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Ropes on linear subspaces of a projective space

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Abstract. In this paper, we study good ropes supported by linear spaces of dimension ≥ 2. At first, we show that these schemes have some nice properties (for example, they all are locally Cohen-Macaulay), then we investigate the problem of extending a rope supported by a line to a good rope supported by a linear space of dimension ≥ 2. In particular, when the linear space is a plane, we study the problem of extending a rope supported by a line to a good rope with stable conormal bundle.

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1 Introduction

Let Y be a smooth, integral curve in $\mathbb{P}^n$, where with curve we mean a closed, locally Cohen-Macaulay subscheme of pure dimension 1. Multiple structures C supported by Y, that is to say, non-reduced curves such that $C_{\text{red}} = Y$ have been studied with respect to various properties. In particular, ropes supported by a smooth, integral curve Y have been investigated in [4], [15], [6], where a rope is

1 Definition. Fix an integer b such that $b \geq 2$. A *b-rope* C is a projective scheme of degree $\deg(C) = b\deg(Y)$ which verifies

(1) $C_{\text{red}} = Y$ is an irreducible, nondegenerate smooth curve;

(2) the ideal sheaf $\mathcal{I}_{Y,C}$ has $\mathcal{I}_{Y,C}^2 = 0$ and so it is an $\mathcal{O}_Y$-module;

(3) $\mathcal{I}_{Y,C}$ is locally free of rank $b-1$ over Y.

In particular, if the smooth curve Y is a line L, the ropes supported by L can be studied via a naturally associated exact sequence

$$0 \to \oplus_{j=1}^{n-b}\mathcal{O}_L(-\beta_j - 1) \to \mathcal{O}_L(-1)^{\oplus(n-1)} \to \oplus_{i=1}^{b-1}\mathcal{O}_L(\alpha_i - 1) \to 0 \qquad (1)$$

where $\alpha_i, 1 \leq i \leq b-1$, and $\beta_j, 1 \leq j \leq n-b$, are non-negative integers (see [16]). Conversely, each exact sequence (1) with arbitrary non-negative integers α_i and β_j uniquely determines a rope $C \subset \mathbb{P}^n$ supported by the line L (see [16]).

Ropes supported by lines were studied in detail in [16] and [18] (their cohomological properties, their moduli spaces, components of the Hilbert scheme $\text{Hilb}(\mathbb{P}^n)$ of $\mathbb{P}^n$ whose general member is a rope, and so on). In [18] the integers $\alpha_i, 1 \leq i \leq b-1$, (resp. $\beta_j, 1 \leq j \leq n-b$) are called the $\underline{\alpha}$-type (resp. $\underline{\beta}$-type) of C.

It is possible to study multiple structures supported by schemes of dimension ≥ 2 but then it is very important to specify the properties the resulting scheme has to verify (for example, see [14], [13]).

In this paper we study "good" ropes $X \subset \mathbb{P}^{n+x-1}$ such that X_{red} is a linear space $Z \subset \mathbb{P}^{n+x-1}$ of dimension x (see Definition 3 below). We describe some properties they have, and we consider the problem of extending ropes supported by lines to ropes supported by linear spaces of higher dimension, that is to say, we consider a b-rope $C \subset \mathbb{P}^n \cong H \subset \mathbb{P}^{n+x-1}$ supported by the line L, and we look for a good b-rope X supported by a linear space Z of dimension x such that $L = Z \cap H$ and $C = X \cap H$.

The plan of the paper is the following.

In Section 2, we define good ropes supported by linear spaces, both abstract and embedded, and we show that they are always split ropes, and locally Cohen-Macaulay schemes. Moreover, we construct a natural parameter space for the good embedded ropes and we compute the cohomology of such ropes in terms of the parameters.

In Section 3, we study the problem of extending ropes supported by lines to good ropes supported by linear spaces of dimension $x \geq 2$. In particular, we prove that there exist good extensions under suitable hypotheses on x and on the degree of the rope we extend (see Theorem 15 below).

In Section 4, we focus on the case $x = 2$. Using results on the existence of vector bundles of rank 2 on $\mathbb{P}^2$ with expected properties, we construct corresponding good embedded ropes of degree 3 supported by a plane. Furthermore, we prove the following theorem about the extension of ropes

2 Theorem. *Assume $0 < b < n$ and $n - b \geq 3$. Fix $b - 1$ non-negative integers $\alpha_1, \ldots, \alpha_{b-1}$, and a b-rope $C \subset \mathbb{P}^n \subset \mathbb{P}^{n+1}$ of $\underline{\alpha}$-type $\alpha_1, \ldots, \alpha_{b-1}$ supported by a line L. There exists an integer γ depending only on the integer $\max\{\alpha_i | 1 \leq i \leq b-1\}$ and on the integer b such that for every integer $c_2 \geq \gamma$*

there is a good b-rope X supported by a plane Z containing L and a hyperplane H of $\mathbb{P}^{n+1}$ such that $C = X \cap H, L = Z \cap H$ and the conormal module $\mathcal{E}^*$ of X is a stable vector bundle with $c_2(\mathcal{E}^*) = c_2$.

