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A nuclear Fréchet space of C^∞-functions which has no basis

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Abstract. An easy example is presented of a nuclear Fréchet space which consists of C^∞-functions and has no basis.

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Dedicated to the memory of Klaus Floret

The aim of this paper is to present an easy example of a nuclear Fréchet space without basis, consisting of C^∞-functions. Of course, there are several examples of nuclear Fréchet spaces without basis. The first one is due to Mitjagin and Zobin [3] (see also [4,5]). Then there is a simpler one of Djakov and Mitjagin [1]. The present note owes much to the example of Moscatelli [6, 7]. It is based on essentially the same idea.

Here is our example: Set

$$M = \{\, (x,y) \in \mathbb{R}^2 : x \geq 0,\ |\sin y| \leq 2\, e^{-\frac{1}{x}} \,\}$$

and

$$E = \{\, f \in C^\infty(\mathbb{R}^2) : f|_M \in \mathscr{S}(M) \,\}$$

with the seminorms

$$\|f\|_k = \sup_{\substack{|x|\leq k \\ |\alpha|\leq k}} |f^{(\alpha)}(x)| + \sup_{\substack{x\in M \\ |\alpha|\leq k}} (1+|x|)^k |f^{(\alpha)}(x)|.$$

Here $\mathscr{S}(M)$ denotes the space of all C^∞-functions on M which are rapidly decreasing for $|x| \longrightarrow \infty$ with all their derivatives and we set $\exp(-1/0) = 0$.

1 Theorem. *E is a nuclear Fréchet space without basis.*

The plan of the paper is the following: first we write down a necessary condition for the existence of a basis in a nuclear Fréchet space, then we use it to prove Theorem 1. Finally we give some theoretical background. It would also be easy to develop a scheme how to construct many such examples.

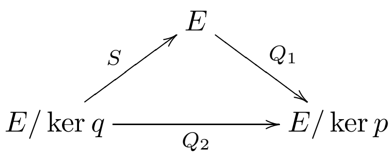
In the following seminorm always means a continuous seminorm.

2 Definition. E has property (SpA) if for every seminorm p there is a seminorm $q \geq p$ and $S_0 \in L(E)$ so that $\ker q \subset \ker S_0$ and $x - S_0 x \in \ker p$ for all $x \in E$.

3 Remark. S_0 with the described properties corresponds to

$$S \in L\left(E/\ker q, E\right)$$

so that the following diagram commutes



where Q_1 and Q_2 are the canonical quotient maps.

We have the following easy lemma:

4 Lemma.

(1) If $E = \Pi_k E_k$ and every E_k has a continuous norm, then E has property (SpA).

(2) Property (SpA) *is inherited by complemented subspaces.*

(3) Every complemented subspace of a Köthe space has property (SpA).

Proof. (1) and (2) are immediate. (3) follows since every Köthe space fulfills the assumption of (1). QED

5 Proposition. *Every nuclear Fréchet space with basis has property* (SpA) *and also each of its complemented subspaces.*

Proof. This follows from Lemma 4 (3) and the Dynin-Mityagin theorem (see [2, 28.12]). QED

6 Lemma. *The space of our example does not have* (SpA).

Proof. Assume that for $\| \, \|_0$ we find $\| \, \|_k$ and $S_0 \in L(E)$ so that $S_0|_{\ker \| \, \|_k} = 0$ and $\|S_0 f - f\|_0 = 0$, *i.e.* $S_0 f|_M = f$.

We set $D = \{ (x, y) : x^2 + (y - k\pi)^2 \leq 1 \}$ and $A = \{ (x, y) : \frac{1}{2} \leq x^2 + (y - k\pi)^2 \leq 1 \}$. Then we put $K = D \cap M$, $K_0 = A \cap M$. Due to [9, VI, 3.1, Theorem 5] (or e.g. [10, Satz 4.6]) there is a continuous linear extension operator $\mathscr{E}(K_0) \longrightarrow C^\infty(\mathbb{R}^2)$ and, in consequence, a continuous linear extension operator $L_0 \colon \mathscr{E}(K) \longrightarrow \mathscr{E}(M)$.

