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Spatial and structural stability in thermoelastodynamics on a half-cylinder

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Abstract. The linear nonhomogeneous thermoelastodynamic problem in a half-cylinder is considered subject to assigned initial conditions, and to the displacement and temperature being specified over the base, and vanishing on the lateral boundary. Spatial stability, derived from a differential inequality, establishes that the mean-square volume integrals of displacement and temperature are bounded above by a decaying function of axial distance for each finite positive time instant. Structural stability, which here relates to continuous dependence of the displacement on the thermal coupling, depends upon the construction of further differential inequalities

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Dedicated to W. A. Day on the occasion of his 65*th birthday.*

1 Introduction

This paper treats the spatial and structural stability of a half-cylinder composed of linear thermoelastic material and which is moving subject to homogeneous lateral Dirichlet boundary conditions, specified displacement and temperature distributed over the cylinder's base, and with prescribed initial data. For spatial stability, we construct a decaying upper bound for certain mean-square measures of the solution. Extension of the analysis to include homogeneous mechanical traction and thermal flux prescribed on the lateral boundary is possible but is not pursued here. It is known for homogeneous lateral and initial data (see, e.g., [2]) and for given base traction and thermal flux that a suitable measure of the total energy is bounded above by a function that decays like

$\exp(-x_3/\nu(t)\sqrt{t})$, where x_3 is the axial distance, and $\nu(t)$ is a positive function of time t. Our result is similar but is valid for non-zero initial data and with the base traction and thermal flux replaced by prescribed distributions of displacement and temperature.

The aspect of structural stability that we consider is with respect to thermal coupling. We retain the same data as for spatial stability and discuss the effect on the displacement when the thermal coupling coefficients and their derivatives tend to zero. In the limit the isothermal theory is recovered, thermal dissipation, caused by the coupling, disappears, and effects due to the specified base displacement propagate with finite speed, as demonstrated in [3]. By contrast, in the corresponding coupled thermoelastic problem effects due to base data propagate with infinite speed and are simultaneously experienced at all points of the cylinder. Nevertheless, we are able to prove that the mean-square difference between the displacement in the thermal and isothermal problems vanishes in the limit as the thermal coupling tends to zero. Ames and Payne [1] consider the complementary problem, but backward in time, of convergence to the ordinary heat problem.

The method of proof for both spatial and structural stability relies upon the derivation and integration of differential inequalities for time integrals of the total energy contained in a volume of the cylinder. Section 2 describes the notation together with certain other preliminaries, while Section 3, devoted to spatial stability in the non-homogeneous thermoelastic initial boundary value problem, establishes the decay estimate. The final Section discusses structural stability and related issues.

Throughout, a solution of sufficient smoothness to justify the following operations is assumed to exist. The comma notation to indicate partial differentiation and the summation convention are adopted with Greek suffixes, apart from η, ranging over $[1,2]$ and Latin suffixes over $[1,2,3]$.

2 Notation and other preliminaries

We consider a prismatic cylinder $B \subset \mathbb{R}^3$ of semi-infinite length whose plane base lies in the x_1x_2- coordinate plane of a three-dimensional Cartesian coordinate system whose positive x_3- axis is directed along that of the cylinder. The uniform plane cross-section $D \subset \mathbb{R}^2$ has piecewise smooth boundary ∂D, and to emphasise that quantities at a distance x_3 from the base are under consideration, we employ the notation $D(x_3)$. We also introduce the notation:

$$B(z) = \{x \in B : z \leq x_3\} \tag{1}$$

to indicate that part of the cylinder whose points are each at an axial distance no less than z from the base.

The cylinder is occupied by a classical linear inhomogeneous anisotropic thermoelastic material in motion subject to prescribed initial Cauchy data, and with the displacement $u_i(x,t)$ and temperature $\theta(x,t)$ supposed zero on the lateral boundary $\partial D\times[0,\infty)$ and prescribed non-zero over the base $D(0)$. The data is assumed always to produce a bounded total energy, implying the asymptotic vanishing with respect to axial distance of the displacement and temperature together with their respective spatial gradients, and time derivative of the displacement.To be explicit, we impose on our solution the asymptotic behaviour:

$$\lim_{x_3\to\infty}\int_0^t\int_{D(x_3)}(u_iu_i+u_{,\eta}u_{,\eta}+u_{i,j}u_{i,j}+\theta^2+\theta_{,i}\theta_{,i})\,dS\,d\eta=0. \tag{2}$$