In last Section 5, we give existence results for good ropes supported by $3-$dimensional linear spaces, using arguments like in Section 4.

We always work over an algebraically closed base field. In Sections 4 and 5 we will assume characteristic zero because we will heavily use [9], §7, and [10].

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2 "Good" abstract and embedded ropes

In this section, we give the definition of abstract and embedded rope supported by a linear space, we prove that these schemes are split and locally Cohen-Macaulay, and that there is a natural parameter space for them. At last, we compute their cohomology in terms of the parameters.

To start with, we define a good abstract rope, analogously to ropes supported by curves (see Definition 1).

3 Definition. Fix integers x and b with $x \geq 1$ and $b \geq 2$. A *good abstract b-rope on $\mathbb{P}^x$* is a scheme X with $X_{\text{red}} = \mathbb{P}^x$ and such that, calling $\mathcal{I}$ the ideal sheaf of X_{red} in X, we have $\mathcal{I}^2 = 0$ and, seeing $\mathcal{I}$ as a coherent sheaf on $\mathbb{P}^x$, the $\mathcal{O}_{\mathbb{P}^x}$-sheaf $\mathcal{I}$ is locally free of rank $b-1$.

It follows from the definition that there is an exact sequence of $\mathcal{O}_X$- modules

$$0 \to \mathcal{E}^* \to \mathcal{O}_X \to \mathcal{O}_Z \to 0 \tag{2}$$

with $Z = X_{\text{red}}$, and $\mathcal{E}^* = \mathcal{I}$. By Definition 3, $\mathcal{E}^*$ is a locally free $\mathcal{O}_Z$-module with $\text{rank}(\mathcal{E}^*) = b-1$.

4 Definition. The vector bundle $\mathcal{E}^*$ is called the *conormal module* of the good rope X.

Among the good abstract ropes, we will focus on the split ones.

5 Definition. A good abstract b-rope X is a *split b-rope* if there exists a retraction $\psi : X \to Z$ of the inclusion of Z in X.

Now, we define the good embedded b-ropes supported by linear spaces.

6 Definition. Fix an integer $n \geq b+2$. A closed subscheme $X \subset \mathbb{P}^{n+x-1}$ is a *good b-rope supported by the x-dimensional linear space $Z \subset \mathbb{P}^{n+x-1}$* if there exists a good abstract b-rope X' on $\mathbb{P}^x$ and an emdedding $j : X' \to \mathbb{P}^{n+x-1}$ such that $j(X'_{\text{red}}) = Z$ and $j(X') = X$.

It is clear from its definition that a good rope is always embedded.

Of course, the previous definition is equivalent to say that there is a very ample line bundle $\mathcal{O}_X(1)$ on X with $\deg(\mathcal{O}_X(1)|Z) = 1$. As in the case of a good abstract b-rope, we call $\mathcal{I}_{Z,X}$ the conormal module of X and we denote it as $\mathcal{E}^*$. Notice that $\deg(X) = \operatorname{rank}(\mathcal{E}^*) + 1 = b$.

As first result on such linear ropes, we prove that they are always split ropes.

7 Lemma. *Let $X \subset \mathbb{P}^{n+x-1}$ be a good b-rope supported by the x-dimensional linear space Z. Then X is a good split b-rope supported by Z.*

Proof. At first, we construct a retraction of the inclusion of Z in X.

To this end, take a linear subspace $W \subset \mathbb{P}^{n+x-1}$ with $\dim(W) = n-2$ and $W \cap Z = \emptyset$. Let $h : \mathbb{P}^{n+x-1} \setminus W \to \mathbb{P}^x$ be the linear projection from W. The morphism h induces an isomorphism $h_{|Z} : Z \to \mathbb{P}^x$.

Set $\psi = (h_{|Z})^{-1} \circ (h_{|X}) : X \to Z$.

From its definition, it follows that $\psi_{|Z} = id_Z$, and so the constant function 1 on Z is the restriction of the corresponding one on X. Furthermore, if $j : Z \to X$ is the inclusion, then $\psi \circ j = id_Z$, and so ψ is a retraction of the inclusion j. QED

8 Remark. Using ψ, we can define an $\mathcal{O}_Z$-module structure on $\mathcal{O}_X$. But, $\mathcal{E}^*$ is a locally free $\mathcal{O}_Z$-module, and so the sequence (2) becomes an exact sequence of $\mathcal{O}_Z$-modules. Because of the properties of ψ, the sequence (2) is split exact.

9 Corollary. *Let $X \subset \mathbb{P}^{n+x-1}$ be a good b-rope supported by the x-dimensional linear space Z. Then X is locally Cohen-Macaulay.*

Proof. By definition, Z is a linear space of dimension x, and so it is arithmetically Cohen-Macaulay. Then, Z has depth x at each of its closed points. In the proof of Lemma 7, we constructed a retraction $\psi : X \to Z$ with $\psi_*(\mathcal{O}_X) \cong \mathcal{O}_Z \oplus \mathcal{E}^*$, where $\mathcal{E}^*$ is a locally free $\mathcal{O}_Z$-module. Then, $\psi_*(\mathcal{O}_X)$ is a locally free $\mathcal{O}_Z$-module, and so X has depth x at each closed point of Z. But X and Z have the same closed points and so we have the claim. QED

It follows from the definition of good rope and from the sequence (2) that there is another exact sequence of $\mathcal{O}_Z$-modules associated to a good rope. In fact, we have the following remark.

