Note di Matematica **27,** n. 2, 2007, 55–67.

A new class of fit regions

Gianpietro Del Piero
Dipartimento di Ingegneria, Università di Ferrara,
Via Saragat 1, 44100 Ferrara, Italy
dlpgpt@unife.it

Received: 09/10/2006; accepted: 30/10/2006.

Abstract. A new set of regularity assumptions is proposed for the regions of space which can be occupied by the continuous bodies of continuum mechanics. The new assumptions are more restrictive than those made in the preceding proposals [3, 4, 8]. This has the favorable effect of excluding some pathological regions which were present in all classes of fit regions proposed earlier.

Keywords: foundations of continuum mechanics, fit regions

MSC 2000 classification: primary 74A05, secondary 74A99

Dedicated to W. A. Day on the occasion of his 65th birthday

1 Introduction

The subject of the regularity of the regions of space occupied by continuous bodies attracted the attention of a number of researchers interested in the foundations of continuum mechanics [2, 4, 7]. In the paper [8] the term *fit region* was coined, and the following requirements for a class of fit regions were fixed:

(F1) The set of all fit subregions of a given fit region should satisfy the axioms of a *material universe*.

(F2) A class of fit regions should be invariant under transplacements, which should include adjustments to fit regions of smooth diffeomorphisms from one Euclidean space to another.

(F3) Each region should have a *surface-like boundary* for which a form of the integral-gradient (Gauss-Green) theorem should be valid.

A *material universe* is a pair $(\Psi, \prec)$, with Ψ a set and $\prec$ a partial ordering on Ψ. The elements $A, B, \dots$ of Ψ are called *bodies*, and $A \prec B$ is to be read as *A is a part of B*. A list of axioms for a material universe was given in Noll's paper [6] and in Truesdell's book [9], and the following modified list was proposed in my paper [3]. The first axiom in the list is the existence of the *null body*

(B1) There is an element $\emptyset$ in Ψ such that $\emptyset \prec A$ for all $A \in \Psi$,

and the second is the existence of the join

(B2) For every A, B in Ψ there is a $C \in \Psi$ such that

(i) A and B are parts of C,

(ii) if A and B are parts of D, then C is a part of D.

The body C is called the *join* of A and B and is denoted by $A \vee B$.[1] Two bodies are *separate* if the only body which is a part of both is the null body:

$$A \text{ and } B \text{ are separate} \qquad \Longleftrightarrow \qquad C \prec A \text{ and } C \prec B \;\Rightarrow\; C = \emptyset\,.$$

The third axiom is a *separation postulate*:

(B3) If A and C are separate and if B and C are separate, then $A \vee B$ and C are separate,

and the fourth, and last, axiom is a *partition postulate*

(B4) If $A \prec C$, there is a part A_C of C such that

(i) A and A_C are separate,

(ii) $A \vee A_C = C$.

A_C is called the *complementary part* of A in C.[2]

A basic point made in [8] is that the class of the fit regions should be as small as possible, in order to "include all that can be possibly imagined by an engineer but exclude those that can be dreamt up *only* by an ingenious mathematician". With this goal in mind, Noll and Virga restricted the class of the *d-regular* open regions, bounded and with finite perimeter, proposed in [4].[3] They gave the following definition of a fit region Ω of a N-dimensional Euclidean space:

(NV1) Ω is bounded,

(NV2) Ω is regularly open,

(NV3) Ω has finite perimeter,

[1]For a proof of the uniqueness of the null body and of the join see [3].

[2]The name of the partition postulate comes from the fact that A and A_C form a partition of C in the standard sense of set theory, see e.g. [5].

[3]For the terminology used here and below see the following section.

(NV4) Ω has negligible boundary.