Accordingly, the thermoelastic problem is specified by:

$$(c_{ijkl}e_{kl})_{,j}+(\beta_{ij}\theta)_{,j}=\rho\ddot{u}_i,\qquad (x,t)\in B\times[0,T), \tag{3}$$
$$-\beta_{ij}\dot{u}_{i,j}+\dot{\theta}=(k_{ij}\theta_{,i})_{,j},\quad (x,t)\in B\times[0,T), \tag{4}$$

where a superposed dot indicates time differentiation, $[0,T]$ is the maximal time interval of existence of a sufficiently smooth solution, $\rho(x)$ is the mass density, $\beta_{ij}(x)$ are the thermal (stress-temperature) coupling coefficients, $k_{ij}(x)$ are the thermal conductivities, the linear strain components e_{ij} are given by

$$e_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}), \tag{5}$$

and by suitable rescaling the positive specific heat in (4) is taken as unity. The nonhomongeous mass density is supposed smooth, to possess the bounds:

$$0<\rho_0\le\rho(x)\le\rho_1,\qquad x\in B, \tag{6}$$

where ρ_0,ρ_1 are given uniform positive constants, and for simplicity, to be independent of the axial variable x_3. The elasticities $c_{ijkl}(x)$, similarly assumed independent of x_3, satisfy the symmetries

$$c_{ijkl}=c_{jikl}=c_{klij}, \tag{7}$$

and are positive-definite and bounded in the sense that

$$c_0\psi_{ij}\psi_{ij}\le c_{ijkl}\psi_{ij}\psi_{kl},\quad \psi_{ij}=\psi_{ji},\ x\in B, \tag{8}$$
$$c_{ijkl}\xi_i\xi_k\zeta_j\zeta_l\le c_1\xi_i\xi_i\zeta_j\zeta_j,\qquad x\in B, \tag{9}$$

for uniform positive constants c_0, c_1, and $\xi \in \mathbb{R}$, $\zeta \in \mathbb{R}$. It is straightforward to adapt the following analysis to include generally inhomogeneous mass density and elasticities subject to additional conditions on their derivatives with respect to the axial variable. The thermal conductivities are symmetric and positive-definite so that

$$k_{ij} = k_{ji}, \qquad x \in B, \tag{10}$$

$$k_0 \xi_i \xi_i \leq k_{ij} \xi_i \xi_j, \quad x \in B, \tag{11}$$

where k_0 is a uniform positive constant, while the component $k_{33}(x)$ is assumed to be bounded:

$$\max_{x \in B} k_{33} \leq M_1^2, \tag{12}$$

for given positive constant M_1. The thermal coupling coefficients are supposed to be continuously differentiable in B and consequently there are positive constants M_2, M_3 such that:

$$\sup_{x \in B} \beta_{ij} \beta_{ij} \leq M_2^2, \quad \sup_{x \in B} \beta_{ij,j} \beta_{ik,k} \leq M_3^2. \tag{13}$$

The lateral boundary conditions become

$$u_i(x,t) = \theta(x,t) = 0, \qquad (x,t) \in \partial D \times [0,\infty) \times [0,T], \tag{14}$$

while the base distributions of displacement and temperature are given by:

$$u_i(x,t) = l_i(x_\alpha, t), \quad \theta(x,t) = \Phi(x_\alpha, t), \quad (x,t) \in D(0) \times [0,T), \tag{15}$$

where $l_i(x_\alpha, t), \Phi(x_\alpha, t)$ are prescribed functions. The initial conditions are specified to be:

$$\theta(x,0) = f(x), \quad x \in B \times \{0\}, \tag{16}$$

$$u_i(x,0) = g_i(x), \quad x \in B \times \{0\}, \tag{17}$$

$$\dot{u}_i(x,0) = h_i(x), \quad x \in B \times \{0\}, \tag{18}$$

where f, g_i, h_i are given and for all $t \geq 0$ are assumed to further satisfy the boundedness condition:

$$\int_0^\infty \exp\left(z\gamma^{-1}(t)\right) R(z)\, dz < \infty, \tag{19}$$

where $\gamma(t)$ is defined in (32) and

$$R(z) \equiv \int_{B(z)} [\rho h_i h_i + c_{ijkl} g_{i,j} g_{k,l} + f^2]\, dx. \tag{20}$$

A sufficient condition for (19) is the asymptotic behaviour

$$\lim_{x_3 \to \infty} \exp x_3 \gamma^{-1}(t) \int_{D(x_3)} (\rho h_i h_i + c_{ijkl} g_{i,j} g_{k,l} + f^2)\, dx_1\, dx_2 = 0. \tag{21}$$

Let $v_i(x,t)$ and $e_{ij}(v)$ be the displacement and strain respectively in the corresponding isothermal problem whose governing equations are given by

$$\left(c_{ijkl} e_{kl}(v)\right)_{,j} = \rho \ddot{v}_i, \qquad (x,t) \in B \times [0,T), \tag{22}$$

The isothermal solution is subject to the same smoothness, respective asymptotic behaviour, boundary and initial conditions as are prescribed in the thermoelastic problem.

As previously stated, in Section 4 we establish that vanishingly small β_{ij} together with that of its derivatives imply the convergence of the thermal displacement $u_i(x,t)$ to that in the corresponding isothermal problem, provided convergence is measured in appropriate mean-square norms.

Before, however, discussing convergence with respect to vanishing β_{ij}, we prove that the thermoelastic solution, represented by mean-square measures of $u_i(x,t), \theta(x,t)$, exponentially decays to zero in the limit as $x_3 \to \infty$ for each given finite t.