10 Remark. Let X be a good b-rope supported by the x-dimensional linear space $Z \subset \mathbb{P}^{n+x-1}$ with $\mathcal{E}^*$ as conormal module. Both Z and X are embedded in $\mathbb{P}^{n+x-1}$ and so we have the two exact sequences

$$0 \to \mathcal{I}_Z \to \mathcal{O}_{\mathbb{P}^{n+x-1}} \to \mathcal{O}_Z \to 0$$

and

$$0 \to \mathcal{I}_X \to \mathcal{O}_{\mathbb{P}^{n+x-1}} \to \mathcal{O}_X \to 0.$$

There is a map of complexes induced from the exact sequence (2) that induces a surjective map $\mathcal{I}_Z \to \mathcal{E}^*$. Then, X is equipped with a surjective map

$$\phi_A : \mathcal{O}_Z(-1)^{\oplus(n-1)} \to \mathcal{E}^*,$$

which follows from Z being a complete intersection.

Set $\mathcal{K} = \ker(\phi_A)$ and let $\phi_B : \mathcal{K} \to \mathcal{O}_Z(-1)^{\oplus(n-1)}$ be the associated inclusion. Then, we have the following exact sequence

$$0 \to \mathcal{K} \xrightarrow{\phi_B} \mathcal{O}_Z(-1)^{\oplus(n-1)} \xrightarrow{\phi_A} \mathcal{E}^* \to 0. \tag{3}$$

If Z is a line L, then the sequence (3) becomes the exact sequence (1).

As in the case $\dim(Z) = 1$, we can use the exact sequence (3) to construct a natural parameter space for the good b-ropes.

11 Theorem. *There is a 1-to-1 correspondence between the good b-ropes supported by a linear space $Z \subset \mathbb{P}^{n+x-1}$ of dimension x and the pairs $(\mathcal{E}^*, \phi_A)$ up to isomorphism where $\operatorname{rank}(\mathcal{E}^*) = b-1$ and $\phi_A : \mathcal{O}_Z(-1)^{\oplus(n-1)} \to \mathcal{E}^*$ is a surjective map.*

Proof. If X is a good b-rope supported by Z, we naturally have a surjective map $\phi_A : \mathcal{O}_Z(-1)^{\oplus(n-1)} \to \mathcal{E}^*$, as constructed in previous Remark 10. It is evident that the pair $(\mathcal{E}^*, \phi_A)$ is unique up to isomorphism.

Vice versa, in $\mathbb{P}^{n+x-1}$ we have the isomorphism $\mathcal{I}_Z/\mathcal{I}_Z^2 \cong \mathcal{O}_Z(-1)^{\oplus(n-1)}$ and so using the exact sequence (3) we construct an ideal sheaf $\mathcal{I}_X$ such that $\mathcal{I}_Z^2 \subset \mathcal{I}_X \subset \mathcal{I}_Z$ which defines a good b-rope X supported by Z having $(\mathcal{E}^*, \phi_A)$ as associated pair. QED

12 Remark. Obviously, a rank k vector bundle $\mathcal{K}$ on Z and an injective map $\phi_B : \mathcal{K} \to \mathcal{O}_Z(-1)^{\oplus(n-1)}$ with locally free cokernel (i. e. a map $\phi_B : \mathcal{K} \to \mathcal{O}_Z(-1)^{\oplus(n-1)}$ with rank k at each point of Z) uniquely determines a good $(n-k)$-rope X over Z with $\operatorname{coker}(\phi_B)$ as conormal module.

The pair $(\mathcal{E}^*, \phi_A)$ determines the cohomology of the good rope X. In fact, we have

13 Proposition. *Let $X \subset \mathbb{P}^{n+x-1}$ be a good b-rope supported by the linear space Z of dimension x, and let $(\mathcal{E}^*, \phi_A)$ be a pair associated to X as in Remark 10. Then, $H^i_*(\mathcal{I}_X) \cong H^{i-1}_*(\mathcal{E}^*)$ for $2 \le i \le x$, while $H^1_*(\mathcal{I}_X) \cong \operatorname{coker}(H^0(\phi_A) : S(-1)^{\oplus(n-1)} \to H^0_*(\mathcal{E}^*))$ where $S = R/I_Z$ and the isomorphisms are of R-modules.*

Proof. From the exact sequence (2), we have the exact sequence

$$0 \to \mathcal{I}_X \to \mathcal{I}_Z \xrightarrow{\varphi} \mathcal{E}^* \to 0.$$

Taking the associated long exact cohomology sequence, we get that

$$H^i_*(\mathcal{I}_X) \cong H^{i-1}_*(\mathcal{E}^*) \qquad \text{for } 2 \leq i \leq x$$

because $H^i_*(\mathcal{I}_Z) = 0$ for $i = 1, \dots, x$, and

$$0 \to I_X \to I_Z \xrightarrow{H^0(\varphi)} H^0_*(\mathcal{E}^*) \to H^1_*(\mathcal{I}_X) \to 0,$$

where I_X and I_Z are the saturated ideals defining X and Z, respectively. Then, $H^1_*(\mathcal{I}_X) = \operatorname{coker}(H^0(\varphi))$.

