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Torus cobordant complex projective spaces

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Abstract. In this paper we investigate equivalence of torus cobordant complex projective spaces. In linear case it was shown that torus cobordant complex projective spaces are equivalent.

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1 Introduction

In the study of transformation groups on cohomology projective spaces, those elementary abelian groups (i.e. connected torus and Z_p-tori) play a crucial role.

It is well known that if a connected torus acts on a cohomology complex projective space then connected components of the fixed point set are again cohomology complex projective spaces. See J. C. Su [4]. Furthermore one can define a geometric weight system for torus actions on cohomology complex projective spaces similar to the case of acyclic manifolds. See Hsiang [2].

Let T be a connected torus and $\varphi : T \to U(n+1)$ be a complex representation of T. Then there is an induced T-action on CP^n which is called linear. It is well known that a complex representation of a connected torus is determined by its weight system. Weights can be considered to be elements of $H^2(B_T, Q)$. This weight system can be chosen as a geometric weight system of induced linear action on CP^n.

The cohomological aspects of orbit structures of elementary abelian transformation groups on cohomology projective spaces have been shown to be the same as those of suitable linear models. In this paper all cohomologies are sheaf cohomology over Q.

1 Theorem. *Let T be a connected torus and X compact $T-$cobordism such that $\partial X = X_0 \sqcup X_1 = CP^n \sqcup CP^n$. Assume that inclusions $i : X_0 \hookrightarrow X$ and $j : X_1 \hookrightarrow X$ induce isomorphisms $H^*(X, Q) \approx H^*(X_k, Q), k = \{0, 1\}$ Assume also that the induced actions on X_0 and X_1 are linear. Then these actions are equivalent.*

PROOF. Let X_1, X_2 be compact G manifolds. Let's define $(G, X_1) \sim (G, X_2)$ if there is a compact G manifold (with boundary) X such that $\partial X = X_1 \sqcup X_2$ and $X_1 \hookrightarrow X$ and $X_2 \hookrightarrow X$ induces isomorphisms in cohomology. This is an equivalence relation. This theorem shows that if two linear spaces (T, CP^n) are equivalent in this sense then these two representations are equivalent and thus equivariantly diffeomorphic. Let B_G be the classifying space of G and $X_G = (X \times E_G)/G$ where $E_G \to B_G$ is the universal principal $G-$bundle. Let $H^*_G(X) = H^*(X_G)$ be the equivariant cohomology ring of X. Recall that (X, A) totally nonhomologous to zero in (X_G, A_G) with respect to $H^*(-)$ if $H^*_G(X, A) \to H^*(X, A)$ is surjective. Note that if G a compact connected Lie group, (X, A) a G space, and if $H^*(X, A)$ is zero in odd degrees then the Leray-Serre spectral sequence associated to $(X, A) \to (X_G, A_G) \to B_G$ degenerates for formal reasons. In addition, since G acts trivially on $H^*(X, A)$, (X, A) totally nonhomologous to zero in (X_G, A_G). So (X, X_0) obviously totally non homologous to zero in $(X_T, (X_0)_T)$. Similarly (X, X_1) totally nonhomologous to zero in $(X_T, (X_1)_T)$. Therefore the inclusions $i : X_0^T \hookrightarrow X^T$ and $j : X_1^T \hookrightarrow X^T$ induce isomorphisms $H^*(X_k^T, Q) \approx H^*(X^T, Q), k = 0, 1$.

This is essentially Theorem 1.6 of Bredon [1, Chapter VII]. The proof in Bredon is for the case where $G = S^1$ or Z_p. But one gets the result for higher rank tori and $p-$tori using induction. He must assume finitistic orbit space for S^1 action but this is now known to be true by Deo, Tripathi [3].

Let φ_0 (respectively φ_1) denote the restricted action on X_0 (resp. X_1). Since $\varphi_0(\varphi_1)$ is linear then there is a complex representation $\widetilde{\varphi_0} : T \to U(n+1)$ $(\widetilde{\varphi_1} : T \to U(n+1))$ with the weight system $\Omega(\widetilde{\varphi_0}) = \{ w_i, m_i; 1 \leq i \leq s \}$ $(\Omega(\widetilde{\varphi_1}) = \{ w'_j, m'_j; 1 \leq j \leq r \})$ such that induced action on X_0 (X_1) is φ_0 (φ_1) $(r = s$ because it is the number of connected components of the fixed point set).

