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Flokki planes and cubic polynomials[i]

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Abstract. Non-Desarguesian translation planes of order q^2 are constructed whenever $q = 2^e \geq 16$ and e is not divisible by 3. Each plane has kernel GF(q) and translation complement of order $q(q-1)^2e$, with orbits of lengths 1, q and $q^2 - q$ on the translation line. The planes have elation groups of order q that produce derivable nets, but are not flock planes, semifield planes, or lifted planes.

The same algebraic tools are used to construct non-Desarguesian translation planes of order 2^p for every prime $p > 3$.

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1 Introduction

In the 1970's, Thas and Walker constructed translation planes from flocks of Miquelian Laguerre planes (**flock planes**, surveyed in [13]). Here we twist that construction with a field automorphism, still producing translation planes in Section 3 and new ones in Section 4, in the process recontextualizing the Hughes-Kleinfeld semifield planes [7].

Another impetus for this paper came from an attempt to use cubic polynomials in place of more common uses of quadratics in the study of 2-dimensional translation planes. This produces the planes in Section 4, and additional ones in Section 5. It is surprising, and perhaps even bizarre, that we construct planes of even order q and q^2 using the exact same computations, where q may or may not be a square.

We conclude in Section 6 with some open problems.

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2 Flokki and Flokki planes

Flock planes are related to flocks of a quadratic cone. Each such cone can be transformed into a cone $xy = z^2$ of $\mathrm{PG}(3, q)$. In this section we consider a cone on a plane curve obtained from a conic by twisting by a field automorphism. Recall that a **spread set** on $\mathrm{GF}(q)^2$ is a set $\mathbb{S}$ of q^2 2×2 matrices over $\mathrm{GF}(q)$ such that the difference of any two is invertible; this determines a translation plane $\pi(\mathbb{S})$ in a standard manner [2, p. 220].

1 Proposition. *Consider the cone* $C_\sigma := \{\langle x, y, z, w\rangle \mid x^\sigma y = z^{\sigma+1}\}$ *in* $\mathrm{PG}(3, q)$ *on the plane curve* $x^\sigma y = z^{\sigma+1}$, *where* $\sigma \in Aut(\mathrm{GF}(q))$. *Consider functions* $f, g\colon \mathrm{GF}(q) \to \mathrm{GF}(q)$. *Then the planes* $w = xt - yf(t)^\sigma + zg(t)^\sigma$, $t \in \mathrm{GF}(q)$, *partition* $C_\sigma \backslash \{\langle 0, 0, 0, 1\rangle\}$ *if and only if*

$$\mathbb{S} := \left\{ \begin{pmatrix} u + g(t) & f(t) \\ t & u^\sigma \end{pmatrix} \;\middle|\; t, u \in \mathrm{GF}(q) \right\} \text{ is a spread set.}$$

Proof. We have equations for q planes, each containing $q+1$ points of C_σ, while $|C_\sigma| = 1+q(q+1)$. Thus, in order to have a partition we only need to ensure that, if $t_1 \neq t_2$, then the conditions $w = xt_1 - yf(t_1)^\sigma + zg(t_1)^\sigma = xt_2 - yf(t_2)^\sigma + zg(t_2)^\sigma$ and $x^\sigma y = z^{\sigma+1}$ imply that $x = y = z = 0$. For distinct $t_1, t_2 \in \mathrm{GF}(q)$, write $\Delta t := t_1 - t_2$, $\Delta f := f(t_1) - f(t_2)$ and $\Delta g := g(t_1) - g(t_2)$. Then our requirement states that $x^{\sigma+1}\Delta t - z^{\sigma+1}\Delta f^\sigma + x^\sigma z \Delta g^\sigma = 0 \Rightarrow (x, z) = (0, 0)$. If $z \neq 0$ and $u := (x\Delta t/z)^{\sigma^{-1}}$, then it follows that $u^{\sigma+1} + u^\sigma \Delta g - \Delta t \Delta f = 0$ must have no root $u \in \mathrm{GF}(q)$. This is precisely the condition that $\mathbb{S}$ is a spread set, and this argument reverses. QED

We call a partition of C_σ as in the theorem a **flokki**[1] **of** C_σ, and the corresponding plane $\pi(\mathbb{S})$ is a **flokki plane**. These are (Thas-Walker) flock planes if $\sigma = 1$.

