Note di Matematica **25,** n. 2, 2005/2006, 97–103.

The domain of the Schrödinger operator $-\Delta + x^2y^2$

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Abstract. We compute the domain of the Schrödinger operator $-\Delta + x^2y^2$ in $L^2(\mathbb{R}^2)$.

Keywords: Schrödinger operators

MSC 2000 classification: primary 35J10

1 Introduction

Let V be a nonnegative potential in $\mathbb{R}^d$ which belongs to $L^2_{loc}(\mathbb{R}^d)$. Then the quadratic form

$$a(u,v) = \int_{\mathbb{R}^d} (\nabla u \cdot \nabla \bar{v} + V u \bar{v})\, dx, \quad u, v \in H = \{\, u \in H^1(\mathbb{R}^d) : V^{1/2}u \in L^2(\mathbb{R}^d) \,\}$$

is closed, symmetric and nonnegative in $L^2(\mathbb{R}^d)$. Therefore a defines a self-adjoint operator $(A, D(A))$ in $L^2(\mathbb{R}^d)$ formally given by $A = -\Delta + V$, see e.g. [2, Chapter 8]. Moreover, A can be described by

$$D(A) = \{\, u \in H : \exists f \in L^2(\mathbb{R}^d) \text{ s.t. } a(u,v) = \int_{\mathbb{R}^d} f \bar{v}\, dx \quad \forall v \in H\}, \qquad Au = f. \tag{1}$$

The test function space $C_0^\infty(\mathbb{R}^d)$ is a core for A since $V \in L^2_{loc}(\mathbb{R}^d)$, due to [6, Corollary VII.2.7]. Thus the question arises whether $D(A)$ coincides with the intersection $H^2(\mathbb{R}^d) \cap D(V)$, see [5] where this problem seems to be considered for the first time from the point of view of operator inequalities like 3. Here $H^k(\mathbb{R}^d)$ is the usual Sobolev space and $D(V) = \{\, u \in L^2(\mathbb{R}^d) : Vu \in L^2(\mathbb{R}^d) \,\}$ is the domain of the multiplication operator $V : u \mapsto Vu$. The equality $D(A) = H^2(\mathbb{R}^d) \cap D(V)$ holds if V satisfies the oscillation condition

$$|\nabla V(x)| \le a V(x)^{3/2} + b \tag{2}$$

for $x \in \mathbb{R}^d$ and positive a, b with $a^2 < 2$, see [3] and [4] where also potentials with local singularities are considered. We refer the reader to [1], [10], [11] for results in L^p, $1 < p < \infty$. Examples show that $D(A)$ can be strictly larger than $H^2(\mathbb{R}^d) \cap D(V)$ if (2) does not hold, see again [3] and [4] for counterexamples with singular potentials and [10] for smooth potentials. Surprisingly enough the situation is much better in $L^1(\mathbb{R}^d)$ where the domain of $-\Delta + V$ is always the intersection of the domains of $-\Delta$ and of the potential V, [7].

In this note we prove that $D(A) = H^2(\mathbb{R}^2) \cap D(V)$ for the potential $V(x, y) = x^2y^2$ which, as is easy to see, does not satisfy (2). The same potential was studied in detail in [12] where the compactness of the resolvent was proved, (see also [9] for a characterization of the discreteness of the spectrum for polynomial potentials). We point out that the equality $D(A) = H^2(R^d) \cap D(V)$ holds for every polynomial potential V, see [13] where methods of harmonic analysis are used. Our proof for $V = x^2y^2$ is, on the other hand, elementary and based on explicit computations with Hermite functions.

1 Notation. The norm of $L^p(\mathbb{R}^d)$ is denoted by $\|\cdot\|_p$. $H^k(\mathbb{R}^d)$ is the Sobolev space of all functions in $L^2(\mathbb{R}^d)$ having weak derivatives in $L^2(\mathbb{R}^d)$ up to the order k. $C_0^\infty(\mathbb{R}^d)$ is the space of test functions.

2 The result

We begin with the following elementary lemma.