Set ϕ' the composition of the isomorphism $\mathcal{I}_Z/\mathcal{I}_Z^2 \cong \mathcal{O}_Z(-1)^{\oplus(n-1)}$ and ϕ_A. We have that ϕ' and φ have the same image because $I_Z^2 \subset I_X$ and the map $I_X \to I_Z$ is the inclusion. Hence, $H^0(\phi_A)$ and $H^0(\varphi)$ have the same image, and so we have the claim. QED

3 Extension of ropes supported by lines

In this section we study the problem of extending a rope supported by a line to a good rope supported by a linear space of higher dimension, that is to say, we find conditions to assure that a rope supported by a line is the intersection of a good rope supported by a linear space of dimension ≥ 2 with a linear space of the right dimension. The answer to this problem depends on the constraints imposed to the conormal bundle $\mathcal{E}^*$.

14 Definition. Let $Z \subset \mathbb{P}^{n+x-1}$ be an x-dimensional linear space. A good b-rope X supported by Z and given by an exact sequence (3) will be called $\underline{\alpha}$-*split* where $\underline{\alpha} = (\alpha_1, \dots, \alpha_{b-1})$ (resp. $\underline{\alpha}$-*stable*, or $\underline{\beta}$-*split*, or $\underline{\beta}$-*stable*, where $\underline{\beta} = (\beta_1, \dots, \beta_{n-b})$) if $\mathcal{E}^*(1) = \oplus_{i=1}^{b-1}\mathcal{O}_Z(\alpha_i)$ (resp. $\mathcal{E}^*$ is stable, or $\mathcal{K}(1) = \oplus_{i=1}^{n-b}\mathcal{O}_Z(-\beta_i)$, or $\mathcal{K}$ is stable).

Then, $\underline{\alpha}$ and $\underline{\beta}$ can be seen as the splitting types of $\mathcal{E}^*(1)$ and $\mathcal{K}(1)$, respectively.

Now, we prove that under suitable hypotheses on the dimension of Z and the degree of the rope, every rope supported by a line (and hence $\underline{\alpha}$-split) can be extended to a good $\underline{\alpha}$-split rope supported by a linear space.

15 Theorem. *Fix integers n, x, b with $2 \leq x < n - b < n$. Let $Z \subset \mathbb{P}^{n+x-1}$ be an x-dimensional linear space and let $H \subset \mathbb{P}^{n+x-1}$ be a codimension $x - 1$ linear space such that $L = Z \cap H$ is a line. Let $C \subset H$ be a b-rope supported by the line L, with splitting type $\underline{\alpha} = (\alpha_1, \dots, \alpha_{b-1})$. Then there is a good $\underline{\alpha}$-split b-rope X on Z with $X \cap H = C$.*

Proof. The rope C supported by L is defined via the exact sequence (1)

$$0 \to \oplus_{j=1}^{n-b}\mathcal{O}_L(-\beta_j - 1) \to \mathcal{O}_L(-1)^{\oplus(n-1)} \xrightarrow{\phi_{A,L}} \oplus_{i=1}^{b-1}\mathcal{O}_L(\alpha_i - 1) \to 0.$$

The theorem is equivalent to the existence of a surjective map

$$\phi_A : \mathcal{O}_Z(-1)^{\oplus(n-1)} \to \oplus_{i=1}^{b-1}\mathcal{O}_Z(\alpha_i - 1)$$

such that $\phi_{A,L} = \phi_{A|L}$, that is to say, ϕ_A is an extension of $\phi_{A,L}$. The set of all such extensions is parameterized by the vector space

$$H^0(Z, \mathcal{I}_{L,Z} \otimes \mathrm{Hom}(\mathcal{O}_Z(-1)^{\oplus(n-1)}, \oplus_{i=1}^{b-1}\mathcal{O}_Z(\alpha_i - 1))).$$

Notice that the vector bundle $\mathrm{Hom}(\mathcal{O}_Z(-1)^{\oplus(n-1)}, \oplus_{i=1}^{b-1}\mathcal{O}_Z(\alpha_i - 1))$ is spanned by its global sections. This implies that for every $P \in Z$ the set of all extensions of $\phi_{A,L}$ not surjective at P has codimension at least $n-1-(b-1) = n-b$ in $H^0(Z, \mathcal{I}_{L,Z} \otimes \mathrm{Hom}(\mathcal{O}_Z(-1)^{\oplus(n-1)}, \oplus_{i=1}^{b-1}\mathcal{O}_Z(\alpha_i - 1)))$. Since $\dim(Z) = x < n-b$, we obtain that a general extension of $\phi_{A,L}$ is surjective at every point of Z, and so the claim follows. QED

