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Generalized secant varieties of projective varieties

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Abstract. Let $X \subset \mathbf{P}^r$ be an integral projective variety. For any integer $t > 0$ let $S^{\{t\}}(X)$ be the closure in $\mathbf{P}^r$ of the union of all $t-1$ linear spaces spanned by a length t zero-dimensional subscheme of X (a generalization of the secant variety of X). Usually $S^{\{t\}}(X)$ is reducible when X is singular. Here we prove that quite often $S^{\{t\}}(X)$ is reducible, even if X is smooth (but of dimension at least 3).

Keywords: secant variety, Hilbert scheme, zero-dimensional scheme

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1 Introduction

Let $X, Y \subseteq \mathbf{P}^r$ be integral non-degenerate varieties. Let $[X; Y]$ denote the join of X and Y, i. e. if $X = Y = \{P\}$ are the same point, set $[X; Y] := \{P\}$, while in all other cases $[X; Y]$ denotes the closure in $\mathbf{P}^r$ of the union of all lines $\langle P, Q\rangle$ spanned by $P \in X$, $Q \in Y$ with $P \neq Q$. Hence $[X; Y]$ is always irreducible. Set $S^0(X) := X$. For all integers $t \geq 1$ define inductively the t-secant variety $S^t(X)$ of X by the formula $S^t(X) := [X; S^{t-1}(X)]$. Hence each $S^t(X)$ is irreducible. For more on secant varieties, see [1], [2], [3] and references therein. In this paper we consider a generalized t-secant variety $S^{\{t\}}(X)$. Set $S^0(X) := X$. For all integers t such that $1 \leq t \leq r$ let $S^{\{t\}}(X)$ denote the closure in $\mathbf{P}^r$ of the union of all t-dimensional linear subspaces spanned by a closed subscheme of X. Notice that any such subscheme of X must contain a zero-dimensional subscheme with length at least $t+1$ and spanning that subspace. As in the classical case we take the closure of the union of the linear subspaces spanned by their intersection with X, not the closure of the union of all t-dimensional linear subspaces containing at least a length $t+1$ zero-dimensional subscheme of X for the following reason.

1 Remark. Let $X \subset \mathbf{P}^r$ be an integral variety, X not a line. Assume the existence of a line D such that either $D \subset X$ or $D \cap X$ is a zero-dimensional scheme with length at least 3. For any $P \in \mathbf{P}^r \backslash D$ the plane $\langle \{P\} \cup D \rangle$ contains a subscheme of X with length at least 3.

We have $S^{\{r\}}(X) = \mathbf{P}^r$ because X is assumed to be non-degenerate. Set $S^{\{t\}}(X) := \mathbf{P}^r$ for all $t > r$.

It is easy to see that $S^{\{t\}}(X)$ may be reducible and with dimension not bounded only in terms of the integers t and $\dim(X)$. For instance, if X has a unique singular point, P, then $S^{\{1\}}(X) = S^1(X) \cup T_P X$, where $T_P X \subseteq \mathbf{P}^r$ denotes the embedded Zariski tangent space to X at P. Thus if X is singular, quite often $S^{\{t\}}(X)$ is reducible. The aim of this note is to prove the following theorem which shows that quite often $S^{\{t\}}(X)$ is reducible, even if X is smooth. To avoid misunderstandings for any algebraic scheme T, let $\mathrm{maxdim}(T)$ denote the maximal dimension of an irreducible component of T_{red}.

2 Theorem. *Set $c_3 := 102$, $c_4 := 25$, $c_5 := 35$ and $c_n := (n+1)(1+n/4)$ if $n \geq 5$. Fix integers n, t, r such that $n \geq 3$, $t \geq c_n$ and $r \geq t(n+1)$. Let $X \subset \boldsymbol{P}^r$ be an integral non-degenerate n-dimensional variety such that every length $z \leq 2t$ zero-dimensional subscheme of X spans a $(z-1)$-dimensional linear subspace of $\boldsymbol{P}^r$. Then* $\mathrm{maxdim}(S^{\{t\}}(X)) \geq (n+1)t$ *and in particular $S^{\{t\}}(X)$ is reducible. More precisely, if $t > (2n^2)^n/n!$, then*

$$\mathrm{maxdim}(S^{\{t\}}(X)) \geq \min\{\, r, t-1+(t^{2-2/n}(n!/2)^{-2/n}(n^2/32))\,\} \qquad (1)$$

3 Remark. The assumption " every length $z \leq 2t$ zero-dimensional subscheme of X spans a $(z-1)$-dimensional linear subspace of $\mathbf{P}^r$ " in Theorem 2 is rather restrictive, but it is satisfied when the inclusion $X \subset \mathbf{P}^r$ is obtaining from an inclusion of X in a projective space $\mathbf{P}^m$ with the Veronese embedding of some order $x \geq 2t-1$ of $\mathbf{P}^m$.

4 Remark. When $n = 2$ and X is smooth all $S^{\{t\}}(X)$ are irreducible by [4]. When $n \geq 3$ and X is smooth, $S^t(X)$ contains the closure $S^{\{t\}}(X)_*$ of the union of all $(t-1)$-dimensional linear subspace spanned by a length t curvilinear subscheme of X because any curvilinear subscheme of a smooth variety is the flat limit of sets of t distinct points of X. We believe that it would be interesting to study $S^{\{t\}}(X)_*$ for singular varieties X. It should be a mix of an elementary part of motivic integration and projective properties of X. We think that even the case of normal surfaces may give some tool for the study of normal singularities.

We work over an algebraically closed field $\mathbb{K}$.

PROOF OF THEOREM 2. For $z = t$ the assumption made in the statement of Theorem 2 implies that every length t zero-dimensional subscheme $Z \subset X$ spans a $t-1$ dimensional linear subspace $\langle Z \rangle$. The same assumption for $z = t+1$ implies $X \cap \langle Z \rangle = Z$ (scheme-theoretically). The same assumption for $z = 2t$ implies $\langle Z \rangle \cap \langle A \rangle = \emptyset$ for all length t zero-dimensional subschemes Z, A of X

such that $Z \neq A$. Thus $\mathrm{maxdim}(S^{\{t\}}(X)) = t - 1 + \mathrm{maxdim}(\mathrm{Hilb}^t(X))$. Hence the theorem follows from [5] (see [5], eq. (11), for the inequality (1)). *QED*

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