2 Example. The most obvious flokki are *linear*, where all planes contain a common line. This occurs if and only if f^σ and g^σ are linear. The corresponding flokki planes $\pi(\mathbb{S})$ are semifield planes. In fact, *they have been known for almost a half century*: they are the Hughes-Kleinfeld planes [7]. The flokki setting provides a uniform way to view this class of projective planes as part of a larger object (the cone C_σ) in place of the previous more computational view.

Unlike in the conical flock case there are many different orbits of such lines under the group of collineations preserving C_σ, and hence many "different" linear flokki. Presumably this can be used to explain the isomorphisms among the Hughes-Kleinfeld planes [17].

[1] This is the Finnish word for "flock". Plural: flokki.

3 Example. $\mathbb{S} := \left\{ \begin{pmatrix} u & nt^\tau \\ t & u^\sigma \end{pmatrix} \;\middle|\; t, u \in \mathrm{GF}(q) \right\}$ is a spread set, where $1 \neq \sigma \in Aut(\mathrm{GF}(q))$, $\tau \in Aut(\mathrm{GF}(q))$, and $n \in \mathrm{GF}(q)$ is not of the form $u^{\sigma+1}/t^{\tau+1}$ (i.e., n is not a dth power for some $d|(\sigma+1, \tau+1)$). Here $g \equiv 0$ is linear, but $\mathbb{S}$ is not linear if $\tau \neq \sigma$, in which case all q planes have in common only a single point: $\langle 0, 0, 1, 0 \rangle$. The semifield plane arising from $\mathbb{S}$ is due to Knuth [16].

3 Flokki planes and nets

One starting point of this paper is the following result of Gevaert and Johnson:

4 Theorem. *A translation plane π of order q^2 with kernel* $\mathrm{GF}(q)$ *admits an elation group E of order q one of whose component-orbits union the elation axis is a derivable net if and only if each of the component-orbits of E union the axis is a derivable net, if and only if π is isomorphic to $\pi(\mathbb{S})$, where*

$$\mathbb{S} := \left\{ \begin{pmatrix} u + g(t) & f(t) \\ t & u^\sigma \end{pmatrix} \;\middle|\; t, u \in \mathrm{GF}(q) \right\}$$

for some functions $f, g\colon \mathrm{GF}(q) \to \mathrm{GF}(q)$ and some $\sigma \in Aut(\mathrm{GF}(q))$. Moreover, π is a flock plane if and only if $\sigma = 1$.

Proof. This is a restatement of [3, Theorem 2.2], incorporating Note 2.2(ii) after Theorem 2.2 and the result in [8] that, in this situation, if the union of one of the component-orbits with the elation axis is a derivable net, then the union of each of the component-orbits with the axis is a derivable net. *QED*

The elation groups mentioned above are represented by the matrices

$$X = \begin{pmatrix} I & Y \\ 0 & I \end{pmatrix} \text{ with } Y = \begin{pmatrix} d & 0 \\ 0 & d^\sigma \end{pmatrix}, d \in \mathrm{GF}(q).$$

It follows from Proposition 1 that *a plane satisfies the hypotheses of* Theorem 4 *if and only if it is a flokki plane.* Note that the preceding two results involve the same spread sets.

5 Example. HMO flokki planes
Suppose

$$\mathbb{S} = \left\{ \begin{pmatrix} s & f(s,t) \\ t & g(s,t) \end{pmatrix} \;\middle|\; s, t \in \mathrm{GF}(q) \right\}$$

is a spread set, where q is odd and n is a fixed non-square in $\mathrm{GF}(q)$. Fix $\eta \in \mathrm{GF}(q^2)$ with $\eta^2 = n$. Then

$$\mathbb{S}' = \left\{ \begin{pmatrix} u & f(s,t) + g(s,t)\eta \\ s + t\eta & u^q \end{pmatrix} \;\middle|\; s, t \in \mathrm{GF}(q), u \in \mathrm{GF}(q^2) \right\}$$

is a spread set. The process of moving from the plane given by $\mathbb{S}$ to the plane given by $\mathbb{S}'$ is due to Hiramine-Matsumoto-Oyama [6], and is generalized to characteristic 2 in their Remark 2.4 (see also [9] and [15]), as follows. This time fix a basis $\{1, \eta\}$ for $\mathrm{GF}(q^2)$ over $\mathrm{GF}(q)$, and let $a = \eta + \eta^q$. If

$$\mathbb{S} := \left\{ \begin{pmatrix} s & f(s,t) + ag(s,t) \\ t & g(s,t) \end{pmatrix} \;\middle|\; s, t \in \mathrm{GF}(q) \right\}$$

is a spread set, then so is

$$\mathbb{S}' := \left\{ \begin{pmatrix} u & f(s,t) + g(s,t)\eta \\ s + t\eta & u^q \end{pmatrix} \;\middle|\; s, t \in \mathrm{GF}(q), u \in \mathrm{GF}(q^2) \right\}.$$