16 Remark. A similar computation works for non $\underline{\alpha}$-split extensions. Let $\mathcal{E}^*$ be a vector bundle on Z. If $H^1(Z, \mathcal{I}_{L,Z} \otimes \mathrm{Hom}(\mathcal{O}_Z(-1)^{\oplus(n-1)}, \mathcal{E}^*)) = 0$, i. e. if $H^1(Z, \mathcal{I}_{L,Z} \otimes \mathcal{E}^*(1)) = 0$, we may lift any surjective map $\phi_{A,L} : \mathcal{O}_L(-1)^{\oplus(n-1)} \to \mathcal{E}^*|L$ to a morphism $\phi_A : \mathcal{O}_Z(-1)^{\oplus(n-1)} \to \mathcal{E}^*$. If $x < n-b$ and $\mathcal{E}^*$ is spanned by its global sections, the proof of Theorem 15 shows that the general lifting ϕ_A is surjective and hence it defines a good b-rope supported by Z extending the given b-rope supported by L.

17 Remark. Let $X \subset \mathbb{P}^{n+x-1}$ be a good b-rope supported by an x-dimensional linear space Z and with conormal module $\mathcal{E}^*$. For every line $L \subset X$ the vector bundle $\mathcal{E}^*|L$ is the direct sum of $b-1$ line bundles on $L \cong \mathbb{P}^1$ and hence there are $b-1$ integers $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{b-1}$, with $\mathcal{E}^*|L \cong \oplus_{i=1}^{b-1}\mathcal{O}_L(\alpha_i - 1)$. The ordered integers $\alpha_i, 1 \leq i \leq b-1$, are uniquely determined by X and L. They are called the $\underline{\alpha}$-type of X with respect to the line L. Of course, we have $c_1(\mathcal{E}^*) = \alpha_1 + \cdots + \alpha_{b-1} - b + 1$, where $c_1(\mathcal{E}^*)$ is the first Chern class of $\mathcal{E}^*$. Moreover, if X is an $\underline{\alpha}$-split good b-rope, then the Chern classes of X are uniquely determined by the integers $x, \alpha_1, \ldots, \alpha_{b-1}$.

Analogously, if (3) is the exact sequence defining X, the splitting type $\beta_j, 1 \leq j \leq n-b$, of $\mathcal{K}(1)|L$ is called the $\underline{\beta}$-type of X with respect to L.

If we fix a linear space $H \subset \mathbb{P}^{n+x-1}$ with $\dim(H) = n$ and $Z \cap H = L$, and set $C = X \cap H$, the integers $\alpha_i, 1 \leq i \leq b-1$, and $\beta_j, 1 \leq j \leq n-b$, are the integers appearing in the exact sequence (1) associated to the b-rope C.

18 Remark. Fix integers n, b, x, with $2 \leq x \leq b-2$, and $b < n$. Fix non negative integers $\beta_1, \ldots, \beta_{n-b}$. Let $Z \subset \mathbb{P}^{n+x-1}$ be an x-dimensional linear space and $H \subset \mathbb{P}^{n+x-1}$ a codimension $x-1$ linear space such that $L = Z \cap H$ is a line. Consider an injective map $\phi_{B|L} : \oplus_{j=1}^{n-b}\mathcal{O}_L(-\beta_j - 1) \to \mathcal{O}_L(-1)^{\oplus(n-1)}$ giving the $\underline{\beta}$-type of a b-rope $C \subset H$ supported by L. By Remark 16 the condition

$x \leq b-2$ allows us to extend $\phi_{B|L}$ to an injective map $\phi_B : \oplus_{j=1}^{n-b} \mathcal{O}_Z(-\beta_j - 1) \to \mathcal{O}_Z(-1)^{\oplus(n-1)}$. Notice that $\mathcal{E}^* = \mathrm{coker}(\phi_B)$ is a rank $b-1$ vector bundle on Z equipped with a surjective map $\phi_A : \mathcal{O}_Z(-1)^{\oplus(n-1)} \to \mathcal{E}^*$ and hence it defines a good b-rope X over Z with $\mathcal{E}^*$ as conormal module. By construction X is a $\underline{\beta}$-split extension of C.

The case of 2-ropes (i. e. when the conormal module is a line bundle) is very easy and left to the interested reader. In this case the conormal module is uniquely determined by its restriction to any line. Taking the dual exact sequence as in Remark 18 one easily studies the case $n-b=1$, i. e. $(n-1)$-ropes supported by a codimension $n-1$ linear subspace of $\mathbb{P}^{n+x-1}$.

4 Ropes on a plane

In this section we will consider the case $x = 2$, i. e. the case of good b-ropes on a plane.

