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# A lattice characterization of groups with finite torsion-free rank

**Rocco De Luca**
*Dipartimento di Matematica e Applicazioni, Università Di Napoli Federico II, via Cintia, I - 80126 Napoli (Italy)*
rocco.deluca@dma.unina.it

**Francesco de Giovanni**
*Dipartimento di Matematica e Applicazioni, Università Di Napoli Federico II, via Cintia, I - 80126 Napoli (Italy)*
degiovan@unina.it

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**Abstract.** A group $G$ has finite torsion-free rank if it has a series of finite length whose factors are either infinite cyclic or periodic. A lattice-theoretic characterization of groups with finite torsion-free rank is obtained; it follows in particular that the class of such groups is invariant under projectivities. Moreover, a lattice description of radical groups with finite abelian section rank is given.

**Keywords:** subgroup lattice, torsion-free rank

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## 1 Introduction

Let $G$ and $\bar{G}$ be groups. A *projectivity* from $G$ onto $\bar{G}$ is an isomorphism from the lattice $\mathfrak{L}(G)$ of all subgroups of $G$ onto the subgroup lattice $\mathfrak{L}(\bar{G})$ of $\bar{G}$, and a class $\mathfrak{X}$ of groups is *invariant under projectivities* if all projective images of $\mathfrak{X}$-groups likewise belong to $\mathfrak{X}$. Obviously, if a group class $\mathfrak{X}$ can be described by means of only purely lattice concepts, then $\mathfrak{X}$ is invariant under projectivities. Lattice-theoretic characterizations of many relevant group classes are known; among many others, for a group $G$ each of the properties of being simple, perfect, hyperabelian, polycyclic or supersoluble can be detected from the subgroup lattice. As cyclic subgroups can be recognized in the lattice of all subgroups of a group, it is also clear that groups with finite Prüfer rank have a lattice theoretic description, and the aim of this short paper is to prove that other finiteness conditions on the ranks produce group classes for which it is possible to give a lattice-theoretic characterization.

A group $G$ is said to have *finite torsion-free rank* if there exists a series of finite length

$$\{1\} = G_0 < G_1 < \cdots < G_t = G$$

whose factors are either infinite cyclic or periodic; the number $r_0(G)$ of infinite cyclic factors in such a series is an invariant, called the *torsion-free rank* of $G$. Clearly, if $G$ is a group and $N$ is any normal subgroup of $G$, then $G$ has finite torsion-free rank if and only if both groups $N$ and $G/N$ have finite torsion-free rank, and in this case the equality $r_0(G) = r_0(N) + r_0(G/N)$ holds. We shall prove that groups with finite torsion-free rank can be described by the behaviour of their subgroup lattices, and hence they form a group class which is invariant under projectivities. As an application of this result, we will also give a lattice characterization of radical groups with finite abelian section rank. Recall here that a group $G$ has *finite abelian section rank* if no infinite abelian groups of prime exponent occur as sections of $G$.

Most of our notation is standard and can be found in [4]; for definitions and properties concerning arbitrary lattices and lattices of subgroups, we refer to the monograph [5].

## 2 Some lattice preliminaries

Let $\mathfrak{L}$ be a lattice with least element 0 and greatest element $I$. Recall that an element $c$ of $\mathfrak{L}$ is *cyclic* if the interval $[c/0]$ is a distributive lattice satisfying the maximal condition. Moreover, an element $x$ of $\mathfrak{L}$ is *covered irreducibly* by elements $x_1, \dots, x_t$ of the interval $[x/0]$ if, for each cyclic element $c$ of $[x/0]$, there is $i \leq t$ such that $c \leq x_i$ and the set $\{x_1, \dots, x_t\}$ is minimal with respect to such property. Clearly a subgroup $H$ of a group $G$ is covered irreducibly in the lattice $\mathfrak{L}(G)$ by its subgroups $H_1, \dots, H_t$ if and only if $H$ is the set-theoretic union of $H_1, \dots, H_t$ and none of these subgroups can be omitted from the covering.