In [3], I proposed the following class:

(D1) Ω is bounded,

(D2) $\operatorname{int}\Omega$ is regularly open,

(D3) $\operatorname{clo}\Omega$ is regularly closed,

(D4) $\mathcal{H}^{N-1}(\operatorname{bdy}\Omega)$ is finite,

where $\mathcal{H}^{N-1}$ is the $(N-1)$-dimensional Hausdorff measure, and $\operatorname{int}\Omega$, $\operatorname{clo}\Omega$, $\operatorname{bdy}\Omega$ are the topological interior, closure, and boundary of Ω, respectively. There is no inclusion relation between this class and the class of the NV-regions. Indeed, a D-region need not be a NV-region because it need not be open. Conversely, a NV-region need not be a D-region because, as we shall see, condition (D4) is more strict than (NV4).[4]

It is shown in [3] that the D-regions form equivalence classes, such that all regions in the same equivalence class have the same interior and the same closure. Therefore, one can take the open set $\operatorname{int}\Omega$ as the representative for the equivalence class of a region Ω. The class formed by the open representatives of D-regions is strictly included in the class of the NV-regions, and an example of a NV-region which is not a D-region will be given in Section 5 below.

In this paper I restrict my analysis to the open representatives of fit regions, for which condition (D3) is automatically satisfied. Moreover, I impose a supplementary restriction (E4) involving the essential boundary $\operatorname{eby}\Omega$ of Ω. So, I define the following class

(E1) Ω is bounded,

(E2) Ω is regularly open,

(E3) $\mathcal{H}^{N-1}(\operatorname{bdy}\Omega)$ is finite,

(E4) $\mathcal{H}^{N-1}(\operatorname{bdy}\Omega) = \mathcal{H}^{N-1}(\operatorname{eby}\Omega)$,

which I call the class of the E-regions. I prove below that all E-regions are D-regions, while an open D-region need not be an E-region, as shown by an example provided in Section 5.

I close this introduction with a pair of comments.

[4] The possibility of substituting condition (NV4) with (D4) was taken into consideration by Noll and Virga, see Remark 6 in [8].

1 Remark. The boundedness assumption (E1) is not essential and can be removed, see footnote on p.194 of [3]. Here, I keep it for the sake of simplicity.

2 Remark. In [8], Noll and Virga claim that the class of the open d-regular regions with finite perimeter proposed in [4] is "unnecessarily large", and provide an example of a pathological region in this class, see Section 5 below. But the same class of regions has another inconvenience, shown by the following example. Let $N = 2$, and let A and B be two disjoint open circles, whose boundaries have in common a point P. According to the definition in [4], the join of A and B is the set of all density points of $A \cup B$. This is a d-regular region, since it coincides with the set of its own density points, but it is not open, since it includes the boundary point P. On the other hand, according to the definition in [8] the join of A and B is the interior of the closure of $A \cup B$. This set is open, but not d-regular, since it does not include the density point P. Therefore, in neither case the join of an open d-regular region is an open d-regular region, in contrast with the requirement (B2) that the join of fit regions should be a fit region.

2 Technical preliminaries

Here I collect some basic notations and definitions from topology and geometric measure theory. For more details the reader is addressed to the book [1] by Ambrosio, Fusco and Pallara.

Let Ω be a region of $\mathbb{R}^N$, and let $\operatorname{int}\Omega$, $\operatorname{ext}\Omega$, $\operatorname{bdy}\Omega$ be the topological interior, exterior, and boundary of Ω, respectively. The three sets are pairwise disjoint, and their union is $\mathbb{R}^N$. The union of $\operatorname{int}\Omega$ and $\operatorname{bdy}\Omega$ is the closure of Ω and is denoted by $\operatorname{clo}\Omega$. The region Ω is open if $\operatorname{int}\Omega = \Omega$, and is closed if $\operatorname{clo}\Omega = \Omega$. It is *regularly open* if $\operatorname{int}\operatorname{clo}\Omega = \Omega$, and it is *regularly closed* if $\operatorname{clo}\operatorname{int}\Omega = \Omega$. If Ω is regularly open, it is open, and therefore $\operatorname{int}\Omega = \Omega$ is regularly open; moreover, $\operatorname{clo}\operatorname{int}\operatorname{clo}\Omega = \operatorname{clo}\Omega$, so that $\operatorname{clo}\Omega$ is regularly closed. This proves that, for an open region, condition (NV2) in Section 1 implies (D2) and (D3).