Let $v = s + t\eta$ and let $Tr(x) = x + x^q$ be the trace map $\mathrm{GF}(q^2) \to \mathrm{GF}(q)$. For q odd, we have $s = \frac{1}{2}Tr(v)$ and $t = \frac{v - \frac{1}{2}Tr(v)}{\eta}$, so that with

$$F(v) := f\big(Tr(v)/2, \frac{v - \frac{1}{2}Tr(v)}{\eta}\big) + g\big(\frac{1}{2}Tr(v)\eta, \frac{v - \frac{1}{2}Tr(v)}{\eta}\big)\eta,$$

$$\mathbb{S}' = \left\{ \begin{pmatrix} u & F(v) \\ v & u^q \end{pmatrix} \;\middle|\; u, v \in \mathrm{GF}(q^2) \right\}.$$

For q even, note that $a \neq 0$ as $\{1, \eta\}$ is a basis for $\mathrm{GF}(q^2)$ over $\mathrm{GF}(q)$. This time $t = Tr(v)/a$ and $s = v - Tr(v)\eta/a$. If $F(v) := f\big(v - Tr(v)\eta/a, Tr(v)/a\big) + g\big(v - Tr(v)\eta/a, Tr(v)/a\big)\eta$, then

$$\mathbb{S}' = \left\{ \begin{pmatrix} u & F(v) \\ v & u^q \end{pmatrix} \;\middle|\; u, v \in \mathrm{GF}(q^2) \right\}.$$

For both parities, $\pi(\mathbb{S}')$ is a flokki plane (with $g \equiv 0$ and σ of order 2), called a **lifted plane** in [12].

4 Cubic flokki planes

In the preceding section we were led to the equation $u^{\sigma+1} + u^\sigma \Delta g - \Delta t \Delta f = 0$. We do not know of any formula for its roots in general. However, in characteristic 2, when σ is the Frobenius automorphism $x^\sigma = x^2$ cubic equations naturally arise in constructing flokki planes, and there is such a classical formula. The **Hessian** of a cubic polynomial $F(x) = a_0x^3 + a_1x^2 + a_2x + a_3$ over any field is the polynomial $H(x) := (3a_0a_2 - a_1^2)x^2 + (9a_0a_3 - a_1a_2)x + (3a_1a_3 - a_2^2)$. The formula for the roots of cubic polynomials yields the following

6 Lemma. [5, Theorem 1.34][2] *Let F be a cubic polynomial over* $\mathrm{GF}(q)$ *with distinct roots, where q is not a power of* 3. *Assume that its Hessian H has degree* 2, *with distinct roots β_1, β_2 in some extension field and $\beta_2 \neq 0$. Then F has no roots in* $\mathrm{GF}(q)$ *if and only if either*

(i) $q \equiv 1$ $(mod\ 3)$, $\beta_1, \beta_2 \in \mathrm{GF}(q)$ *and* $F(\beta_1)/F(\beta_2) \notin \mathrm{GF}(q)^3$, *or*

(ii) $q \equiv 2$ $(mod\ 3)$, $\beta_1, \beta_2 \in \mathrm{GF}(q^2)\backslash\mathrm{GF}(q)$ *and* $F(\beta_1)/F(\beta_2) \notin \mathrm{GF}(q^2)^3$.

Recall that the **absolute trace** T maps a given field K of characteristic 2 onto $GF(2)$. If $k \in K$, then $T(k) = 0$ if and only if the quadratic $x^2 + x + k$ is reducible over K.