An element $a$ of $\mathfrak{L}$ is said to be *cofinite* if there exists in $\mathfrak{L}$ a finite chain

$$a = a_0 < a_1 < \cdots < a_n = I$$

such that (for every $i = 0, 1, \dots, n-1$) $a_i$ is a maximal element of the lattice $[a_{i+1}/0]$ and satisfies one of the following conditions:

- $a_{i+1}$ is covered irreducibly by finitely many elements $b_1, \dots, b_{n_i}$ of $\mathfrak{L}$ such that $b_1 \wedge \cdots \wedge b_{n_i} \leq a_i$;
- for every automorphism $\varphi$ of the lattice $[a_{i+1}/0]$, the element $a_i \wedge a_i^{\varphi}$ is modular in $[a_{i+1}/0]$ and the lattice $[a_{i+1}/a_i \wedge a_i^{\varphi}]$ is finite.

An element $a$ of $\mathfrak{L}$ is said to be *permodular* if it satisfies the following conditions:

- $a$ is a modular element of $\mathfrak{L}$;

- if $c$ is a cyclic element of $\mathfrak{L}$ and $b$ is an element of $[a \vee c/a]$ such that the lattice $[a \vee c/b]$ is finite, then $b$ is a cofinite element of $[a \vee c/0]$.

The above definition is suggested by the behaviour of images of normal subgroups of a group $G$ under a projectivity $\varphi : \mathfrak{L}(G) \to \mathfrak{L}(\bar{G})$. All such images are *permodular subgroups* of $\bar{G}$ (in the sense that they are permodular elements of the lattice $\mathfrak{L}(\bar{G})$). The concept of a permodular subgroup was introduced by Zacher [8], and has played a central role in most lattice characterizations of classes of infinite groups closed under projectivities. It is important to observe that, within the universe of locally (soluble-by-finite) groups, modular and permodular subgroups coincide (see [6, Theorem 1]). In the lattice $\mathfrak{L}(G)$ of all subgroups of a group $G$ an element $X$ is cofinite if and only if the index $|G : X|$ is finite (see [5, Theorem 6.1.10]); thus if $X$ is a permodular subgroup of a group $G$, for each element $g$ of $G$ and for each subgroup $Y$ of $G$ such that $X \leq Y \leq \langle g, X\rangle$, the finiteness of the lattice $[\langle g, X\rangle/Y]$ is equivalent to that of the index $|\langle g, X\rangle : Y|$. It follows that if $X$ is a permodular subgroup of $G$ and $H$ is any subgroup of $G$ containing $X$ and such that the interval $[H/X]$ is finite, then $X$ has finite index in $H$.

We shall say that a lattice $\mathfrak{L}$ with $0, I$ is *periodic* if the lattice $[a/0]$ is finite for each cyclic element $a$ of $\mathfrak{L}$; clearly, a group has periodic subgroup lattice if and only if it is periodic.

**1 Lemma.** *Let $G$ be a group and let $X$ be a permodular subgroup of $G$. The lattice $[G/X]$ is periodic if and only if for each element $g$ of $G$ there is a positive integer $k$ such that $g^k \in X$.*