We choose $\varphi \in \mathscr{D}(\mathbb{R}^2)$, so that $\varphi \equiv 1$ in a neighborhood of K and $\operatorname{supp}\varphi \cap \{ (x,y) : x^2+y^2 \le k^2 \} = \emptyset$. For $f \in \mathscr{E}(K)$ we choose any extension F of $L_0 f$ to $C^\infty(\mathbb{R}^2)$. We set $Lf := S_0(\varphi F)$.

L is well defined: if F_1 and F_2 are extensions then $\varphi F_1 - \varphi F_2 \in \ker \| \ \|_k$. Moreover $Lf = \varphi(L_0 f)$ on M, hence $Lf = f$ on K.

So we have an extension operator $L\colon \mathscr{E}(K) \longrightarrow C^\infty(\mathbb{R}^2)$. Since K is locally diffeomorphic at the point $(0, k\pi)$ to $\{ (x,y) : |y| \le e^{-\frac{1}{x}},\ 0 \le x \le \varepsilon \}$ the map L cannot exist by Tidten [10, Beispiel 2].

Since obviously E is a nuclear Fréchet space Theorem 1 is proved. QED

We continue with a few comments on property (SpA). First we exhibit its theoretical relevance.

7 Theorem. *A Fréchet space E has property* (SpA) *if and only if it is isomorphic to a complemented subspace of a countable product of Fréchet spaces with continuous norm.*

Proof. One direction of the proof is given by Lemma 4. For the other we may assume that E has no continuous norm. We choose a fundamental system of seminorms $\| \ \|_1 \le \| \ \|_2 \le \cdots$ for E and set $E_k = E/\ker \| \ \|_k$ with the quotient topology. We consider the exact sequence

$$0 \longrightarrow E \xrightarrow{\ j\ } \prod_k E_k \xrightarrow{\ \sigma\ } \prod_k E_k \longrightarrow 0$$

where $jx = (j^k x)_k$, $\sigma x = (j^k_{k+1} x_{k+1} - x_k)_k$ and j^k, j^k_{k+1} are the natural quotient maps.

Since E has property (SpA) we may assume the fundamental system of seminorms chosen so that for every $k = 2, 3, \ldots$ there is a map $S_k \in L(E_k, E)$ with $j^{k-1} \circ S_k = j^{k-1}_k$. We set

$$Rx := \left(\sum_{\nu=2}^{k} j^k S_\nu x_\nu - x_k \right)_{k\in\mathbb{N}} .$$

It is easily verified that R is a continuous linear right inverse for σ. Therefore E is isomorphic to a complemented subspace of $\prod_k E$. QED

In Moscatelli [6] there is mentioned the problem of Dubinsky, whether every Fréchet space is isomorphic to a product of Fréchet spaces having a continuous norm. This, of course, is solved in the negative in [6]. However a slightly more sophisticated version of the problem remains interesting. To formulate it we begin with a remark.

8 Remark. E has property (SpA) if for every seminorm p there is a seminorm $q \ge p$ and $T \in L(E)$ so that $T|_{\ker q} = \mathrm{id}$, $R(T) \subset \ker p$.

PROOF. The proof is given by setting $T = \mathrm{id} - S_0$ and vice versa. QED

In view of this remark we could describe a Fréchet space with property (SpA) as a Fréchet space admitting a fundamental system of seminorms with "almost complemented" kernels. A Fréchet space admits a fundamental system of seminorms with complemented kernels if and only if it is isomorphic to the product of Fréchet spaces having a continuous norm. Köthe spaces have this property. We call it property (CSK).

9 Problem. It is not known to the author whether every nuclear Fréchet space with property (SpA) has property (CSK), nor even whether every complemented subspace of a nuclear Köthe space has it. A counterexample to the latter would solve in the negative the problem of Pełczyński [8], whether every complemented subspace of a nuclear Köthe space has a basis.

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