7 Theorem. *For even q and functions $f, g\colon \mathrm{GF}(q) \to \mathrm{GF}(q)$, let*

$$\mathbb{S} := \left\{ \begin{pmatrix} s + g(t) & f(t) \\ t & s^2 \end{pmatrix} \;\middle|\; s, t \in \mathrm{GF}(q) \right\}.$$

Write $\Delta t = t_1 + t_2$, $\Delta f = f(t_1) + f(t_2)$ and $\Delta g = g(t_1) + g(t_2)$ for $t_1, t_2 \in$ $\mathrm{GF}(q)$. *Assume that g is* 1-1 (*or, more generally, that $\Delta t \neq 0 = \Delta g \Rightarrow \Delta t \Delta f \notin \mathrm{GF}(q)^3$*). *If $\Delta f \Delta g \neq 0$ let $\beta_i = \beta_i(t_1, t_2), i = 1, 2$, be the roots of the quadratic $[\Delta g]^3 x^2 + \Delta t \Delta f\, x + \Delta t \Delta f = 0$. Then $\mathbb{S}$ is a spread set if and only if f is* 1-1 *and, for any distinct $t_1, t_2 \in \mathrm{GF}(q)$, either*

(i) $q \equiv 1\,(mod\ 3)$, *and* $T\Big(\frac{[\Delta g]^3}{\Delta t \Delta f}\Big) = 0$ *and* $\beta_1/\beta_2 \notin \mathrm{GF}(q)^3$ *whenever* $\Delta g \neq 0$, *or*

(ii) $q \equiv 2$ $(mod\ 3)$, *and* $T\Big(\frac{[\Delta g]^3}{\Delta t \Delta f}\Big) = 1$ *and* $\beta_1/\beta_2 \notin \mathrm{GF}(q^2)^3$ *whenever* $\Delta g \neq 0$.

Proof. Assume that $\mathbb{S}$ is a spread set. Then

$$(u_1 + u_2)^3 + (u_1 + u_2)^2 \Delta g + \Delta t \Delta f = 0 \Rightarrow (t_1, t_2) = (u_1, u_2),$$

and we must deduce (i) or (ii). Putting $U = u_1 + u_2$ yields the cubic equation

$$U^3 + U^2 \Delta g + \Delta t \Delta f = 0.$$

We may assume that $\Delta t \neq 0$. Then f must be 1-1 (as otherwise $\Delta f = 0$ for some distinct t_1, t_2, and then $U = \Delta g$ would yield a contradiction), and hence also $U \neq 0$.

By hypothesis, $\Delta g \neq 0$. Then $h = h(t_1, t_2) := \frac{\Delta t \Delta f}{[\Delta g]^3} \neq 0$ and

$$[U/\Delta g]^3 + [U/\Delta g]^2 + h = 0.$$

[2] There are slight errors in [5, Theorem 1.34], see the Errata included in reference [5].

The cubic $F(x) = F_{t_1,t_2}(x) := x^3 + x^2 + h \in \mathrm{GF}(q)[x]$ has Hessian $H(x) = x^2+hx+h$, with roots β_i satisfying $0 \neq h\beta_i = \beta_i^2+h$. Then $F(\beta_i) = \beta_i^3+\beta_i^2+h = \beta_i(\beta_i^2 + h) = \beta_i h\beta_i$, so $F(\beta_1)/F(\beta_2) = \beta_1^2/\beta_2^2$.

Clearly $(\beta_i/h)^2 + (\beta_i/h) = 1/h$. If $q \equiv 1 \pmod 3$, by Lemma 6 we need to have $\beta_1^2/\beta_2^2 \notin \mathrm{GF}(q)^3$ and $\beta_i \in \mathrm{GF}(q)$, so that $T(1/h) = 0$. The case $q \equiv 2 \pmod 3$ is similar. In either case F has distinct roots: a multiple root of F would be a root of its derivative x^2, whereas $F(0) = h \neq 0$.

Finally, this argument reverses, yielding the converse. *QED*

We call the planes $\pi(\mathbb{S})$ arising from Theorem 7 **cubic flokki planes**.

8 Example. $g \equiv 0$ and $f(t) = kt^2$ with k a non-cube. The cubic is $U^3 + \Delta tk[\Delta t]^2$, and its roots are not in $\mathrm{GF}(q)$. This is a special case of Example 3.