Denote by $\mathcal{L}^N$ the N-dimensional Lebesgue measure, and for any integer M with $0 \le M \le N$ denote by $\mathcal{H}^M$ the M-dimensional Hausdorff measure. Recall[5] that $\mathcal{H}^N = \mathcal{L}^N$ and that, for all $L < M \le N$,

$$\mathcal{H}^M > 0 \implies \mathcal{H}^L = +\infty\,, \quad \mathcal{H}^L < +\infty \implies \mathcal{H}^M = 0\,.$$

For a region Ω of $\mathbb{R}^N$ and for any $x \in \mathbb{R}^N$, consider the limit

$$\lim_{r\to 0} \frac{\mathcal{L}^N(B(x,r)\cap\Omega)}{\mathcal{L}^N(B(x,r))}\,,$$

[5] See [1, Section 2.8].

where $B(x,r)$ is the open ball centered at x with radius r. Then x is said to be a *point of density* for Ω if the above limit is equal to one, a *point of rarefaction* if it is equal to zero, and a point in the *essential boundary* of Ω in any other case. This defines three pairwise disjoint sets $\operatorname{dns}\Omega$, $\operatorname{rar}\Omega$, $\operatorname{eby}\Omega$, whose union is $\mathbb{R}^N$. For every $\Omega \subset \mathbb{R}^N$, the following inclusions hold:

$$\operatorname{int}\Omega \subset \operatorname{dns}\Omega \subset \operatorname{clo}\Omega, \quad \operatorname{ext}\Omega \subset \operatorname{rar}\Omega, \quad \operatorname{eby}\Omega \subset \operatorname{bdy}\Omega\,. \tag{1}$$

Regions such that $\Omega = \operatorname{dns}\Omega$ are called *d-regular regions* [4].

The *perimeter* of a set can be identified with the $(N-1)$-dimensional Hausdorff measure of the essential boundary:

$$\operatorname{per}\Omega = \mathcal{H}^{N-1}(\operatorname{eby}\Omega)\,,$$

see [1, Section 3.5]. Therefore, the condition (E3) that Ω has a finite $(N-1)$-dimensional Hausdorff measure is equivalent to the condition (NV3) that Ω is a set with finite perimeter.

For a NV-region Ω, conditions (NV3) and (NV4) require that

$$\operatorname{per}\Omega \le +\infty\,, \quad \mathcal{L}^N(\operatorname{bdy}\Omega) < +\infty\,. \tag{2}$$

Therefore, $\mathcal{L}^N(\operatorname{bdy}\Omega)$ is allowed to be greater than zero, and in this case one has $\mathcal{H}^{N-1}(\operatorname{bdy}\Omega) = +\infty$. An example of a NV-region with $\mathcal{L}^N(\operatorname{bdy}\Omega) > 0$ is given in [8], and will be recalled in Section 5 below. For a D-region, this possibility is excluded by condition (D4). Moreover, from the inclusion $(1)_3$ it follows that

$$\operatorname{per}\Omega \le \mathcal{H}^{N-1}(\operatorname{bdy}\Omega)\,, \tag{3}$$

so that condition (D4) implies (NV3). Together with the fact that for an open region (D2) is the same as (NV2), this proves that the set of all open D-regions is a subclass of the NV-regions. That the inclusion is strict will be proved by an example in Section 5.

For an open region, conditions (E1)–(E3) are equivalent to (D1)–(D4). Therefore, every E-region is an open D-region. Again, strict inclusion will be proved by an example in Section 5.

I recall that a *homeomorphism* of $\mathbb{R}^N$ is a bijection from $\mathbb{R}^N$ onto $\mathbb{R}^N$, continuous and with a continuous inverse, and that a *smooth diffeomorphism* of $\mathbb{R}^N$ is a C^1 bijection from $\mathbb{R}^N$ onto $\mathbb{R}^N$ with a C^1 inverse. A locally bi-Lipschitz function, or *locally bi-Lipschitz homeomorphism*, is a homeomorphism f of $\mathbb{R}^N$ with the following property: for every compact set K of $\mathbb{R}^N$ there are positive constants c_K, m_K such that

$$c_K|x-y| \le |f(x)-f(y)| \le m_K|x-y| \quad \forall x,y \in K\,. \tag{4}$$

The smooth diffeomorphisms of $\mathbb{R}^N$ are particular locally bi-Lipschitz homeomorphisms of $\mathbb{R}^N$. A reason for considering bi-Lipschitz homeomorphisms instead of smooth diffeomorphisms is that the deformations of continuum mechanics need not be continuously differentiable, but only differentiable almost everywhere.