2 Lemma. *Let $0 \le V \in L^2_{loc}(\mathbb{R})$. Assume that there exists a constant $C > 0$ such that*

$$\|Vu\|_2 \le C\,\| - u'' + Vu\|_2 \tag{3}$$

for every $u \in C_0^\infty(\mathbb{R})$. Then the potential $V_\lambda(x) = \lambda^{-2}V(x/\lambda)$ satisfies (3) with the same constant C for every $\lambda > 0$.

Proof. Applying (3) to the function $v(x) = u(\lambda x)$, we obtain

$$\int_{\mathbb{R}} |V(x)u(\lambda x)|^2\,dx \le C^2 \int_{\mathbb{R}} | - \lambda^2 u''(\lambda x) + V(x)u(\lambda x)|^2\,dx.$$

Setting $y = \lambda x$, this inequality leads to

$$\int_{\mathbb{R}} |V(y/\lambda)u(y)|^2\,dy \le C^2 \int_{\mathbb{R}} | - \lambda^2 u''(y) + V(y/\lambda)u(y)|^2\,dy,$$

which implies the assertion. QED

In order to compute the domain of $-\Delta + x^2y^2$ we have to estimate the constant C in (3) for the potential $V(x) = x^2$.

3 Proposition. *The estimate*

$$\|x^2u\|_2 \le C\,\| -u'' + x^2u\|_2$$

holds for every $u \in C_0^\infty(\mathbb{R})$ *and a constant* $C > 0$ *satisfying* $C^2 < 2$.

Before proving this proposition, we show how the announced domain characterization follows from Proposition 3 and Lemma 2.

4 Theorem. *The domain of* $-\Delta + x^2y^2$ *in* $L^2(\mathbb{R}^2)$ *coincides with* $H^2(\mathbb{R}^2) \cap D(V)$.

Proof. The representation (1) of A implies that $H^2(\mathbb{R}^2) \cap D(V)$ is contained in $D(A)$ and that $Au = -\Delta u + x^2y^2u$ for $u \in H^2(\mathbb{R}^2) \cap D(V)$. Since $C_0^\infty(\mathbb{R}^2)$ is a core for $D(A)$, see [6, Corollary VII.2.7], it suffices to prove that the graph norm and the canonical norm of $H^2(\mathbb{R}^2) \cap D(V)$ are equivalent on $C_0^\infty(\mathbb{R}^2)$. Clearly, $\|u\|_2 + \|Au\|_2 \le \|u\|_{H^2} + \|x^2y^2u\|_2$ for $u \in C_0^\infty(\mathbb{R}^2)$.

Thus it remains to establish the converse inequality. To estimate the H^1-norm of $u \in C_0^\infty(\mathbb{R}^2)$, we note that

$$\int_{\mathbb{R}^2} (u + Au)\bar{u}\,dx\,dy = \int_{\mathbb{R}^2} (|u|^2 + |\nabla u|^2 + x^2y^2|u|^2)\,dx\,dy,$$

Hence, $\|u\|_{H^1} \le c\,(\|u\|_2 + \|Au\|_2)$ for a suitable $c > 0$. We next treat the L^2-norms of the functions x^2y^2u and D^2u. We set $f = -\Delta u + x^2y^2u$ for $u \in C_0^\infty(\mathbb{R}^2)$. Then $-u_{xx} + x^2y^2u = f + u_{yy}$. Fix $y \in \mathbb{R} \setminus \{0\}$. Proposition 3 and Lemma 2 with $\lambda^4 = y^{-2}$ show that

$$\int_{\mathbb{R}} x^4y^4u(x,y)^2\,dx \le C^2 \int_{\mathbb{R}} |f(x,y) + u_{yy}(x,y)|^2\,dx,$$

where C is the constant from Proposition 3. Integrating this estimate with respect to y, we obtain

$$\int_{\mathbb{R}^2} x^4y^4u^2\,dx\,dy \le C^2 \int_{\mathbb{R}^2} |f + u_{yy}|^2\,dx\,dy.$$