Proof. Suppose first that $[G/X]$ is a periodic lattice, and let $g$ be any element of $G$. As $X$ is a modular subgroup of $G$, the intervals $[\langle g, X\rangle/X]$ and $[\langle g\rangle/\langle g\rangle \cap X]$ are isomorphic, and hence $\langle g, X\rangle$ is a cyclic element of $[G/X]$. Thus $[\langle g, X\rangle/X]$ is finite, so that the group $\langle g\rangle/\langle g\rangle \cap X$ is likewise finite and there is a positive integer $k$ such that $g^k$ belongs to $X$. Assume by contradiction that the converse statement is false, and let $C$ be a cyclic element of the lattice $[G/X]$ such that $[C/X]$ is infinite. Consider in $C$ a subgroup $M$ containing $X$ which is maximal with respect to the condition that $[M/X]$ is finite, and let $c$ be an element of $C \setminus M$; put $K = \langle c, M\rangle$. As $[M/X]$ is finite, there exist finitely many elements $a_1, \dots, a_t$ of $M$ such that $M = \langle a_1, \dots, a_t, X\rangle$. Then $K = \langle c, a_1, \dots, a_t, X\rangle$ and so the index $|X^K : X|$ is finite (see [5, Lemma 6.2.8]). Moreover, the group $K/X^K$ is cyclic, since its subgroup lattice is isomorphic to a sublattice of $[C/X]$, and hence it is finite. Therefore the index $|K : X|$ is finite, and so the interval $[K/X]$ is finite, contradicting the choice of $M$. This contradiction proves the lemma. QED

Our next two results are direct consequences of Lemma 1.

**2 Corollary.** *Let $G$ be a group and let $X$ be a permodular subgroup of $G$ such that $[G/X]$ is periodic. If $Y$ is any permodular subgroup of $G$ containing $X$, then also the lattice $[G/Y]$ is periodic.*

**3 Corollary.** *Let $G$ be a group and let $X$ and $Y$ be permodular subgroups of $G$ such that $X \leq Y$. If $[G/Y]$ and $[Y/X]$ are periodic lattices, then also the interval $[G/X]$ is periodic.*

A lattice $\mathfrak{L}$ with $0, I$ is called *torsion-free* if $[a/0]$ is infinite for every cyclic non-zero element $a$ of $\mathfrak{L}$; also in this case it follows directly from the definition that a group is torsion-free if and only if it has torsion-free subgroup lattice.

**4 Lemma.** *Let $G$ be a group and let $X$ be a permodular subgroup of $G$. If the lattice $[G/X]$ is torsion-free, then $X$ is normal in $G$.*

Proof. Let $g$ be any element of $G \setminus X$. As $[\langle g\rangle/\langle g\rangle \cap X] \simeq [\langle g, X\rangle/X]$ is infinite, we have that $g$ has infinite order and $\langle g\rangle \cap X = \{1\}$. Thus $X^g = X$ (see [5, Lemma 6.2.3]), and hence $X$ is normal in $G$. QED

## 3 Lattice characterizations

It follows from the definition that if $N$ is a normal subgroup with finite torsion-free rank of a group $G$ and $G/N$ is periodic, then $G$ has finite torsion-free rank and $r_0(G) = r_0(N)$. The first lemma of this section extends such property to modular subgroups of soluble groups.

**5 Lemma.** *Let $G$ be a soluble group and let $X$ be a modular subgroup of $G$ with finite torsion-free rank. If the lattice $[G/X]$ is periodic, then $G$ has finite torsion-free rank and $r_0(G) = r_0(X)$.*

Proof. The statement is obvious if $X$ is normal in $G$, so that without loss of generality it can be assumed that $G$ is not abelian. Let $K$ be the smallest non-trivial term of the derived series of $G$. As the lattice $[G/XK]$ is periodic by Corollary 2, by induction on the derived length of $G$ we have that $G/K$ has finite torsion-free rank and $r_0(G/K) = r_0(XK/K)$. On the other hand, $K/K \cap X$ is periodic, so that $K$ has finite torsion-free rank and $r_0(K) = r_0(K \cap X)$. It follows that $G$ has finite torsion-free rank and

$$r_0(G) = r_0(K) + r_0(G/K) = r_0(K \cap X) + r_0(XK/K) = r_0(X).$$

The lemma is proved. QED

Let $G$ be a group and let $X$ be a subgroup of $G$. Recall that the *isolator* $I_G(X)$ of $X$ in $G$ is the set of all $g \in G$ such that $g^n \in X$ for some positive integer $n$. It is well known that the isolator of any subgroup of a locally nilpotent group is a subgroup; for this and other properties of isolators in locally nilpotent groups we refer to [3, Section 2.3]. For our pourposes, we need a result on permutable subgroups of locally nilpotent groups. Recall that a subgroup $X$ of a group $G$ is *permutable* if $XH = HX$ for each subgroup $H$ of $G$. Clearly, all permutable subgroups are permodular and in any locally nilpotent group permutable and permodular subgroups coincide (see [5, Theorem 6.2.10]).