9 Corollary. *If the functions f, g produce a cubic flokki plane then so do the functions f^{-1}, gf^{-1}.*

Proof. Set $z_i = f(t_i)$ in (i) and (ii). (This produces the transpose of the spread set $\mathbb{S}$.) *QED*

10 Remark (Another variant)**.** If

$$\mathbb{S} := \left\{ \begin{pmatrix} s+g(t) & f(t) \\ t & s^{1/2} \end{pmatrix} \;\middle|\; s,t \in \mathrm{GF}(q) \right\},$$

let $u = s^{1/2}$ to put this into the form

$$\mathbb{S} = \left\{ \begin{pmatrix} u^2+g(t) & f(t) \\ t & u \end{pmatrix} \;\middle|\; u,t \in \mathrm{GF}(q) \right\},$$

where a cubic is evident. Now use a variant of the calculations above in order to obtain conditions involving $T\left(\frac{[\Delta g]^3}{\Delta t[\Delta f]^2}\right)$ for $\mathbb{S}$ to be a spread set (cf. Lemma 17). When $q > 4$ we suspect that the flokki plane $\pi(\mathbb{S})$ is not isomorphic to a flokki plane arising from Theorem 7 with $\sigma = 2$.

The remainder of this section concerns the planes $\pi(\mathbb{S})$ obtained as follows:

11 Theorem. *Let $q = 2^e$, where e is not divisible by 3. Then*

$$\mathbb{S} := \left\{ \begin{pmatrix} s+t^5 & t^{14} \\ t & s^2 \end{pmatrix} \;\middle|\; s,t \in \mathrm{GF}(q) \right\}$$

is a spread set.

Proof. Since $(7, q-1) = (2^3-1, 2^e-1) = 1$, t^{14} is $1-1$. If $t_1, t_2 \in \mathrm{GF}(q)$ with $t_2, \Delta t \neq 0$ but $\Delta g = 0$, then $t := t_1/t_2$ satisfies $g(t) = 0$ and hence has order 5. We must show that $U^3 + \Delta t\Delta f = 0$ has no root in $\mathrm{GF}(q)$, that is, $(t+1)(t^{14}+1)$ is a non-cube in $\mathrm{GF}(q)$. This can be checked by a calculation in

$GF(2)[t] = GF(16)$ since $GF(q)^*/GF(16)^*$ does not have order divisible by 3 in view of our hypothesis on e.

Fix t_1, t_2 in Theorem 7; we may assume that $t := t_2/t_1 \neq 0, 1$. As above we will use $h = h(t) := \frac{\Delta t \Delta f}{[\Delta g]^3} = \frac{(t+1)(t^{14}+1)}{(t^5+1)^3}$.

Let $\omega \in GF(4)$ with $\omega + \omega^2 = 1$; here ω might lie in $GF(q^2)\backslash GF(q)$. Since

$$\frac{1}{h} = \Big(\omega + \frac{t^6+t^4+t^3+t}{t^7+1}\Big) + \Big(\omega + \frac{t^6+t^4+t^3+t}{t^7+1}\Big)^2, \tag{1}$$

the trace condition in Theorem 7 is magically satisfied, and the roots of the quadratic $x^2 + hx + h$ are

$$\beta_1 = \Big(\omega + \frac{t^6+t^4+t^3+t}{t^7+1}\Big)h, \quad \beta_2 = \Big(\omega^2 + \frac{t^6+t^4+t^3+t}{t^7+1}\Big)h.$$

These lie in $K := GF(q^i)$, where $q \equiv i \pmod 3$ for $i = 1$ or 2, and not in $GF(q)$ if $i = 2$. Again magically,

$$\beta_1/\beta_2 = \frac{\omega + \frac{t^6+t^4+t^3+t}{t^7+1}}{\omega^2 + \frac{t^6+t^4+t^3+t}{t^7+1}} = \frac{\omega(t+1)(t^2+\omega t+1)^3}{\omega^2(t+1)(t^2+\omega^2 t+1)^3} \notin K^3, \tag{2}$$

as $\omega \notin K^3$ since $2e$ is not divisible by 6, so that the non-cube condition in Theorem 7 is satisfied. QED

12 Corollary. *If $q \geq 16$ then $\pi(\mathbb{S})$ is a translation plane of order q^4 that is not a semifield plane.*

PROOF. $\pi(\mathbb{S})$ is not a semifield plane since $\mathbb{S}$ contains 0 and is not closed under addition [2, 5.1.2]. QED

13 Remark. If $q = 4$, then $\pi(\mathbb{S})$ is the semifield plane of order 16 with kernel $GF(4)$ [16, p. 209].

14 Theorem. *For $q = 2^e \geq 16$, the translation complement of $\pi = \pi(\mathbb{S})$ has order $q(q-1)^2 e$, with orbits-lengths $1, q$ and $q^2 - q$ on the translation line.*