All homeomorphisms of $\mathbb{R}^N$ have the property that

$$f(\operatorname{int} A) = \operatorname{int} f(A)\,, \quad f(\operatorname{clo} A) = \operatorname{clo} f(A)\,, \quad f(\operatorname{bdy} A) = \operatorname{bdy} f(A) \tag{5}$$

for every bounded region A of $\mathbb{R}^N$, and all locally bi-Lipschitz homeomorphisms have the property that

$$c_K^M \mathcal{H}^M(A) \leq \mathcal{H}^M(f(A)) \leq m_K^M \mathcal{H}^M(A)\,, \tag{6}$$

for all M such that $0 \leq M \leq N$, and with the constants c_K, m_K relative to any compact K of $\mathbb{R}^N$ containing A. The second inequality is proved in [1, Prop. 2.49], and the first follows from inequality $(4)_1$ rewritten in the form

$$\left|f^{-1}(f(x)) - f^{-1}(f(y))\right| \leq c_K^{-1} \left|f(x) - f(y)\right|\,. \tag{7}$$

From it, we have

$$\mathcal{H}^M(A) = \mathcal{H}^M\left(f^{-1}(f(A))\right) \leq c_K^{-M} \mathcal{H}^M(f(A))\,, \tag{8}$$

and this is inequality $(6)_1$.

Let $BLip_{loc}(\mathbb{R}^N)$ denote the set of all locally bi-Lipschitz homeomorphisms of $\mathbb{R}^N$. For this set I prove here some elementary properties, whose proofs I did not find in the literature.

3 Lemma. *For all homeomorphisms f of $\mathbb{R}^N$ and for all $A, B \subset \mathbb{R}^N$,*

$$f(A \cup B) = f(A) \cup f(B)\,, \quad f(A \cap B) = f(A) \cap f(B)\,. \tag{9}$$

Proof. The first equality holds for all $f : \mathbb{R}^N \to \mathbb{R}^N$, see e.g. [5, Section 1.3]. For the second equality, the inclusion $A \cap B \subset A$ proves that $f(A \cap B) \subset f(A)$. Because this also holds with A and B interchanged, one gets $f(A \cap B) \subset f(A) \cap f(B)$. If f is a homeomorphism, one also has

$$f^{-1}(f(A) \cap f(B)) \subset f^{-1}(f(A)) \cap f^{-1}(f(B)) = A \cap B\,.$$

By applying f, one gets the reversed inclusion $f(A) \cap f(B) \subset f(A \cap B)$ and, therefore, the desired equality.

$\boxed{QED}$

4 Lemma. *For any bounded subset* Ω *of* $\mathbb{R}^N$ *and for every* $f \in BLip_{loc}(\mathbb{R}^N)$,

$$f(\operatorname{dns}\Omega) = \operatorname{dns} f(\Omega)\,, \quad f(\operatorname{rar}\Omega) = \operatorname{rar} f(\Omega)\,, \quad f(\operatorname{eby}\Omega) = \operatorname{eby} f(\Omega)\,. \tag{10}$$

Proof. Let $x \in \operatorname{rar}\Omega$, so that

$$\lim_{r\to 0} \frac{\mathcal{L}^N(\Omega \cap B(x,r))}{\mathcal{L}^N(B(x,r))} = 0\,. \tag{11}$$

Let K be a compact containing Ω. From (6) with $M = N$, for $A = \Omega \cap B(x,r)$ we have

$$c_K^N \mathcal{L}^N(\Omega \cap B(x,r)) \le \mathcal{L}^N(f(\Omega \cap B(x,r))) \le m_K^N \mathcal{L}^N(\Omega \cap B(x,r))\,, \tag{12}$$

and for $A = B(x,r)$ we have

$$c_K^N \mathcal{L}^N(B(x,r)) \le \mathcal{L}^N(f(B(x,r))) \le m_K^N \mathcal{L}^N(B(x,r))\,, \tag{13}$$

so that

$$\lim_{r\to 0} \frac{\mathcal{L}^N(f(\Omega \cap B(x,r)))}{\mathcal{L}^N(f(B(x,r)))} \le \frac{m_K^N}{c_K^N} \lim_{r\to 0} \frac{\mathcal{L}^N(\Omega \cap B(x,r))}{\mathcal{L}^N(B(x,r))} = 0\,. \tag{14}$$

