overline{\beta_j} = \sum_{i=1}^m \overline{\gamma}^*_{i,i}(2j)$*.*

It is interesting to evaluate powers of $CPQP^TC$ since one can obtain them efficiently from these results using the notation in the previous result.

26 Corollary. *Let* $J_r(Q) = CPQP^TC$*. For* $1 \le n < t(1)$*,*

$$(J_r(Q))^n = \mathrm{Re}(\lambda^n)\alpha + \mathrm{Im}(\lambda^n)\overline{\alpha} + \sum_{j=1}^{n} \binom{n}{j} \mathrm{Re}(\lambda^{n-j})\beta_j + \sum_{j=1}^{n} \binom{n}{j} \mathrm{Im}(\lambda^{n-j})\overline{\beta}_j$$

and for $t(1) \le n$*,*

$$(J_r(Q))^n = \mathrm{Re}(\lambda^n)\alpha + \mathrm{Im}(\lambda^n)\overline{\alpha} + \sum_{j=1}^{t(1)} \binom{n}{j} \mathrm{Re}(\lambda^{n-j})\beta_j + \sum_{j=1}^{t(1)} \binom{n}{j} \mathrm{Im}(\lambda^{n-j})\overline{\beta}_j.$$

This is what one does for the two blocks in the Jordan canonical form of a real matrix associated with eigenvalues λ and $\overline{\lambda}$. One carries this out for all the blocks in the Jordan canonical form $J = SAS^{-1}$ of a real matrix A and combining them with the natural direct sum. In this way we obtain **the canonical basis for $C_r(CPJP^TC)$ over the real numbers**. By a similarity map we obtain **the canonical basis for $C_{\mathbb{R}}(A)$ for a real matrix** A which has the property that a nonzero product of elements of this basis is a unit in $\mathbb{R}$ times another basis element. In view of Proposition 17 this is the best result one could hope for.

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