For all integers c_1, c_2, r with $r \geq 2$, let $\mathcal{M}(\mathbb{P}^2; c_1, c_2, r)$ be the moduli scheme of stable vector bundles on $\mathbb{P}^2$ with rank r and Chern classes c_1 and c_2. Assume $c_1 \geq -2$. It is known (see [3], [5] or [7]) that $\mathcal{M}(\mathbb{P}^2; c_1, c_2, 2)$ is smooth and irreducible or empty and that $\mathcal{M}(\mathbb{P}^2; 0, c_2, 2) \neq \emptyset$ if and only if $c_2 \geq 2$ while $\mathcal{M}(\mathbb{P}^2; -1, c_2, 2) \neq \emptyset$ if and only if $c_2 > 0$ (see [3], [12] or [8, Lemma 3.2]).

Now, we want to compute the twists for a rank 2 stable vector bundle $\mathcal{F}$ on $\mathbb{P}^2$ to be the conormal module of a good rope, of course of degree 3.

Recall that, if $\mathcal{F}$ is a vector bundle on $\mathbb{P}^2$, with Chern classes $c_1 = c_1(\mathcal{F})$, and $c_2 = c_2(\mathcal{F})$, then the Chern classes of $\mathcal{F}(x)$ are

$$c_1(\mathcal{F}(x)) = c_1 + 2x \qquad \text{and} \qquad c_2(\mathcal{F}(x)) = c_2 + xc_1 + x^2$$

(see e.g. [5, bottom of p. 469], or [9, Lemma 2.1], or [8, proof of 7.1]), while the Euler characteristic of $\mathcal{F}(x)$ is

$$\chi(\mathcal{F}(x)) = (c_1 + 2x + 1)(c_1 + 2x + 2)/2 + c_2 + xc_1 + x^2$$

(use Riemann-Roch or read [5, p. 470]).

Hence, to get a normalized vector bundle from $\mathcal{F}$ we have to twist $\mathcal{F}$ by $-a$, where $a = (c_1 - d_1)/2$, and $d_1 = 0$ if c_1 is even, $d_1 = -1$ otherwise. We set $d_1 = c_1(\mathcal{F}(-a))$ and $d_2 = c_2(\mathcal{F}(-a))$.

For a normalized rank 2 bundle $\mathcal{F}$ on $\mathbb{P}^2$ with Chern classes d_1, d_2 (and so $-1 \leq d_1 \leq 0$), we have that $H^0(\mathcal{F}(y)) \neq 0$ for every $y \geq y(d_1, d_2)$ where

$$y(d_1, d_2) = \min\{y | y \geq 0, (y+1)(y+2+d_1) > d_2\}$$

(see [8, Prop. 7.1]).

Now, we can state some lemmas.

19 Lemma. *Let $\mathcal{F}$ be a rank 2 stable vector bundle on $\mathbb{P}^2$ with Chern classes c_1, c_2. Assume $d_2 \geq 2 + d_1$ for the normalized bundle $\mathcal{F}(-a)$. For every integer $y \geq y(d_1, d_2) - a$ we have $h^0(\mathbb{P}^2, \mathcal{F}(y)) \neq 0$. Let t be the least integer with $0 \leq t \leq y(d_1, d_2)$ such that $h^0(\mathbb{P}^2, \mathcal{F}(t-a)) \neq 0$. We have $h^1(\mathbb{P}^2, \mathcal{F}(y-a)) = 0$ for every integer $y \geq a + d_2 - t^2 - 1$ and $\mathcal{F}(z-a)$ is spanned by its global sections for every integer $z \geq a + d_2 - t^2$.*

Proof. The lower bound for d_2 is equivalent to the non-emptiness of $\mathcal{M}(\mathbb{P}^2; d_1, d_2, 2)$ (see [3], [12] or [8, Lemma 3.2]). The inequality $h^0(\mathbb{P}^2, \mathcal{F}(t-a)) \neq 0$ is [8, Prop. 7.1]. The vanishing of $h^1(\mathbb{P}^2, \mathcal{F}(y-a))$ is [8, Theorem 7.4]. The spannedness of $\mathcal{F}(z-a)$ follows from Castelnuovo - Mumford's Lemma because we just proved that $h^1(\mathbb{P}^2, \mathcal{F}(z-a-1)) = 0$, while $h^2(\mathbb{P}^2, \mathcal{F}(z-a-2)) = 0$ by Serre duality and the stability of $\mathcal{F}$. QED

Now, we recall a result on general rank 2 stable vector bundles on $\mathbb{P}^2$. By [5, Corollary 5.2], if c_1 is even or its proof if c_1 is odd (see [11] or [7] or [2] for much more) for every integer x with $c_1 + 2x \geq -2$ we have $h^2(\mathbb{P}^2, \mathcal{F}(x)) = 0$ and at most one of the cohomology groups $h^0(\mathbb{P}^2, \mathcal{F}(x))$ and $h^1(\mathbb{P}^2, \mathcal{F}(x))$ is non-zero, i. e. we have $h^0(\mathbb{P}^2, \mathcal{F}(x)) = \max(0, \chi(\mathcal{F}(x)))$ and $h^1(\mathbb{P}^2, \mathcal{F}(x)) = \max(0, -\chi(\mathcal{F}(x)))$. Then, for a general rank 2 stable vector bundle on $\mathbb{P}^2$ we set

$$x(c_1, c_2) = \min\{x | c_1 + 2x > -2 \text{ and } \chi(\mathcal{F}(x)) \geq 0\}.$$