3 Spatial Stability

We consider the thermoelastic initial boundary value problem specified in Section 2 and first obtain a decaying upper bound for the total energy $E(z,t)$, defined by expression

$$E(z,t) = \int_0^t \int_{B(z)} \left[\rho u_{i,\eta} u_{i,\eta} + c_{ijkl} e_{ij} e_{kl} + \theta^2 + 2(t-\eta) k_{ij} \theta_{,i} \theta_{,j}\right] dx\, d\eta. \tag{23}$$

We record here for subsequent use the following result which is immediate from (23):

$$\frac{dE(z,t)}{dz} = -\int_0^t \int_{D(z)} \left[\rho u_{i,\eta} u_{i,\eta} + c_{ijkl} e_{ij} e_{kl} + \theta^2 + 2(t-\eta) k_{ij} \theta_{,i} \theta_{,j}\right] dS\, d\eta, \tag{24}$$

for each fixed t, where dS is the element of cross-sectional area.

To obtain the decay estimate, we first note that (3)-(7), the lateral boundary conditions (14), and the divergence theorem enable the energy $E(z,t)$ to be represented alternatively as:

$$E(z,t) = I(z,t) + tR(z), \tag{25}$$

where $R(z)$ is defined in (20) and

$$I(z,t) = 2\int_0^t \int_{D(z)} (t-\eta)\left[c_{ijkl}e_{ij}u_{k,\eta}n_l + \beta_{ij}u_{i,\eta}\theta n_j + k_{ij}\theta\theta_{,i}n_j\right] dS\, d\eta, \quad (26)$$

in which n_i are the components of the unit outward normal on $\partial B(z)$. We next apply Schwarz's inequality, followed by the arithmetic-geometric mean inequality, to derive the bounds:

$$\begin{aligned} I(z,t) \leq\, & 2\left[\int_0^t\int_{D(z)}(t-\eta)c_{ijkl}e_{ij}e_{kl}\, dS\, d\eta \int_0^t\int_{D(z)}(t-\eta)c_{ijkl}u_{i,\eta}u_{k,\eta}n_jn_l\, dS\, d\eta\right]^{\frac{1}{2}} \\ & +2\left[\int_0^t\int_{D(z)}(t-\eta)\theta^2\, dS\, d\eta \int_0^t\int_{D(z)}(t-\eta)\left(\beta_{ij}u_{i,\eta}n_j\right)^2\, dS\, d\eta\right]^{\frac{1}{2}} \\ & +2\left[\int_0^t\int_{D(z)}(t-\eta)\theta^2 k_{ij}n_in_j\, dS\, d\eta \int_0^t\int_{D(z)}(t-\eta)k_{ij}\theta_{,i}\theta_{,j}\, dS\, d\eta\right]^{\frac{1}{2}} \\ \leq\, & t\sqrt{\frac{c_1}{\rho}}\left[\gamma_1\int_0^t\int_{D(z)} c_{ijkl}e_{ij}e_{kl}\, dS\, d\eta + \mathsf{M}_1\int_0^t\int_{D(z)}\rho u_{i,\eta}u_{i,\eta}\, dS\, d\eta\right] \\ & +\mathsf{M}_2\int_0^t\int_{D(z)}\theta^2\, dS\, d\eta + \frac{M_1t^{1/2}}{\gamma_3}\int_0^t\int_{D(z)}(t-\eta)k_{ij}\theta_{,i}\theta_{,j}\, dS\, d\eta, \end{aligned} \quad (27)$$

where $\gamma_i(t)$, $i=1,2,3$ are positive functions of time to be chosen, M_1, M_2 are given respectively by (12) and (13), whereas

$$\mathsf{M}_1 = \frac{1}{\gamma_1} + \frac{M_2}{c_1^{1/2}\gamma_2},\ \mathsf{M}_2 = \frac{M_2t\gamma_2}{\rho_0^{1/2}} + M_1t^{1/2}\gamma_3.$$

Indeed, on setting for each fixed t:

$$\gamma_1 = \epsilon p, \quad \gamma_2 = \frac{\epsilon M_2 p}{(\epsilon^2p^2-1)c_1^{1/2}}, \quad \gamma_3 = p/\epsilon, \quad (28)$$

where $\epsilon(t)$ and $p(t)$ are given by

$$\epsilon^2 = \frac{1}{2p^2}\left\{(1+q^2+2p^4) \pm \sqrt{(1-q^2-2p^4)^2+4q^2}\right\}], \quad (29)$$

and

$$p^2 = \frac{M_1\rho_0^{1/2}}{2(c_1t)^{1/2}}, \quad q^2 = \frac{M_2^2}{c_1}, \quad (30)$$

we conclude after noting (24) that substitution of inequality (27) in (25) gives

$$E(z,t) \leq -\gamma(t)\frac{dE(z,t)}{dz} + tR(z), \tag{31}$$

where $\gamma(t)$ is given by

$$\gamma(t) = \frac{\epsilon}{\sqrt{2}} M_1^{1/2} \left(\frac{c_1}{\rho_0}\right)^{1/4} t^{3/4}. \tag{32}$$