PROOF. Let $\mathcal{S}$ be the spread of $PG(3, q)$ given by $\mathbb{S}$; it contains the line $l(\infty) := \{(0, 0, s, t) \mid s, t \in GF(q)\}$. Then the translation complement $Aut(\pi)_0$ is the stabilizer C of $\mathcal{S}$ in $\Gamma L(4, q)$. Suppose $X \in \Gamma L(4, q)$ stabilizes $\mathcal{S}$ and $l(\infty)$. Then X can be written as the product of a 2×2 block matrix $X = \begin{pmatrix} X_1 & X_2 \\ 0 & X_3 \end{pmatrix}$ and a field automorphism α such that

$$X_1^{-1}X_2 + X_1^{-1}A^\alpha X_3 \in \mathbb{S} \quad \text{for all } A \in \mathbb{S}.$$

A straightforward calculation then shows that the stabilizer G of $l(\infty)$ in C consists of all matrices $X = \begin{pmatrix} X_1 & X_2 \\ 0 & X_3 \end{pmatrix}$ for which $X_1 = \begin{pmatrix} a^{-14} & 0 \\ 0 & l^4a^{10} \end{pmatrix}$, $X_2 = \begin{pmatrix} d & 0 \\ 0 & d^2 \end{pmatrix}$ and $X_3 = \begin{pmatrix} l^5a^{-9} & 0 \\ 0 & l^{14} \end{pmatrix}$, with $a, l \in \mathrm{GF}(q)^*$, $d \in \mathrm{GF}(q)$, $\alpha \in Aut(\mathrm{GF}(q))$. Thus G has order $q(q-1)^2e$.

Another straightforward calculation shows that G has orbit-lengths $1, q$ and $q^2 - q$ on $\mathcal{S}$. Using [4] and the elations in Theorem 4, we find that $l(\infty)$ is fixed by C, so that $C = G$. QED

15 Theorem. *$\pi(\mathbb{S})$ is a flokki plane that is not a flock plane, a semifield plane or a lifted plane.*

Proof. $\pi(\mathbb{S})$ is in the form given in Theorem 4. It is not a flock plane by [11], since $q+1$ does not divide the order of the translation complement, nor is it a lifted plane by [10]. It is not a semifield plane by Corollary 12. QED

5 Additional planes

The same calculations as in the proof of Theorem 11 produce translation planes in an entirely different manner using prequasifields. A finite **prequasifield** is a finite vector space K over the prime field, together with a binary operation $*$ on K that is left distributive and such that $x * t_1 = x * t_2 \Rightarrow x = 0$ or $t_1 = t_2$. This produces a translation plane $\pi(*)$ in a standard manner (compare [2, Sec. 3.1]).

Let $K = \mathrm{GF}(q)$ with $q = 2^e$. Consider functions $a, b, c\colon \mathrm{GF}(q) \to \mathrm{GF}(q)$, and the operation $*$ defined on $\mathrm{GF}(q)$ given by the formula

$$x * t = x^4a(t) + x^2b(t) + xc(t). \tag{3}$$

With notation as in Section 3, this defines a prequasifield if and only if

$$x^3\Delta a + x\Delta b + \Delta c = 0 \Rightarrow t_1 = t_2. \tag{4}$$

16 Example. $a(t) = t$, $b \equiv 0$, $c(t) = kt^{4^j}$ with $k \in K$ a non-cube, so that we must require that $q \equiv 1 \pmod 3$. Then (3) define a twisted field [1]. Note that this involves a relationship between some planes of Albert and some planes of Knuth given in Example 3.

17 Lemma. *Assume that b is 1-1. If $\Delta a\Delta c \neq 0$ let β_1, β_2 denote the roots of the quadratic $[\Delta b]^3x^2+\Delta a[\Delta c]^2x+\Delta a[\Delta c]^2 = 0$. Then (3) defines a prequasifield if and only if a and c are 1-1 and, for any distinct $t_1, t_2 \in \mathrm{GF}(q)$, either*

(i) *$q \equiv 1$ (mod 3), $T\Big(\frac{[\Delta b]^3}{\Delta a[\Delta c]^2}\Big) = 0$ and $\beta_1/\beta_2 \notin \mathrm{GF}(q)^3$, or*

(ii) $q \equiv 2 \ (mod\ 3)$, $T\Big(\frac{[\Delta b]^3}{\Delta a[\Delta c]^2}\Big) = 1$ *and* $\beta_1/\beta_2 \notin \mathrm{GF}(q^2)^3$.