20 Lemma. *Let $\mathcal{F}$ be the general rank 2 stable vector bundle on $\mathbb{P}^2$, with Chern classes c_1, c_2. Assume $d_2 \geq d_1 + 2$ for the normalized vector bundle $\mathcal{F}(-a)$. We have $h^1(\mathbb{P}^2, \mathcal{F}(y)) = 0$ if and only if $y \geq x(c_1, c_2)$. The vector bundle $\mathcal{F}(y)$ is spanned by its global sections if and only if either $y > x(c_1, c_2)$ or $y = x(c_1, c_2)$ and $\chi(\mathcal{F}(x(c_1, c_2))) \geq 4$.*

Proof. The first assertion is [5, Corollary 5.2], if c_1 is even while if c_1 is odd the proof of [5, Corollary 5.2], works verbatim. The second assertion is well-known; the spannedness of $\mathcal{F}(y)$ for $y > x(c_1, c_2)$ also follows from the first assertion and Castelnuovo-Mumford's Lemma. QED

From previous Lemmas, we obtain at once the following results.

21 Theorem. *Let $Z \subset \mathbb{P}^{n+1}$ be a plane. Fix integers d_1 and d_2 with $-1 \leq d_1 \leq 0$ and $d_2 \geq d_1 + 2$. Let $\mathcal{F}$ be a rank 2 stable vector bundle on $\mathbb{P}^2$ with $c_1(\mathcal{F}) = d_1$ and $c_2(\mathcal{F}) = d_2$. Let t be the first integer with $0 \leq t \leq y(d_1, d_2)$ such that $h^0(\mathbb{P}^2, \mathcal{F}(t)) \neq 0$. Then, for every integer $y \geq d_2 - t^2$ there is a good 3-rope on Z with $\mathcal{F}(y)$ as conormal module.*

22 Theorem. *Let $Z \subset \mathbb{P}^{n+1}$ be a plane. Fix integers d_1 and d_2 with $-1 \leq d_1 \leq 0$ and $d_2 \geq d_1 + 2$. Then, for a general $\mathcal{F} \in \mathcal{M}(\mathbb{P}^2; d_1, d_2, 2)$ and every integer $t > x(d_1, d_2)$ there is a good 3-rope on Z with $\mathcal{F}(t)$ as conormal module.*

In principle one could use the non-emptiness results given in [7] and [2, Theorem 1.7], and the cohomological properties of general stable bundles proved in [2] and [11] to extend the previous results to the case of b-ropes on $\mathbb{P}^2$ with $b \geq 3$. However, the numerology seems to be intractable (at least to us).

Proof of Theorem 2. Assume the existence of a vector bundle $\mathcal{E}^*$ on a plane Z containing L such that $\mathcal{E}^*(1)|L$ has splitting type $\alpha_1, \ldots, \alpha_{b-1}$. If $h^1(Z, \mathcal{E}^*) = 0$, then $h^1(Z, \mathcal{I}_{L,Z} \otimes \mathrm{Hom}(\mathcal{O}_Z(-1)^{\oplus(n-1)}, \mathcal{E}^*)) = 0$ and hence the restriction map

$$H^0(Z, \mathrm{Hom}(\mathcal{O}_Z(-1)^{\oplus(n-1)}, \mathcal{E}^*)) \to H^0(L, \mathrm{Hom}(\mathcal{O}_L(-1)^{n-1}, \oplus_{j=1}^{b-1}\mathcal{O}_L(\alpha_j - 1)))$$

is surjective.

Thus by Remark 16, it is sufficient to prove the existence of a rank $(b-1)$ stable bundle $\mathcal{E}^*$ with $c_2(\mathcal{E}^*) = c_2$, splitting type $\alpha_j - 1, 1 \leq j \leq b-1$, and with $h^1(Z, \mathcal{E}^*) = 0$.

Let A be any vector bundle on Z. The tangent space of the deformation functor Def of A on Z is $H^1(Z, \mathrm{End}(A))$, the obstruction space of the functor Def is contained in $H^2(Z, \mathrm{End}(A))$ and the automorphism group of the functor Def is $H^0(Z, \mathrm{End}(A))$.

Now consider only the deformations of A which keep constant the splitting type of $A|L$; we call L-trivial any such deformation and we obtain in this way a functor $\mathrm{Def}(*, L)$. Since L is the zero-locus of a section of $\mathcal{O}_Z(1)$, the vector spaces $H^i(Z, \mathrm{End}(A)(-1)), 0 \leq i \leq 2$, are respectively the automorphism group, the tangent space and a space containing all the obstructions for the functor $\mathrm{Def}(*, L)$.

Now we drop the assumption that A is locally free, but we require that A is locally free in a neighborhood of L. For the general deformation theory of A, see [1]. For the functor of L-trivial deformations of A we must use the vector space $\mathrm{Ext}^i(Z; A(-1), A)$ instead of the vector space $H^i(Z, \mathrm{End}(A)(-1))$.