Integration of (31) for each fixed t then yields:

$$E(z,t) \leq \left(E(0,t) + t\gamma^{-1}(t)\int_0^z \exp\left(y\gamma^{-1}(t)\right)R(y)\,dy\right)\exp(-z\gamma^{-1}(t)), \tag{33}$$

in which the decay rate $\gamma^{-1}(t)$ depends on the thermal conductivity k_{33} only through the bound M_1 given by (12). Furthermore, we notice that the decay rate becomes respectively small or large for large or small but fixed values of t, and that for finite $t \geq 0$ the energy $E(z,t)$ decays exponentially with respect to axial distance.

All terms, except $E(0,t)$, on the right of (33) depend explicitly upon the data, but without a similar dependence being established for $E(0,t)$ the bound is incomplete. Before discussing this topic, however, we discuss spatial stability with respect to the temperature and displacement. This is easily demonstrated for the temperature since (33) immediately delivers an estimate for its mean-square measure. To obtain a corresponding estimate for the displacement requires separate discussion of the transverse and axial components. Consequently, let us first remark that Poincaré's inequality yields:

$$\int_0^t \int_{D(z)} u_\alpha u_\alpha \, dS\, d\eta \leq \lambda^{-1} \int_0^t \int_{D(z)} u_{\alpha,\beta} u_{\alpha,\beta} \, dS\, d\eta, \tag{34}$$

where λ is the smallest positive eigenvalue of

$$\phi_{,\alpha\alpha} + \lambda\phi = 0, \quad x_\alpha \in D, \quad \phi = 0, \quad x_\alpha \in \partial D. \tag{35}$$

An obvious algebraic identity and the divergence theorem lead to:

$$\begin{aligned} \int_0^t \int_{D(z)} u_{\alpha,\beta} u_{\alpha,\beta} \, dS\, d\eta &= 2\int_0^t \int_{D(z)} \left(e_{\alpha\beta}e_{\alpha\beta} - u_{\alpha,\alpha}u_{\beta,\beta}\right) dS\, d\eta \\ &\leq 2\int_0^t \int_{D(z)} e_{\alpha\beta}e_{\alpha\beta} \, dS\, d\eta \\ &\leq -(2/c_0)\frac{dE(z,t)}{dz}, \end{aligned} \tag{36}$$

where we recall (24). Integration of (36) and an appeal to either (2) or (33) then yield:

$$\begin{aligned}\int_0^t\int_{B(z)} u_\alpha u_\alpha \leq & (2/\lambda c_0)E(z,t)\\ \leq & (2/\lambda c_0)\Big(E(0,t)+t\gamma^{-1}(t)\int_0^\infty \exp(y\gamma^{-1}(t))R(y)\,dy\Big)\exp(-z/\gamma(t)),\end{aligned} \tag{37}$$

which is the required estimate for the transverse displacement.

For the axial displacement, we employ the argument presented in [4], which is summarised here for completeness. On letting

$$V(z,t)=\int_0^t\int_{B(z)} u_3^2\,dx\,d\eta, \tag{38}$$

for fixed t, and on using Schwarz's inequality together with (2), we have:

$$\begin{aligned}-\frac{dV(z,t)}{dz} &= \int_0^t\int_{D(z)} u_3^2\,dS\,d\eta\\ &= -2\int_0^t\int_{B(z)} u_3u_{3,3}\,dx\,d\eta\\ &\leq (2/c_0^{1/2})V^{1/2}(z,t)E^{1/2}(z,t).\end{aligned} \tag{39}$$

Integration of the last inequality and use of (33) enables us to conclude that

$$V(z,t)\leq 4\gamma^2(t)c_0^{-1}\left(E(0,t)+t\gamma^{-1}\int_0^\infty \exp(y\gamma^{-1})R(y)\,dy\right)\exp(-z\gamma^{-1}(t)), \tag{40}$$

which completes the bounds for the displacement.

It remains to provide an upper bound for $E(0,t)$ in terms of the data. It follows from (25) and (26) that we have:

$$\begin{aligned}E(0,t)=-2\int_0^t\int_{D(0)}&(t-\eta)\,\{c_{i3k3}u_{i,3}l_{k,\eta}+c_{i\alpha k3}l_{i,\alpha}l_{k,\eta}\\ &+\beta_{i3}l_{i,\eta}\Phi+k_{ij}\theta\theta_{,i}n_j\}\;dS\,d\eta+tR(0).\end{aligned} \tag{41}$$

Let us consider the first term on the right of (41). By Schwarz's inequality, we obtain:

$$\begin{aligned}-2\int_0^t\int_{D(0)}(t-\eta)c_{i3k3}u_{i,3}l_{k,\eta}\,dS\,d\eta \leq 2\Bigg[&\int_0^t\int_{D(0)}(t-\eta)c_{i3k3}u_{i,3}u_{k,3}\,dS\,d\eta\\ &\times\int_0^t\int_{D(0)}(t-\eta)c_{i3k3}l_{i,\eta}l_{k,\eta}\,dS\,d\eta\Bigg]^{1/2}.\end{aligned} \tag{42}$$