In the same way one deduces that

$$\int_{\mathbb{R}^2} x^4y^4u^2\,dx\,dy \le C^2 \int_{\mathbb{R}^2} |f + u_{xx}|^2\,dx\,dy.$$

Summing the last two inequalities and using $f = -\Delta u + x^2y^2u$, we conclude

$$\begin{aligned}\int_{\mathbb{R}^2} x^4y^4u^2\,dx\,dy &\le C^2 \int_{\mathbb{R}^2} (f^2 + f\Delta u + \frac{1}{2}u_{xx}^2 + \frac{1}{2}u_{yy}^2)\,dx\,dy \\ &= C^2 \int_{\mathbb{R}^2} (f^2 - |\Delta u|^2 + x^2y^2u\Delta u + \frac{1}{2}u_{xx}^2 + \frac{1}{2}u_{yy}^2)\,dx\,dy.\end{aligned}$$

On the other hand, we compute

$$\int_{\mathbb{R}^2} |\Delta u|^2\,dx\,dy = \int_{\mathbb{R}^2} (u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2)\,dx\,dy \tag{4}$$

integrating by parts twice, which leads to

$$\int_{\mathbb{R}^2} x^4y^4u^2\,dx\,dy \le C^2 \int_{\mathbb{R}^2} (f^2 - \frac{1}{2}\,|\Delta u|^2 + x^2y^2u\Delta u)\,dx\,dy.$$

Young's inequality then implies

$$\int_{\mathbb{R}^2} x^4y^4u^2\,dx\,dy \le C^2 \int_{\mathbb{R}^2} (f^2 + \frac{1}{2}x^4y^4u^2)\,dxdy,$$

$$\|x^2y^2u\|_2^2 \le \frac{C^2}{1-C^2/2}\|f\|_2^2\,,$$

since $1 - C^2/2 > 0$ by Proposition 3. This estimate and (4) further yield

$$\|D^2u\|_2^2 \le C_1\,\|\Delta u\|_2^2 = C_1\,\|x^2y^2u - f\|_2^2 \le C_2\,\|f\|_2^2\,.$$

As a result, $\|u\|_{H^2}+\|x^2y^2u\|_2 \le C_3\,(\|u\|_2+\|Au\|_2)$ for some constant C_3. QED

In order to prove Proposition 3 we need some elementary properties of the Hermite functions

$$H_n(x) = \frac{(-1)^n}{\sqrt{2^n n!\sqrt{\pi}}}\,e^{x^2/2}\,\frac{d^n}{dx^n}e^{-x^2} =: \frac{1}{\sqrt{2^n n!\sqrt{\pi}}}\,\psi_n(x), \qquad n \in \mathbb{N}_0\,,$$

for which we refer to [8, §5.6.2]. The Hermite functions are an orthonormal basis of $L^2(\mathbb{R})$ and $-H_n'' + x^2H_n = (2n+1)H_n$. The functions ψ_n satisfy the identity $\psi_{n+1} = 2x\psi_n - 2n\psi_{n-1}$ for $n \in \mathbb{N}_0$, where $\psi_{-1} = 0$. Using this recursion formula, one easily computes the integrals

$$c_{n,m} = \int_{\mathbb{R}} x^2H_n(x)H_m(x)\,dx, \qquad n,m \in \mathbb{N}_0,$$

obtaining

$$c_{n,n-2} = \frac{1}{2}\sqrt{n(n-1)} \quad (n \ge 2) \tag{5}$$

$$c_{n,n} = \frac{1}{2}(2n+1) \tag{6}$$

$$c_{n,n+2} = \frac{1}{2}\sqrt{(n+2)(n+1)} \tag{7}$$

$$c_{n,m} = 0 \quad \text{if } m \ne n, n-2, n+2. \tag{8}$$

Proof of Proposition 3. Let $u \in C_0^\infty(\mathbb{R})$ and expand $f = -u'' + x^2u$ with respect to the orthonormal basis (H_n), i.e.,

$$f = \sum_{m=0}^{\infty} \langle f, H_m \rangle \, H_m = \sum_{m=0}^{\infty} f_m H_m$$

where the brackets denote the inner product of $L^2(\mathbb{R})$ and $f_m = \langle f, H_m \rangle$. Then we obtain

$$u = \sum_{m=0}^{\infty} (2m+1)^{-1} f_m H_m \qquad \text{and} \qquad x^2u = \sum_{m=0}^{\infty} (2m+1)^{-1} f_m x^2 H_m.$$