From inequality (4) it follows that

$$B(f(x), c_K r) \subset f(B(x,r)) \subset B(f(x), m_K r)\,. \tag{15}$$

Then, by the preceding lemma,

$$f(\Omega) \cap B(f(x), c_K r) \subset f(\Omega) \cap f(B(x,r)) = f(\Omega \cap B(x,r))\,, \tag{16}$$

so that $\mathcal{L}^N(f(\Omega) \cap B(f(x), c_K r)) \le \mathcal{L}^N(f(\Omega \cap B(x,r)))$. Moreover, by (13),

$$\begin{aligned}\mathcal{L}^N(f(B(x,r))) &\le m_K^N \mathcal{L}^N(B(x,r)) \\ &= m_K^N \mathcal{L}^N(B(f(x),r)) = \frac{m_K^N}{c_K^N} \mathcal{L}^N(B(f(x), c_K r))\,.\end{aligned} \tag{17}$$

Then,

$$\lim_{c_K r\to 0} \frac{\mathcal{L}^N(f(\Omega) \cap B(f(x), c_K r))}{\mathcal{L}^N(B(f(x), c_K r))} \le \frac{m_K^N}{c_K^N} \lim_{r\to 0} \frac{\mathcal{L}^N(f(\Omega \cap B(x,r)))}{\mathcal{L}^N(f(B(x,r)))}\,, \tag{18}$$

with the right-hand side equal to zero by (14). This proves that f maps rarefaction points of Ω into rarefaction points of $f(\Omega)$:

$$f(\operatorname{rar}\Omega) \subset \operatorname{rar} f(\Omega)\,.$$

Since this property also holds for the inverse mapping f^{-1}, one has

$$f^{-1}(\operatorname{rar} A) \subset \operatorname{rar} f^{-1}(A)$$

for every bounded $A \subset \mathbb{R}^N$. For $A = f(\Omega)$, one gets $\operatorname{rar} f(\Omega) \subset f(\operatorname{rar}\Omega)$, and $(10)_2$ follows.

To prove $(10)_1$, it is sufficient to recall that the density points of Ω are the rarefaction points of $\mathbb{R}^N \setminus \Omega$. Finally, to prove $(10)_3$ we remark that, by $(10)_1$ and $(10)_2$,

$$f(\operatorname{dns}\Omega) \cup f(\operatorname{rar}\Omega) = \operatorname{dns} f(\Omega) \cup \operatorname{rar} f(\Omega) = \mathbb{R}^N \setminus \operatorname{eby} f(\Omega)\,.$$

Then f surjective implies

$$\mathbb{R}^N = f(\operatorname{eby}\Omega) \cup f(\operatorname{dns}\Omega) \cup f(\operatorname{rar}\Omega) = f(\operatorname{eby}\Omega) \cup \big(\mathbb{R}^N \setminus \operatorname{eby} f(\Omega)\big)\ ,$$

so that $\operatorname{eby} f(\Omega) \subset f(\operatorname{eby}\Omega)$, and f injective implies

$$\emptyset = f(\operatorname{eby}\Omega) \cap (f(\operatorname{dns}\Omega) \cup f(\operatorname{rar}\Omega)) = f(\operatorname{eby}\Omega) \cap \big(\mathbb{R}^N \setminus \operatorname{eby} f(\Omega)\big)\ ,$$

so that $\operatorname{eby} f(\Omega) \supset f(\operatorname{eby}\Omega)$. $\boxed{QED}$

3 One-dimensional fit regions

As a preparation to the case $N > 1$, consider first the one-dimensional case. It is known that an open set in $\mathbb{R}$ is a finite or countable union of pairwise disjoint open intervals, see e.g. [5, Section 6.6]. A *regularly open* set is then a finite or countable union of open intervals whose closures are pairwise disjoint.