**6 Lemma.** *Let $G$ be a torsion-free locally nilpotent group, and let $X$ be an abelian permutable subgroup of $G$. Then $X$ is normal in $G$.*

Proof. Consider the isolator $I_G(X)$ of $X$ in $G$. If $I_G(X) = G$, the group $G$ is abelian (see [3, 2.3.9]) and the statement is obvious. Suppose that $I_G(X)$ is a proper subgroup of $G$, so that $G$ is generated by the set of all elements of infinite order $g$ of $G$ such that $\langle g\rangle \cap X = \{1\}$; since $X^g = X$ for all such elements $g$ (see [5, Lemma 5.2.7]), it follows that $X$ is normal in $G$. QED

**7 Corollary.** *Let $G$ be a group with no periodic non-trivial normal subgroups, and let $X$ be an abelian permutable subgroup of $G$. Then $X$ is subnormal in $G$ with defect at most* 2.

Proof. As $X$ is ascendant in $G$, its normal closure $X^G$ is a torsion-free locally nilpotent group (see [4, Part 1, Theorem 2.31]). Thus $X$ is normal in $X^G$ by Lemma 6, and hence it is subnormal in $G$ with defect at most 2. QED

Our main result provides a lattice-theoretic characterization of groups with finite torsion-free rank, and so it shows in particular that the class of such groups is invariant under projectivities.

**8 Theorem.** *A group $G$ has finite torsion-free rank if and only if there exists in $G$ a finite chain*

$$\{1\} = X_0 < X_1 < \cdots < X_t = G$$

*of permodular subgroups such that each interval $[X_{i+1}/X_i]$ is either periodic or a torsion-free distributive lattice with the maximal condition. In this case, the torsion-free rank of $G$ is the number of torsion-free intervals in such a chain.*

Proof. Every group with finite torsion-free rank has of course the property described in the statement. Conversely, suppose first that $G$ is a soluble group with this property. Let $s$ be the number of torsion-free intervals in the chain

$$\{1\} = X_0 < X_1 < \cdots < X_t = G,$$

and let $k \leq t$ be the smallest non-negative integer such that the interval $[G/X_k]$ is a periodic lattice. Clearly, it can be assumed that $G$ is not periodic, i. e. that $k > 0$. Then $[X_k/X_{k-1}]$ is torsion-free by Corollary 3, so that $X_{k-1}$ is a normal subgroup of $X_k$ by Lemma 4 and $X_k/X_{k-1}$ is infinite cyclic. By induction on $s$ we have that $X_{k-1}$ has finite torsion-free rank and $r_0(X_{k-1}) = s - 1$. It follows that $X_k$ has finite torsion-free rank and $r_0(X_k) = s$. Application of Lemma 5 yields finally that also $G$ has finite torsion-free rank and $r_0(G) = r_0(X_k) = s$.

In the general case, observe that if $N$ is any normal subgroup of $G$ and $[X_{i+1}/X_i]$ is a torsion-free lattice, then the interval $[X_{i+1}N/X_iN]$ is infinite if and only if $X_i \cap N = X_{i+1} \cap N$ (because Lemma 4 yields that $X_i$ is normal in $X_{i+1}$ and $X_{i+1}/X_i$ is infinite cyclic); thus $s = s_1 + s_2$, where $s_1$ is the number of torsion-free non-trivial intervals in the chain