Consider the Serre-spectral sequence of fibration $S^1 \to S^{2n+1}_T \to CP^n_T$. Let α be a generator of $H^1(S^1)$, then α is transgressive. Let $\tau(\alpha) = a \in H^2_T(CP^n)$ be the transgression of α then $H^*_T(X_0) \simeq R[a]/ < f(a) >, R = H^*(B_T)$ where $f(a) = \prod_{i=1}^s (a - w_i)^{m_i}$ is the defining polynomial of $H^*((X_0)_T)$ as an algebra over $R = H^*(B_T)$. See Hsiang [2].

Let $F_1, \ldots, F_s$ be the connected components of fixed point set X_0^T, it is well known that $H^*(F_i) = H^*(CP^{m_i - 1})$. Consider the commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 0 & \to & H^2(B_T) & \to & H^2_T(X_0) & \to & H^2(X_0) & \to & 0 \\ & & \| & & \simeq\uparrow i^2_T & & \simeq\uparrow i^2 & & \\ 0 & \to & H^2(B_T) & \to & H^2_T(X) & \to & H^2(X) & \to & 0 \end{array}$$

Let $b = (i^2_T)^{-1}(a)$ then $H^*_T(X) \simeq R[b]/ < g(b) >$ where $g(b) = \prod_{i=1}^s (b - w''_i)^{m''_i}$ is defining polynomial of $H^*(X_T)$

Since the inclusion $X_0^T \hookrightarrow X^T$ induces the isomorphism $H^*(X_0^T) \approx H^*(X^T)$ there is one to one correspondence of connected components. Furthermore if $F_1'', \dots, F_s''$ connected components of X^T then $H^*(F_i'') = H^*(CP^{m_i-1})$. So: $g(b) = \prod_{i=1}^{s}(b - w_i'')^{m_i}$.

Let $q_j \in F_j \subseteq F_j''$ be an arbitrary point. This inclusion induces an isomorphism $H^*(F_j) \approx H^*(F_j'') = H^*(CP^{m_j-1})$. In addition $q_j \in F_j \subset X_0^T \subset X_0$ induces homomorphism, $H_T^*(X_0) \to H_T^*(q_j)$, such that the image of a is w_j. So consider the commutative diagram

$$\begin{array}{ccc} H_T^*(X_0) & \rightarrow & H_T^*(q_j) \\ \simeq\uparrow i_T^* & & \| \\ H_T^*(X) & \rightarrow & H_T^*(q_j) \end{array}$$

It is easily seen that $w_j = w_j''$. So:

$$f(a) = \prod_{i=1}^{s}(a - w_i)^{m_i}$$

and

$$g(b) = \prod_{i=1}^{s}(b - w_i)^{m_i}$$

On the other hand consider the commutative diagram

$$\begin{array}{ccc} H_T^2(X) & \underset{\simeq}{\overset{j_T^2}{\rightarrow}} & H_T^2(X_1) \\ i_T^2\downarrow\simeq & & \tau\uparrow \\ H_T^2(X_0) & \overset{\tau}{\leftarrow} & H^1(S^1) \end{array}$$

Let $c = j_T^2(b) \in H_T^2(X_1)$ then $H_T^*(X_1) \simeq R[c]/ < h(c) >$, $h(c) = \prod_{i=1}^{s}(c - w_i')^{m_i}$. Similarly it can be seen that $h(c) = \prod_{i=1}^{s}(c - w_i)^{m_i}$. Thus $\Omega(\widetilde{\varphi_0}) = \Omega(\widetilde{\varphi_1}) = \{\, w_i, m_i; 1 \le i \le s \,\}$. So $\widetilde{\varphi_0}$ and $\widetilde{\varphi_1}$ are linearly equivalent. Therefore φ_0 and φ_1 are equivalent. QED

References

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