For a bounded regularly open set, the perimeter is twice the number of the disjoint intervals. Therefore, a bounded regularly open set with finite perimeter is a *finite* union of open intervals with pairwise disjoint closures. For such a set, the essential boundary coincides with the topological boundary, and conditions (NV4), (D4), (E3), (E4) are all satisfied. Therefore, the classes of NV-regions, open D-regions, and E-regions coincide when $N = 1$.

Let us verify the conditions (F1)–(F3) for a fit region. Condition (F1) requires that the axioms (B1)–(B4) for a material universe $(\Psi, \prec)$ be satisfied. Let us take as Ψ the set whose elements $A, B, C \ldots$ are finite unions of bounded open intervals of the real line with pairwise disjoint closures, and as $\prec$ the set inclusion $\subset$. Then axiom (B1) is satisfied by taking as the null body the empty set, and axiom (B2) is satisfied by defining the join of A and B as

$$A \vee B = \operatorname{int} \operatorname{clo} A \cup B\,. \tag{19}$$

Two regions A, B are separate if and only if their intersection is the empty set. The separation axiom (B3) then reduces to the implication

$$A \cap C = B \cap C = \emptyset \quad \Longrightarrow \quad (\operatorname{int}\operatorname{clo} A \cup B) \cap C = \emptyset\,,$$

and the separation axiom (B4) is verified by taking

$$A_C = \operatorname{int}\left(C \setminus A\right)\,,$$

for all A, C such that $A \subset C$.

The condition (F2) on fit regions is verified if the image $f(\Omega)$ of a fit region Ω under a locally bi-Lipschitz homeomorphism f of $\mathbb{R}$ is a fit region. In fact, because f is a homeomorphism, each of the open intervals which form Ω is mapped into an open interval. Thus, $f(\Omega)$ is a finite union of open intervals. Moreover, two intervals have disjoint closures if and only if their distance $d(I_i, I_j)$ is strictly positive; for $f \in BLip_{loc}(\mathbb{R})$, from $(4)_1$ we have

$$d(f(I_i),\, f(I_j)) \geq c_K d(I_i,\, I_j) > 0\,.$$

Then $f(\Omega)$ is a finite union of open intervals with disjoint closures. Finally, each interval $f(I_i)$ is bounded, because $|f(x) - f(y)| \leq m_K|x - y| < +\infty$ for all $x, y \in I_i$. Then, $f(\Omega)$ is a fit region.

The requirement (F3) is trivially verified, since for $N = 1$ the Gauss-Green formula reduces to the integration by parts formula at each of the bounded intervals which form a fit region.

4 The multi-dimensional case

Let now $N > 1$. That NV-regions and D-regions satisfy the requirements (F1)–(F3) for a fit region is proved in [8] and in [3], respectively. Here we prove that the same requirements are satisfied by E-regions. For the condition (F1) on a material universe $(\Psi, \prec)$, we take as Ψ the set of all E-regions and for $\prec$ the set inclusion $\subset$. The proof that axioms (B1)–(B4) for a material universe are satisfied may be achieved by repeating the arguments used in the one-dimensional case.

For the requirement (F2), let us prove that the image of an E-region under a locally bi-Lipschitz homeomorphism f is an E-region. From (6) with $M = N$ we have $\mathcal{L}^N(f(\Omega)) \leq m_K^N \mathcal{L}^N(\Omega)$, and this proves the condition (E1) of the boundedness of $f(\Omega)$. Moreover, from (5) we have

$$\operatorname{int}\operatorname{clo} f(\Omega) = \operatorname{int} f(\operatorname{clo}\Omega) = f(\operatorname{int}\operatorname{clo}\Omega) = f(\Omega)\,, \tag{20}$$

and this proves condition (E2).

From $(5)_3$ and from $(6)_2$ with $M = N-1$ and $A = \mathrm{bdy}\,\Omega$, we have

$$\mathcal{H}^{N-1}(\mathrm{bdy}\, f(\Omega)) = \mathcal{H}^{N-1}(f(\mathrm{bdy}\,\Omega)) \le m_K^{N-1}\mathcal{H}^{N-1}(\mathrm{bdy}\,\Omega) < +\infty\,, \qquad (21)$$

so that condition (E3) is satisfied as well.