We bound the first term on the right by noting that (3) implies:

$$\int_0^t \int_B (t-\eta) u_{i,3} \left[\rho u_{i,\eta\eta} - (c_{ijkl} u_{k,l})_{,j} - (\beta_{ij}\theta)_{,j}\right] \, dx \, d\eta = 0. \tag{43}$$

An application of the divergence theorem and subsequent rearrangement yield the identity:

$$\begin{aligned}\int_0^t \int_{D(0)} (t-\eta) c_{i3k3} u_{i,3} u_{k,3} \, dS \, d\eta = & \int_0^t \int_{D(0)} (t-\eta) c_{i\alpha k\beta} l_{i,\alpha} l_{k,\beta} \, dS \, d\eta \\ & - 2\int_0^t \int_B (t-\eta) u_{i,\eta\eta} u_{i,3} \, dx \, d\eta \\ & + 2\int_0^t \int_B (t-\eta) u_{i,3} \{\beta_{ij,j}\theta + \beta_{ij}\theta_{,j}\} \, dx \, d\eta.\end{aligned} \tag{44}$$

We next seek bounds for the last two integrals on the right in the previous identity, and for this purpose let $\gamma_4 \ldots \gamma_6$ be positive functions of time that are to be determined. By successive application of the Schwarz and arithmetic-geometric mean inequalities, and appeal to the bound (13), we obtain

$$2\int_0^t \int_B (t-\eta) u_{i,3} \beta_{ij,j} \theta \, dx \, d\eta \le (M_3 t \gamma_4) \int_0^t \int_B u_{i,j} u_{i,j} \, dx \, d\eta + \frac{M_3 t}{\gamma_4} \int_0^t \int_B \theta^2 \, dx \, d\eta, \tag{45}$$

and

$$\begin{aligned}2\int_0^t \int_B (t-\eta) u_{i,3} \beta_{ij} \theta_{,j} \, dx \, d\eta \le & M_2 \gamma_5 \left(\frac{t}{k_0}\right)^{1/2} \int_0^t \int_B u_{i,j} u_{i,j} \, dx \, d\eta \\ & + \frac{M_2}{\gamma_5} \left(\frac{t}{k_0}\right)^{1/2} \int_0^t \int_B (t-\eta) k_{ij} \theta_{,i} \theta_{,j} \, dx \, d\eta.\end{aligned} \tag{46}$$

The second integral on the right of (44) is treated by rearrangement of terms succeeded by integration, use of the data (15)-(18), together with standard inequalities. Because we have that

$$2\int_0^t \int_B (t-\eta) \rho u_{i,\eta\eta} u_{i,3} \, dx \, d\eta = 2\int_0^t \int_B (t-\eta) \left[(\rho u_{i,\eta} u_{i,3})_{,\eta} - \rho u_{i,\eta} u_{i,\eta 3}\right] \, dx \, d\eta,$$

we obtain:

$$
\begin{aligned}
-2\int_0^t\int_B (t-\eta)\rho u_{i,\eta\eta}u_{i,3}\,dx\,d\eta \le \gamma_6\int_0^t\int_B \rho u_{i,\eta}u_{i,\eta}\,dx\,d\eta \\
+\frac{\rho_1}{\gamma_6}\int_0^t\int_B u_{i,j}u_{i,j}\,dx\,d\eta + 2t\int_B \rho h_i g_{i,3}\,dx \\
-\int_0^t\int_{D(0)}(t-\eta)\rho l_{i,\eta}l_{i,\eta}\,dS\,d\eta. \quad (47)
\end{aligned}
$$

We now derive a bound for the first integral on the right of (45), which appears also in (46) and (47), and again apply an argument presented in [4, eqn.(5.7) ff.]. We commence with the identity

$$\int_0^t\int_B u_{i,j}u_{i,j}\,dx\,d\eta = \int_0^t\int_B (2e_{ij}e_{ij} - 2u_{\alpha,3}u_{3,\alpha} - u_{\alpha,\beta}u_{\beta,\alpha} - u_{3,3}^2)\,dx\,d\eta, \quad (48)$$

which after an integration by parts and standard inequalities yields

$$\int_0^t\int_B u_{i,j}u_{i,j}\,dx\,d\eta \le 2c_0^{-1}\int_0^t\int_B c_{ijkl}e_{ij}e_{kl}\,dx\,d\eta - 2\int_0^t\int_{D(0)} l_3 l_{\alpha,\alpha}\,dS\,d\eta. \quad (49)$$