$$\{1\} = X_0 \cap N \leq X_1 \cap N \leq \cdots \leq X_t \cap N = N$$

and $s_2$ is the number of torsion-free non-trivial intervals in the chain

$$\{1\} = X_0N/N \leq X_1N/N \leq \cdots \leq X_t/N = G/N.$$

Consider the second derived subgroup $G''$ of $G$. Since any permodular subgroup $X$ of $G$ is permutable in $XG''$ (see [5, Theorem 6.2.19]), we have that $X_i \cap G''$ is a permutable subgroup of $G''$ for each $i = 0, 1, \ldots, t$. As by the soluble case the group $G/G''$ has finite torsion-free rank and $r_0(G/G'')$ is the number of torsion-free non-trivial intervals in the chain

$$\{1\} = X_0G''/G'' \leq X_1G''/G'' \leq \cdots \leq X_t/G'' = G/G'',$$

it is enough to show that the statement holds for the group $G''$ with the chain

$$\{1\} = X_0 \cap G'' \leq X_1 \cap G'' \leq \cdots \leq X_t \cap G'' = G''.$$

Thus replacing $G$ by $G''$ it can be assumed without loss of generality that the subgroups $X_0, X_1, \ldots, X_t$ are permutable in $G$. Clearly, we can also factor out the largest periodic normal subgroup of $G$, and suppose that $G$ has no periodic non-trivial normal subgroups. Thus $X_1$ must be infinite cyclic, and so $X_1$ is normal in $X_1^G$ by Corollary 7. By induction on $s$, it can be assumed that the groups $X_1^G/X_1$ and $G/X_1^G$ have finite torsion-free rank and

$$r_0(X_1^G/X_1) + r_0(G/X_1^G) = s - 1.$$

Therefore $G$ itself has finite torsion-free rank and $r_0(G) = s$. The theorem is proved. QED

Recall that a group $G$ is called *radical* if it has an ascending series with locally nilpotent factors, or equivalently if the upper Hirsch–Plotkin series of $G$ terminates with $G$. It has been proved by Baer and Heineken [2] that a radical group has finite abelian section rank if and only if it is hyperabelian and all its abelian subgroups have finite $p$-rank for $p = 0$ or a prime. On the other hand, the class of hyperabelian groups has a lattice description (see [5, Theorem 6.4.7]), and hence by Theorem 8 the following result provides a lattice characterization of the class of radical groups with finite abelian section rank.

**9 Corollary.** *Let $G$ be a hyperabelian group and let $T$ be the join of all its periodic permodular subgroups. Then $G$ has finite abelian section rank if and only if it has finite torsion-free rank and all indecomposable modular intervals $[X/\{1\}]$ of $\mathfrak{L}(T)$ satisfy the minimal condition.*

PROOF. Suppose first that $G$ has finite abelian section rank, so that in particular $G$ has finite torsion-free rank (see [4, Part 2, Lemma 9.34]). Let $X$ be any subgroup of $T$ such that $\mathfrak{L}(X)$ is an indecomposable modular lattice. As $T$ is the largest periodic normal subgroup of $G$ (see [5, Theorem 6.5.17]), the subgroup $X$ is periodic and so it contains a primary subgroup of finite index (see [5, Theorem 2.4.13]). On the other hand, every primary subgroup of $G$ is a Černikov group (see [7]), and hence the lattice $\mathfrak{L}(X)$ satisfies the minimal condition. Conversely, if the group $G$ has the property described in the statement, every primary abelian subgroup of $T$ satisfies the minimal condition and so $T$ has finite abelian section rank. Moreover, the factor group $G/T$ has finite Prüfer rank (see [4, Part 2, Lemma 9.34]), and hence $G$ itself has finite abelian section rank. QED

A hyperabelian group $G$ is said to be an $\mathcal{S}_1$*-group* if it has finite abelian section rank and $\pi(G)$ is finite (here $\pi(G)$ denotes the set of all prime numbers $p$ for which $G$ has elements of order $p$). It is well known that any $\mathcal{S}_1$-group has a characteristic series $\{1\} \leq T \leq K \leq G$ such that $T$ is a Černikov group, $K/T$ is torsion-free and $G/K$ is finite (see [4, Part 2, Theorem 10.33]). Therefore $\mathcal{S}_1$-groups admit a lattice description, which depends on the following result.