Less quick is the proof of condition (E4). I begin with the observation that, due to the inclusion $\mathrm{eby}\,\Omega \subset \mathrm{bdy}\,\Omega$, condition (E4) means that the set $\mathrm{bdy}\,\Omega \setminus \mathrm{eby}\,\Omega$ has zero $(N-1)$-dimensional measure. Then I prove that this property holds for all regions $f(\Omega)$ with Ω an E-region and $f \in BLip_{loc}(\mathbb{R}^N)$.

5 Proposition. *For every E-region Ω of $\mathbb{R}^N$ and for every $f \in BLip_{loc}(\mathbb{R}^N)$,*

$$\mathcal{H}^{N-1}\left(\mathrm{bdy}\, f\left(\Omega\right) \setminus \mathrm{eby}\, f\left(\Omega\right)\right) = 0\,. \qquad (22)$$

Proof. I recall that for every $A \subset \mathbb{R}^N$ and for every $B \subset A$, A is the disjoint union of B and $A \setminus B$. Then $\mathrm{eby}\, f(\Omega) \subset \mathrm{bdy}\, f(\Omega)$ implies

$$\mathrm{bdy}\, f(\Omega) = \mathrm{eby}\, f(\Omega) \cup (\mathrm{bdy}\, f(\Omega) \setminus \mathrm{eby}\, f(\Omega))\,, \qquad (23)$$

and $\mathrm{eby}\,\Omega \subset \mathrm{bdy}\,\Omega$ together with $(9)_1$ implies

$$f\left(\mathrm{bdy}\,\Omega\right) = f\left(\mathrm{eby}\,\Omega \cup (\mathrm{bdy}\,\Omega \setminus \mathrm{eby}\,\Omega)\right) = f\left(\mathrm{eby}\,\Omega\right) \cup f\left(\mathrm{bdy}\,\Omega \setminus \mathrm{eby}\,\Omega\right)\,. \quad (24)$$

In the two equalities, $\mathrm{bdy}\, f(\Omega) = f(\mathrm{bdy}\,\Omega)$ by $(5)_3$ and $\mathrm{eby}\, f(\Omega) = f(\mathrm{eby}\,\Omega)$ by $(10)_3$. Because the union in (23) is disjoint, it follows that

$$\mathrm{bdy}\, f(\Omega) \setminus \mathrm{eby}\, f(\Omega) \subset f(\mathrm{bdy}\,\Omega \setminus \mathrm{eby}\,\Omega)\,, \qquad (25)$$

and, by $(6)_2$ with $K = N-1$ and $A = \mathrm{bdy}\,\Omega \setminus \mathrm{eby}\,\Omega$,

$$\begin{aligned}&\mathcal{H}^{N-1}(\mathrm{bdy}\, f(\Omega) \setminus \mathrm{eby}\, f(\Omega)) \\ &\qquad \le \mathcal{H}^{N-1}(f(\mathrm{bdy}\,\Omega \setminus \mathrm{eby}\,\Omega)) \le m_K^{N-1}\mathcal{H}^{N-1}(\mathrm{bdy}\,\Omega \setminus \mathrm{eby}\,\Omega)\,. \quad (26)\end{aligned}$$

But $\mathcal{H}^{N-1}(\mathrm{bdy}\,\Omega \setminus \mathrm{eby}\,\Omega) = 0$ because Ω is an E-region, and (22) follows. $\boxed{QED}$

It remains to prove that $f(\Omega)$ has a *surface-like boundary* as required by condition (F3). In [3] this has been proved to be true for all D-regions. Then, in particular, this is true for all E-regions, which form a subclass of the D-regions.