Set $B = \oplus_{j=1}^{b-1}\mathcal{O}_Z(\alpha_j - 1)$. Fix an integer $x \geq 0$ and x general points $P_1, \ldots, P_x$ of $Z \setminus L$. Fix any surjective map $u : B \to \oplus_{j=1}^{x}\mathcal{O}_{Z,P_j}$ and set $A = \ker(u)$. The torsion free sheaf A has rank $b-1$, $c_1(A) = c_1(B), c_2(A) = c_2(B) + x$(see e.g. [1]) and $A|L \cong \oplus_{j=1}^{b-1}\mathcal{O}_L(\alpha_j - 1)$ because $P_i \notin L$ for every i. In [2] it was considered the case $\alpha_j = 1$ for every j. Following [2] we will say that A is a quasi B-trivial sheaf. The proofs in [2], §3, show that for $x \gg 0$ the sheaf A is stable and deformable to a locally free sheaf and that we may do the same with L-trivial deformations; here we use that $\omega_Z \cong \mathcal{O}_Z(-3)$ and hence $h^0(Z, \omega_Z(1)) = 0$. QED

23 Remark. From the proof, it comes out that it is in general impossible to deform A to a locally free sheaf keeping fixed its restriction to a degree 3 (or higher) plane curve.

5 3-ropes on a 3-dimensional linear space

In this section we will give a few remarks for the case $x = 3$ and $b - 1 = 2$. To apply Remark 16, we will always assume $n \geq 6$.

Let $\mathcal{M}(\mathbb{P}^3; c_1, c_2, 2)$ be the moduli space of stable rank two vector bundles on $\mathbb{P}^3$ with Chern classes c_1 and c_2. The main difference with respect to the case $x = 2$ considered in previous section is that $\mathcal{M}(\mathbb{P}^3; c_1, c_2, 2)$ may be reducible and even not equidimensional. Hence for the integers c_1 and c_2 for which $\mathcal{M}(\mathbb{P}^3; c_1, c_2, 2) \neq \emptyset$ one usually identifies an interesting irreducible component T of $\mathcal{M}(\mathbb{P}^3; c_1, c_2, 2)$ and studies the cohomological properties of the elements of T or at least of the general $F \in T$. Following [10] in the case c_1 even we will call $T(c_1, c_2)$ the irreducible component of $\mathcal{M}(\mathbb{P}^3; c_1, c_2, 2)$ containing an instanton bundle obtained from skew lines using a construction due to Serre (see [8, Example 3.1.1], or [10, Example 1.6.1]). For $c_1 = 2c \geq 0$, we have $T(c_1, c_2) \neq \emptyset$ if and only if $c_2 > 0$; if $c_1 = -2$ we have $T(c_1, c_2) \neq \emptyset$ if and only if $c_2 \geq 2$. Similarly, if $c_1 = 2c - 1 \geq -1$ and $c_2 \geq 6$, there is a "cohomologically good" generically smooth irreducible component $T(c_1, c_2)$ of $\mathcal{M}(\mathbb{P}^3; c_1, c_2, 2)$ described in [10] and whose general member has natural cohomology. Using Castelnuovo - Mumford's Lemma and [10], Th. 0.1, instead of the quoted results in [5] and [12], we obtain as in the previous section the following result; to help the reader with the numerology of its statement we recall that $c_1(\mathcal{E}^*(-1)) = c_1 - 2$ and $c_2(\mathcal{E}^*(-1)) = c_2 - c_1 + 1$ and hence Riemann - Roch gives $\chi(\mathcal{E}^*(-1)) = (c_1 + 1)c_1(c_1 - 1)/6 - (c_1 + 2)(c_2 - c_1 + 1)/2 + 1$ (see [9, Th. 2.3]).

24 Proposition. *Fix integers c_1, c_2 with $c_1 \geq -1$. Set $a = [(c_1 + 1)/2], d_1 = c_1 - 2a$ and $d_2 = c_2 - c_1 a + a^2$. Assume $d_2 \geq 6$ if c_1 is odd and $d_2 > 0$ if c_1 is even. Assume $(c_1 + 1)c_1(c_1 - 1)/6 - (c_1 + 2)(c_2 - c_1 + 1)/2 + 1 \geq 0$. Let $\mathcal{E}^*$ be a general member of $T(c_1, c_2)$. Then there is a good 3-rope $X \subset \mathbb{P}^{n+2}$ supported by a 3-dimensional linear space with $\mathcal{E}^*$ as conormal module.*

25 Remark. There is a fundamental difference for $\underline{\alpha}$-stable extensions of ropes between extensions from a line to a plane and from a plane to a 3-dimensional linear space. In the previous section we gave an $\underline{\alpha}$-stable extension of any rope supported by a line to a good rope supported by a plane with stable conormal module. The corresponding result from a plane to a 3-dimensional linear space is false because if $c_1 \geq -2$ and $c_2 \gg 0$ a general element of $\mathcal{M}(\mathbb{P}^2; c_1, c_2, 2)$ is not the restriction of a vector bundle on $\mathbb{P}^3$, just for dimensional reasons.

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