Substitution of (45)-(49) in (44) then leads to

$$
\begin{aligned}
\int_0^t\int_{D(0)}(t-\eta)c_{i3k3}u_{i,3}u_{k,3}\,dS\,d\eta \le \frac{2}{c_0}\left[\frac{\rho_1}{\gamma_6} + \gamma_5 M_2\left(\frac{t}{k_0}\right)^{1/2} + \gamma_4 M_3 t\right]\times \\
\times\int_0^t\int_B c_{ijkl}e_{ij}e_{kl}\,dx\,d\eta + \gamma_6\int_0^t\int_B \rho u_{i,\eta}u_{i,\eta}\,dx\,\eta + \frac{M_3 t}{\gamma_4}\int_0^t\int_B \theta^2\,dx\,d\eta \\
+\frac{M_2}{\gamma_5}\left(\frac{t}{k_0}\right)^{1/2}\int_0^t\int_B (t-\eta)k_{ij}\theta_{,i}\theta_{,j}\,dx\,d\eta + Q_1(t), \quad (50)
\end{aligned}
$$

where $Q_1(t)$, dependent upon the data, is expressed by

$$
\begin{aligned}
Q_1(t) = \int_0^t\int_{D(0)}(t-\eta)c_{i\alpha k\beta}l_{i,\alpha}l_{k,\beta}\,dS\,d\eta + 2t\int_B \rho h_i g_{i,3}\,dx \\
-\int_0^t\int_{D(0)}(t-\eta)l_{i,\eta}l_{i,\eta}\,dS\,d\eta \qquad (51) \\
-2\left[(\rho_1/\gamma_6) + \gamma_5 M_2\left(\frac{t}{k_0}\right)^{1/2} + \gamma_4 M_3 t\right]\int_0^t\int_{D(0)} l_3 l_{\alpha,\alpha}\,dS\,d\eta.
\end{aligned}
$$

In order to bound the final term in the integral appearing in (41), we introduce the auxiliary function defined by:

$$\psi(x,t) = \Phi(x_\alpha,t)\exp(-\delta x_3), \quad (52)$$

where δ is an arbitrary positive constant. Note that $\psi(x_\alpha, 0, t) = \theta(x_\alpha, 0, t)$, so that we have

$$\begin{aligned}
&\int_0^t \int_{D(0)} (t-\eta) k_{ij}\theta\theta_{,i} n_j \, dS\, d\eta = \int_0^t \int_{D(0)} (t-\eta) k_{ij}\psi\theta_{,i} n_j \, dS\, d\eta \\
&= \int_0^t \int_B (t-\eta)[\psi(k_{ij}\theta_{,i})_{,j} + k_{ij}\theta_{,i}\psi_{,j}]\, dx\, d\eta = \int_0^t \int_{D(0)} (t-\eta)\Phi\beta_{i3} l_{i,\eta}\, dS\, d\eta \\
&+ \int_0^t \int_B (t-\eta)(\beta_{ij,j}\psi + \beta_{ij}\psi_{,j}) u_{i,\eta}\, dx\, d\eta + \int_0^t \int_B \psi\theta \, dx\, d\eta \\
&- \int_0^t \int_B (t-\eta)\psi_{,\eta}\theta\, dx\, d\eta + \int_0^t \int_B (t-\eta) k_{ij}\theta_{,i}\psi_{,j}\, dx\, d\eta \\
&- t\int_B f\Phi(x_\alpha, 0)\exp(-\delta x_3)\, dx, \qquad (53)
\end{aligned}$$

where the last equation is obtained from (4), the divergence theorem, and the asymptotic assumption (2). Let $\gamma_7, \ldots, \gamma_{11}$ be positive functions of time that are to be chosen. Standard inequalities applied to (53) give:

$$\begin{aligned}
&\int_0^t \int_{D(0)} (t-\eta) k_{ij}\theta\theta_{,i} n_j dS d\eta \\
&\qquad \le \frac{t}{2(2\delta\rho_0)^{1/2}} [\gamma_7 M_3 + \gamma_8 M_2] \int_0^t \int_B \rho u_{i,\eta} u_{i,\eta} dx d\eta \\
&\qquad + \frac{1}{2(2\delta)^{1/2}} [\gamma_9 + \gamma_{10} t] \int_0^t \int_B \theta^2 \, dx\, d\eta \\
&\qquad + \frac{\gamma_{11}}{2\delta^{1/2}} \int_0^t \int_B (t-\eta) k_{ij}\theta_{,i}\theta_{,j}\, dx\, d\eta + Q_2(t), \qquad (54)
\end{aligned}$$

where the data term $Q_2(t)$ is given by:

$$\begin{aligned}
Q_2(t) \; &= \int_0^t \int_{D(0)} (t-\eta)\Phi\beta_{i3} l_{i,\eta}\, dS\, d\eta - t\int_B f\Phi(x_\alpha, 0)\exp(-\delta x_3)\, dx \\
&+ \frac{1}{2}\left[\frac{M_3 t}{\gamma_7\sqrt{2\delta\rho_0}} + \frac{M_2 t\delta^{3/2}}{\gamma_8\sqrt{2\rho_0}} + \frac{1}{\gamma_9\sqrt{2\delta}} + \frac{\delta^{3/2} M_1^2 t}{2\gamma_{11}}\right] \int_0^t \int_{D(0)} \Phi^2 \, dS\, d\eta \\
&+ \frac{t}{4\gamma_{11}\delta^{1/2}} \left[\int_0^t \int_{D(0)} (k_{\alpha\beta}\Phi_{,\alpha}\Phi_{,\beta} - 2\delta k_{\alpha 3}\Phi\Phi_{,\alpha})\, dS\, d\eta\right] \\
&+ \frac{M_2 t}{2\gamma_8\sqrt{2\delta\rho_0}} \int_0^t \int_{D(0)} \Phi_{,\alpha}\Phi_{,\alpha}\, dS\, d\eta + \frac{t}{2\gamma_{10}\sqrt{2\delta}} \int_0^t \int_{D(0)} \Phi_{,\eta}^2\, dS\, d\eta. \qquad (55)
\end{aligned}$$

On replacing the respective terms in (41) by the bounds (50) and (54) and after further use of the arithmetic-geometric mean inequality, we obtain for the arbitrary positive function $\gamma_{12}(t)$ the inequality:

$$E(0,t) \leq \left[\gamma_6\gamma_{12} + t(2\delta\rho_0)^{-1/2}\{\gamma_7 M_3 + \gamma_8 M_2\}\right] \int_0^t \int_B \rho u_{i,\eta} u_{i,\eta}\, dx\, d\eta$$
$$+2\gamma_{12} c_0^{-1} \left[\gamma_6^{-1}\rho_1 + \gamma_5 M_2 (t/k_0)^{1/2} + \gamma_4 M_3 t\right] \int_0^t \int_B c_{ijkl} e_{ij} e_{kl}\, dx\, d\eta$$
$$+ \left[\gamma_4^{-1}\gamma_{12} M_3 t + (2\delta)^{-1/2}\{\gamma_9 + \gamma_{10} t\}\right] \int_0^t \int_B \theta^2\, dx\, d\eta$$
$$+ \left[\gamma_{12}\gamma_5^{-1}(t/k_0)^{1/2} + \gamma_{11}\delta^{-1/2}\right] \int_0^t \int_B (t-\eta) k_{ij}\theta_{,i}\theta_{,j}\, dx\, d\eta + Q(t), \tag{56}$$

where $Q(t)$ is expressed in terms of data by:

$$Q(t) = \gamma_{12} Q_1(t) + 2Q_2(t) - 2\int_0^t \int_{D(0)} (t-\eta) c_{i\alpha k3} l_{i,\alpha} l_{k,\eta}\, dS\, d\eta$$
$$-2\int_0^t \int_{D(0)} (t-\eta)\beta_{i3} l_{i,\eta} \Phi\, dS\, d\eta$$
$$+\frac{1}{2\gamma_{12}} \int_0^t \int_{D(0)} (t-\eta) c_{i3k3} l_{i,\eta} l_{k,\eta}\, dS\, d\eta + tR(0). \tag{57}$$

Finally, on selecting the functions $\gamma_4, \ldots, \gamma_{12}$ to be

$$\gamma_4 = (c_0/2)\left(1 + \sqrt{8c_0^{-1} + 1}\right), \qquad \gamma_5 = \frac{1}{2}\sqrt{(c_0/M_2)}, \tag{58}$$
$$\gamma_6 = \sqrt{(2\rho_1/c_0)}, \qquad \gamma_9 = 2\gamma_{12}\sqrt{(2\rho_1\delta/c_0)}, \tag{59}$$
$$\gamma_7 = \gamma_8 = \frac{2(2\delta\rho_0)^{1/2} M_3 \gamma_{12}\gamma_4}{c_0(M_2 + M_3)}, \tag{60}$$
$$\gamma_{10} = \gamma_{12} M_3 \sqrt{(2\delta)}, \qquad \gamma_{11} = 2\sqrt{(2\delta)}\gamma_{12}\sqrt{(\rho_1/c_0)}, \tag{61}$$

and

$$\gamma_{12} = \frac{c_0^{1/2} r}{\sqrt{2\left[\rho_1^{1/2} + \sqrt{(M_2 t/2k_0)} + M_3 t\gamma_4\sqrt{(2/c_0)}\right]}}, \tag{62}$$

where r is any positive number with value in the open range $(0,1)$, we conclude that the estimate (56) reduces to

$$E(0,t) \leq \frac{Q(t)}{1-r}, \tag{63}$$

which is the required upper bound for the amplitude $E(0,t)$ in the decay estimate (33). Observe that $Q(t) \to 0$ as $t \to 0$ and that δ may be chosen to optimise $Q(t)$ for a given t.

In the next section, we demonstrate structural stability, or equivalently continuous dependence, with respect to the thermal coupling coefficients β_{ij}.