PROOF. Suppose that (3) defines a prequasifield. Let $t_1 \neq t_2$ behave as in the first part of (4). Since $\Delta b \neq 0$, (4) implies that a and c are 1-1. If $U := \big(\frac{\Delta a}{\Delta b}x^2\big) + 1$ and $h := \frac{\Delta a[\Delta c]^2}{[\Delta b]^3}$, then (4) becomes $F(U) = 0$ for the same cubic $F(X) := X^3 + X^2 + h$ as in Section 4. The β_i are the roots of the Hessian $H(x) = x^2 + hx + h$ of F. We can now imitate the remainder of the proof of Theorem 7. Once again the argument reverses. QED

18 Theorem. *Let $q = 2^e$, where e is not divisible by 3. Then each of the following triples of functions defines a prequasifield using* (3), *and the corresponding translation plane is not a semifield plane if $q \geq 16$:*

(i) $a(t) = t$, $b(t) = t^5$, $c(t) = t^7$, *and*

(ii) $a(t) = t^{14}$, $b(t) = t^5$, $c(t) = t^{1/2}$.

PROOF. (i) As in the preceding section we assume that $t := t_2/t_1 \neq 0, 1$ and use $h := \frac{\Delta a[\Delta c]^2}{[\Delta b]^3} = \frac{(t+1)(t^{14}+1)}{(t^5+1)^3}$. Then $1/h$ and the ratio β_1/β_2 in the preceding lemma are *exactly* as in the proof of Theorem 11. Thus, using (1) and (2) we obtain a prequasifield by the preceding lemma. The plane $\pi(*)$ is not a semifield plane for $q \geq 16$ since the set of functions $x \to x * t$ is not closed under addition [2, 5.1.2].

In detail: if the above set is closed under addition then, for each t, u, there is $v = v(t, u)$ such that $(x^4t + x^2t^5 + xt^7) + (x^4u + x^2u^5 + xu^7) + (x^4v + x^2v^5 + xv^7) = 0$ for all x. Write $a_i = t^i + u^i + v^i$. Then $x^4a_1 + x^2a_5 = xa_7$, and the case $x = 1$ gives $x^4a_1 + x^2a_5 = x(a_1 + a_5)$. Since b is not additive, we can choose t, u such that $a_1 \neq 0$, but then the function $(x^3 + 1)/(x + 1)$ is constant for $x \neq 0, 1$, which is a contradiction since $q > 4$.

(ii) Here $h := \frac{\Delta a[\Delta c]^2}{[\Delta b]^3} = \frac{(t^{14}+1)(t+1)}{(t^5+1)^3}$ is the same as in (i). Since h determines the β_i (for given $t_1 \neq t_2$), (ii) follows as in (i). QED

It seems amazing that the same calculations are used in this theorem as in Theorem 11, so it is natural to wonder if there might be a relationship between the resulting planes. However, the present planes have order q, which might not even be a square, whereas the earlier planes had order q^2.

We suspect that $\pi(*)$ is a new plane in both (i) and (ii). These two planes are evidently related in some formal algebraic sense, but the planes probably are not isomorphic.

See [14, 16] for other translation planes of order 2^e for odd e; *those include the only other non-Desarguesian planes we know of having order 2^e when e is prime.* However, those are semifield planes or flag-transitive planes and hence admit far more automorphisms than the above ones appear to. (The only obvious automorphisms of $\pi(*)$ fixing the vector 0 are powers of $(x, y) \to (x^2, y^2)$.)

6 Open problems

We conclude with open problems concerning the planes in this paper, which neither author plans to work on.

(1) For given q and σ, do linear flokki (Example 2) determine non-isomorphic Hughes-Kleinfeld planes if and only if they are inequivalent under the group of the cone C_σ?

(2) Are the planes in Example 3 the only non-linear ones in which the q planes in Proposition 1 all meet?

(3) Does the use of the automorphism 1/2 in Remark 10 produce a different plane when applied to the example in Theorem 7 if $q > 4$?

(4) Does Corollary 9 usually produce a different plane from Theorem 11 when applied to the example in that theorem?

(5) Are the planes in Theorem 18(i) and (ii) non-isomorphic?

(6) Are there any more cubic flokki planes that are not semifield planes with f and g monomial?

(7) Is there any relationship between flokki and generalized quadrangles?

(8) Is there any relationship between flokki and hyperovals?

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