From the identities (5) it follows that

$$\langle x^2u, H_n \rangle = \alpha_n f_{n-2} + \frac{1}{2} f_n + \beta_n f_{n+2}$$

for $n \in \mathbb{N}_0$, where

$$\alpha_n = \frac{\sqrt{n(n-1)}}{2(2n-3)}, \qquad \beta_n = \frac{\sqrt{(n+2)(n+1)}}{2(2n+5)}, \qquad f_{-2} = f_{-1} = 0.$$

These equalities yield

$$x^2u = \frac{1}{2} f + \sum_{n=0}^{\infty} (\alpha_n f_{n-2} + \beta_n f_{n+2}) H_n =: \frac{1}{2} f + g. \tag{9}$$

We further estimate

$$\begin{aligned}
\|g\|_2^2 &= \sum_{n=0}^{\infty} (\alpha_n f_{n-2} + \beta_n f_{n+2})^2 \\
&= \alpha_2^2 f_0^2 + \alpha_3^2 f_1^2 + 2\alpha_2\beta_2 f_0 f_4 + 2\alpha_3\beta_3 f_1 f_5 + \sum_{n=2}^{\infty} (\alpha_{n+2}^2 + \beta_{n-2}^2) f_n^2 \\
&\quad + 2 \sum_{n=4}^{\infty} \alpha_n f_{n-2} \, \beta_n f_{n+2} \\
&\le \alpha_2^2 f_0^2 + \alpha_3^2 f_1^2 + 2\alpha_2\beta_2 f_0 f_4 + 2\alpha_3\beta_3 f_1 f_5 + 2 \sum_{n=2}^{\infty} (\alpha_{n+2}^2 + \beta_{n-2}^2) f_n^2
\end{aligned}$$

using Hölder's and Young's inequalities. Observe that $\alpha_{n+2}^2 + \beta_{n-2}^2 \le \frac{7}{50}$ for

$n \geq 2$. Hence,

$$\begin{aligned}\|g\|_2^2 &\leq \frac{1}{2}f_0^2 + \frac{1}{6}f_1^2 + \frac{\sqrt{6}}{9}f_0f_4 + \frac{\sqrt{30}}{33}f_1f_5 + \frac{14}{50}\sum_{n=2}^{\infty} f_n^2 \\ &\leq \left(\frac{1}{2} + \frac{\sqrt{6}}{18}\right) f_0^2 + \left(\frac{1}{6} + \frac{\sqrt{30}}{66}\right) f_1^2 + \frac{14}{50}\, f_2^2 + \frac{14}{50}\, f_3^2 + \left(\frac{\sqrt{6}}{18} + \frac{14}{50}\right) f_4^2 \\ &\quad + \left(\frac{\sqrt{30}}{66} + \frac{14}{50}\right) f_5^2 + \frac{14}{50}\sum_{n=6}^{\infty} f_n^2 \\ &\leq \left(\frac{1}{2} + \frac{\sqrt{6}}{18}\right) \|f\|_2^2 .\end{aligned}$$

Together with (9), we conclude

$$\|x^2u\|_2 \leq \left(\frac{1}{2} + \sqrt{\frac{1}{2} + \frac{\sqrt{6}}{18}}\right) \|f\|_2 =: C\, \|f\|_2 .$$

The assertion is established since $C^2 < 2$. QED

5 Remark. As in the proof of Theorem 4 one can establish that $D(-\Delta + bV) = H^2(\mathbb{R}^2) \cap D(V)$ for $b > 0$ and $V(x,y) = x^2y^2$. But it seems that one cannot treat more general potentials by the method used in this paper.

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