**10 Corollary.** *A hyperabelian group $G$ has the property $\mathcal{S}_1$ if and only if the largest periodic permodular subgroup $T$ of $G$ is a Černikov group and $G/T$ contains a torsion-free subgroup $K/T$ of finite Prüfer rank such that the index $|G : K|$ is finite.*

The class of $\mathcal{S}_1$-groups contains of course all soluble minimax groups: a group $G$ is called *minimax* if it has a series of finite length

$$\{1\} = G_0 < G_1 < \cdots < G_t = G$$

whose factors satisfy either the minimal or the maximal condition. We mention here that also the class of soluble minimax groups can be characterized by means of properties of the subgroup lattice. In fact, Baer [1] and Zaicev [9], [10] independently proved that a soluble group $G$ is minimax if and only if it satisfies the *weak minimal condition*, i. e. if in $G$ there are no infinite descending chain of subgroups $X_1 > X_2 > \dots$ in which each index $|X_i : X_{i+1}|$ is infinite; this is actually a lattice condition, because, as we already mentioned, the finiteness of the index of a subgroup can be detected in the subgroup lattice.

The last part of the paper deals with a somewhat different type of group class. Let $\mathfrak{H}$ be the class of all groups whose periodic sections are locally finite.

Clearly, all locally (soluble-by-finite) groups belong to $\mathfrak{H}$, and the imposition of the property $\mathfrak{H}$ seems to be a useful tool in the study of many classes of infinite unsoluble groups. In particular, it turns out that any $\mathfrak{H}$-group with finite torsion-free rank is periodic-by-soluble-by-finite (see for instance [3, p.209]).

It is well known that a finitely generated group $G$ is soluble if and only if it has a finite chain of permodular subgroups

$$\{1\} = X_0 < X_1 < \cdots < X_t = G$$

such that each interval $[X_{i+1}/X_i]$ is a permodular lattice (see [5, Theorem 6.4.8]). It follows that also locally (soluble-by-finite) groups can be described using lattice properties. Our next result provides a lattice-theoretic characterization of the class of groups whose periodic sections are locally finite.

**11 Theorem.** *A group $G$ belongs to the class $\mathfrak{H}$ if and only if whenever $X$ and $Y$ are subgroups of $G$ such that $X$ is permodular in $Y$ and $[Y/X]$ is a periodic lattice, then $[\langle y_1, \dots, y_t, X\rangle/X]$ is finite for all elements $y_1, \dots, y_t$ of $Y$.*

Proof. Clearly, in any group with the lattice property described in the statement all periodic sections are locally finite. Conversely, suppose that the group $G$ satisfies this latter condition, and let $X$ and $Y$ be subgroups of $G$ such that $X$ is permodular in $Y$ and $[Y/X]$ is periodic. Consider elements $y_1, \dots, y_t$ of $Y$, and put $H = \langle y_1, \dots, y_t, X\rangle$. Then $X$ has finite index in its normal closure $K = X^H$ (see [5, Lemma 6.2.8]), so that there is a positive integer $e$ such that $K^e$ is contained in $X$ and in particular $X/X_H$ is periodic. As the interval $[H/X]$ is periodic, it follows from Lemma 1 that $H/X_H$ is likewise periodic, and so locally finite. In particular, the group

$$\langle y_1, \dots, y_t\rangle/\langle y_1, \dots, y_t\rangle \cap X_H \simeq \langle y_1, \dots, y_t\rangle X_H/X_H$$

is finite and hence also the lattice

$$[\langle y_1, \dots, y_t, X\rangle/X] \simeq [\langle y_1, \dots, y_t\rangle/\langle y_1, \dots, y_t\rangle \cap X]$$

is finite. The statement is proved. QED

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