A nuclear Fréchet space of $C^\infty$-functions which has no basis

Dietmar Vogt
Department of Mathematics, University of Wuppertal
dvogt@math.uni-wuppertal.de

Abstract. An easy example is presented of a nuclear Fréchet space which consists of $C^\infty$-functions and has no basis.

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Dedicated to the memory of Klaus Floret

The aim of this paper is to present an easy example of a nuclear Fréchet space without basis, consisting of $C^\infty$-functions. Of course, there are several examples of nuclear Fréchet spaces without basis. The first one is due to Mitjagin and Zobin [3] (see also [4,5]). Then there is a simpler one of Djakov and Mitjagin [1]. The present note owes much to the example of Moscatelli [6,7]. It is based on essentially the same idea.

Here is our example: Set

$$M = \{ (x,y) \in \mathbb{R}^2 : x \geq 0, |\sin y| \leq 2 e^{-\frac{1}{x}} \}$$

and

$$E = \{ f \in C^\infty(\mathbb{R}^2) : f|_M \in \mathcal{S}(M) \}$$

with the seminorms

$$\|f\|_k = \sup_{|x| \leq k} |f^{(\alpha)}(x)| + \sup_{x \in M} (1 + |x|)^k |f^{(\alpha)}(x)|,$$

Here $\mathcal{S}(M)$ denotes the space of all $C^\infty$-functions on $M$ which are rapidly decreasing for $|x| \to \infty$ with all their derivatives and we set $\exp(-1/0) = 0$.

1 Theorem. $E$ is a nuclear Fréchet space without basis.

The plan of the paper is the following: first we write down a necessary condition for the existence of a basis in a nuclear Fréchet space, then we use it to prove Theorem 1. Finally we give some theoretical background. It would also be easy to develop a scheme how to construct many such examples.

In the following seminorm always means a continuous seminorm.
2 Definition. \( E \) has property \((\text{SpA})\) if for every seminorm \( p \) there is a seminorm \( q \geq p \) and \( S_0 \in L(E) \) so that \( \ker q \subset \ker S_0 \) and \( x - S_0 x \in \ker p \) for all \( x \in E \).

3 Remark. \( S_0 \) with the described properties corresponds to

\[ S \in L(E/\ker q, E) \]

so that the following diagram commutes

\[
\begin{array}{ccc}
S & \rightarrow & E \\
\downarrow & & \downarrow \ \ q_1 \\
E/\ker q & \rightarrow & E/\ker p \\
\uparrow & & \uparrow \ \ q_2 \\
& & \\
\end{array}
\]

where \( q_1 \) and \( q_2 \) are the canonical quotient maps.

We have the following easy lemma:

4 Lemma.

(1) If \( E = \Pi_k E_k \) and every \( E_k \) has a continuous norm, then \( E \) has property \((\text{SpA})\).

(2) Property \((\text{SpA})\) is inherited by complemented subspaces.

(3) Every complemented subspace of a Köthe space has property \((\text{SpA})\).

Proof. (1) and (2) are immediate. (3) follows since every Köthe space fulfills the assumption of (1).

5 Proposition. Every nuclear Fréchet space with basis has property \((\text{SpA})\) and also each of its complemented subspaces.

Proof. This follows from Lemma 4 (3) and the Dynin-Mityagin theorem (see [2, 28.12]).

6 Lemma. The space of our example does not have \((\text{SpA})\).

Proof. Assume that for \( \| \|_0 \) we find \( \| \|_k \) and \( S_0 \in L(E) \) so that \( S_0|_{\ker} \|_k = 0 \) and \( S_0 f - f\|_0 = 0 \), i.e. \( S_0 f|_M = f \).