4 Structural stability with respect to thermal coupling

The treatment is developed in terms of the difference displacement $w_i(x,t)$ defined by

$$w_i(x,t) = u_i(x,t) - v_i(x,t), \qquad (x,t) \in B \times [0,T), \tag{64}$$

where we recall from the previous notation that u_i and v_i represent the displacement components in the thermal and isothermal problems respectively.

From (3), (22), (14), (17),(18), and (15) it easily follows that

$$(c_{ijkl}e_{kl}(w))_{,j} + (\beta_{ij}\theta)_{,j} = \rho\ddot{w}_i, \qquad (x,t) \in B \times [0,T), \tag{65}$$

where

$$e_{ij}(w) = \frac{1}{2}(w_{i,j} + w_{j,i}), \tag{66}$$

and the corresponding boundary and initial conditions become:

$$w_i(x,t) = 0, \qquad (x,t) \in \partial D \times [0,T), \tag{67}$$

$$w_i(x,0) = \dot{w}_i(x,0) = 0, \qquad x \in B \times \{0\}, \tag{68}$$

$$w_i(x_\alpha,0,t) = 0, \qquad (x,t) \in D(0) \times [0,T). \tag{69}$$

The aim is to prove that $w_i(x,t)$ tends to zero in mean square measure as $\beta_{ij} \to 0$, and $\beta_{ij,k} \to 0$. For this purpose, we introduce the energy functional

$$W(t) = \int_0^t \int_B (t-\eta)\left[\rho w_{i,\eta}w_{i,\eta} + c_{ijkl}e_{ij}(w)e_{kl}(w)\right]\, dx\, d\eta. \tag{70}$$

By virtue of the governing equations (65), and the homogeneous data (67)-(69), we have that:

$$\dot{W}(t) = 2\int_0^t \int_B (t-\eta)(\beta_{ij}\theta)_{,j}w_{i,\eta}\, dx\, d\eta. \tag{71}$$

Expansion of the integrand followed by application of Schwarz's inequality then gives:

$$\begin{aligned}\dot{W}(t) \leq & 2\left[\frac{M_2^2}{\rho_0 k_0}\int_0^t \int_B (t-\eta)k_{ij}\theta_{,i}\theta_{,j}\, dx\, d\eta \int_0^t \int_B (t-\eta)\rho w_{,\eta}w_{,\eta}\, dx\, d\eta\right]^{1/2} \\ & +2\left[\frac{tM_3^2}{\rho_0}\int_0^t \int_B \theta^2\, dx\, d\eta \int_0^t \int_B (t-\eta)\rho w_{,\eta}w_{,\eta}\, dx\, d\eta\right]^{1/2} \\ \leq & 2A(t)W(t)^{1/2}E(0,t)^{1/2},\end{aligned} \tag{72}$$

where we have appealed to (11),(8),(13), and (70), and where

$$A(t) = \left(M_3(\frac{t}{\rho_0})^{1/2} + \frac{M_2}{\sqrt{(2k_0\rho_0)}} \right). \tag{73}$$

We may then conclude after further integration, standard inequalities, and from (23) that

$$W(t) \leq \int_0^t A^2(s)\, ds \int_0^t E(0,s)\, ds. \tag{74}$$

Now, the uniform convergence to zero of the thermal coupling and its spatial derivative implies that $M_2 \to 0, M_3 \to 0$ and consequently that $A(t) \to 0$, for all finite t. It follows from (74) and the boundedness of $E(0,t)$, established in the previous section, that $W(t) \to 0$ for all finite t. Finally, on recalling the data (67) and (69), we may use an argument similar to that employed for the estimate (49) in order to derive the bound

$$\int_0^t \int_B (t-\eta) w_i w_i \, dx\, d\eta \leq 2(\lambda c_0)^{-1} W(t), \tag{75}$$

and the proof of the main result is complete.

We may deduce from the last inequality that the mean-square *cross-sectional* measure of the difference displacement also vanishes with the thermal coupling and its derivatives. The conclusion follows by virtue of definition (1), the asymptotic behaviour (2), and successive use of standard inequalities, together with (49), (64), and (70). We have:

$$\begin{aligned}
&\int_0^t \int_{D(z)} (t-\eta) w_i w_i \, dS\, d\eta = -2 \int_0^t \int_{B(z)} (t-\eta) w_i w_{i,3} \, dx\, d\eta \\
\leq & 2 \left[\int_0^t \int_{B(z)} (t-\eta) w_i w_i \, dx\, d\eta \int_0^t \int_{B(z)} (t-\eta) w_{i,3} w_{i,3} \, dx\, d\eta \right]^{1/2} \\
\leq & 2 \left[\int_0^t \int_{B} (t-\eta) w_i w_i \, dx\, d\eta \int_0^t \int_{B} (t-\eta) w_{i,j} w_{i,j} \, dx\, d\eta \right]^{1/2} \\
\leq & 4 c_0^{-1} \lambda^{-1/2} W(t),
\end{aligned} \tag{76}$$

and the assertion is proved.

Remark. Detailed consequences for the decay rate of a vanishing thermal coupling may be examined by means of the arguments presented here. An extended analysis will be developed elsewhere.

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