We set \( D = \{ (x, y) : x^2 + (y - k\pi)^2 \leq 1 \} \) and \( A = \{ (x, y) : \frac{1}{2} \leq x^2 + (y - k\pi)^2 \leq 1 \} \). Then we put \( K = D \cap M \), \( K_0 = A \cap M \). Due to [9, VI, 3.1, Theorem 5] (or e.g. [10, Satz 4.6]) there is a continuous linear extension operator \( \mathcal{E}(K_0) \rightarrow C^\infty(\mathbb{R}^2) \) and, in consequence, a continuous linear extension operator \( L_0 : \mathcal{E}(K) \rightarrow \mathcal{E}(M) \).
We choose \( \varphi \in \mathcal{D}(\mathbb{R}^2) \), so that \( \varphi \equiv 1 \) in a neighborhood of \( K \) and \( \text{supp} \varphi \cap \{(x, y) : x^2 + y^2 \leq k^2\} = \emptyset \). For \( f \in \mathcal{E}(K) \) we choose any extension \( F \) of \( L_0f \) to \( C^\infty(\mathbb{R}^2) \). We set \( Lf := S_0(\varphi F) \).

\( L \) is well defined: if \( F_1 \) and \( F_2 \) are extensions then \( \varphi F_1 - \varphi F_2 \in \ker \|\|_k \).

Moreover \( Lf = \varphi(L_0f) \) on \( M \), hence \( Lf = f \) on \( K \).

So we have an extension operator \( L : \mathcal{E}(K) \rightarrow C^\infty(\mathbb{R}^2) \). Since \( K \) is locally diffeomorphic at the point \((0, k\pi)\) to \( \{(x, y) : |y| \leq e^{-\frac{1}{x}}, 0 \leq x \leq \varepsilon\} \) the map \( L \) cannot exist by Tidten [10, Beispiel 2].

Since obviously \( E \) is a nuclear Fréchet space Theorem 1 is proved.

QED

We continue with a few comments on property (SpA). First we exhibit its theoretical relevance.

7 Theorem. A Fréchet space \( E \) has property (SpA) if and only if it is isomorphic to a complemented subspace of a countable product of Fréchet spaces with continuous norm.

Proof. One direction of the proof is given by Lemma 4. For the other we may assume that \( E \) has no continuous norm. We choose a fundamental system of seminorms \( \|\|_1 \leq \|\|_2 \leq \cdots \) for \( E \) and set \( E_k = E/\ker \|\|_k \) with the quotient topology. We consider the exact sequence

\[
0 \rightarrow E \xrightarrow{j} \prod_k E_k \xrightarrow{\sigma} \prod_k E_k \rightarrow 0
\]

where \( jx = (j^k x)_k \), \( \sigma x = (j^k_{k+1} x_{k+1} - x_k)_k \) and \( j^k, j^k_{k+1} \) are the natural quotient maps.

Since \( E \) has property (SpA) we may assume the fundamental system of seminorms chosen so that for every \( k = 2, 3, \ldots \) there is a map \( S_k \in L(E_k, E) \) with \( j^{k-1} \circ S_k = j^k_{k-1} \). We set

\[
Rx := \left( \sum_{\nu=2}^k j^k S_\nu x_\nu - x_k \right)_{k \in \mathbb{N}}.
\]

It is easily verified that \( R \) is a continuous linear right inverse for \( \sigma \). Therefore \( E \) is isomorphic to a complemented subspace of \( \prod_k E_k \).

In Moscatelli [6] there is mentioned the problem of Dubinsky, whether every Fréchet space is isomorphic to a product of Fréchet spaces having a continuous norm. This, of course, is solved in the negative in [6]. However a slightly more sophisticated version of the problem remains interesting. To formulate it we begin with a remark.

8 Remark. \( E \) has property (SpA) if for every seminorm \( p \) there is a seminorm \( q \geq p \) and \( T \in L(E) \) so that \( T|_{\ker q} = \text{id} \), \( R(T) \subset \ker p \).
Proof. The proof is given by setting $T = \text{id} - S_0$ and vice versa. \(\square\)

In view of this remark we could describe a Fréchet space with property (SpA) as a Fréchet space admitting a fundamental system of seminorms with “almost complemented” kernels. A Fréchet space admits a fundamental system of seminorms with complemented kernels if and only if it is isomorphic to the product of Fréchet spaces having a continuous norm. Köthe spaces have this property. We call it property (CSK).

9 Problem. It is not known to the author whether every nuclear Fréchet space with property (SpA) has property (CSK), nor even whether every complemented subspace of a nuclear Köthe space has it. A counterexample to the latter would solve in the negative the problem of Pełczyński [8], whether every complemented subspace of a nuclear Köthe space has a basis.

References