5 An example

For a given point (x_0, y_0) of $\mathbb{R}^2$ and for given positive numbers d, l, consider the two-dimensional region

$$\begin{aligned}\Omega(x_0, y_0, l, d) = \bigcup_{h=1}^{\infty} \bigcup_{k=1}^{2^{h-1}} \Big\{ B\left(x_h^k, y_h, r_h\right) \mid x_h^k = x_0 + l\frac{2k-1}{2^h}, \\ y_h = y_0 + \frac{d}{2^h},\, r_h = \frac{d}{4^h} \Big\}\,, \quad (27)\end{aligned}$$

where $B(x, y, r)$ is the open ball centered at (x, y) with radius r. The region is represented in Fig.1. It is both a NV-region [8] and a D-region [3]. In particular, the essential boundary is the union of the circumferences $\partial B(x_h^k, y_h, r_h)$, and the boundary is the union of the essential boundary plus the segment

$$\{ (x, y) \quad | \quad x_0 \leq x \leq x_0 + l\,, \;\; y = y_0 \}\,.$$

A quick calculation shows that

$$\operatorname{per}\Omega(x_0, y_0, l, d) = \pi d\,, \quad \mathcal{H}^1(\operatorname{bdy}\Omega(x_0, y_0, l, d)) = \pi d + l\,. \tag{28}$$

In [8], Noll and Virga consider the region

$$\mathcal{D}_\varepsilon = \bigcup_{i=1}^{\infty} \Omega(0, y_i, 1, d_i)\,, \tag{29}$$

with $\{y_i, y_i + d_i\}$ a given countable union of pairwise disjoint intervals, dense in (0,1) and with total length less than $\varepsilon > 0$. For this region one finds that

$$\operatorname{per}\mathcal{D}_\varepsilon < \pi\varepsilon\,, \quad 1 - \varepsilon < \mathcal{L}^2(\operatorname{bdy}\mathcal{D}_\varepsilon) < 1\,, \tag{30}$$

so that $\mathcal{H}^1(\operatorname{bdy}\mathcal{D}_\varepsilon) = +\infty$. This is the example given in [8] of a region which satisfies (NV1)–(NV3) but not (NV4).

In [3], I consider a region $\mathcal{D}$ which is the pairwise disjoint, countable union of regions $\Omega(0, y_i, 1, d_i)$, with

$$y_1 = 0\,, \quad y_{i+1} = y_i + d_i\,, \quad d_i = 2^{-i}\,. \tag{31}$$

For this region,

$$\operatorname{per}\mathcal{D} = \pi\,, \quad \mathcal{H}^1(\operatorname{bdy}\mathcal{D}) = +\infty\,, \quad \mathcal{L}^2(\operatorname{bdy}\mathcal{D}) = 0\,. \tag{32}$$

This is a NV-region but not a D-region, because it violates condition (D4). In [3], this example was used to show that the class of the open D-regions is strictly included in the class of the NV-regions, and to suggest that the NV-regions which are not D-regions are somehow far from those “imagined by an engineer”.

Now take the region $\mathcal{D}_0 = \Omega(x_0, y_0, l, d)$. By (28), per $\mathcal{D}_0$ is strictly less than $\mathcal{H}^1(\operatorname{bdy}\mathcal{D}_0)$. Because this contradicts condition (E4), $\mathcal{D}_0$ is not an E-region. Thus, the class of the E-regions is strictly included in the class of the open D-regions.

The region $\mathcal{D}_0$ looks rather pathological for the purpose of describing the shape of a body. This example suggests that the open D-regions which are not E-regions may be of little interest in continuum mechanics.

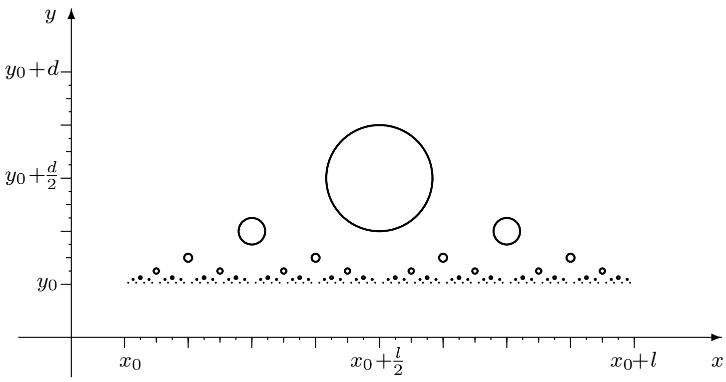

Figure 1. The region $\Omega(x_0, y_0, l, d)$.

Acknowledgements. This work was supported by the Research Project *Nano* & *Nano* of the University of Ferrara.

Added in proof. I recently found in the literature a proof of Lemma 4, see Ref. [10].

References

[1] L. Ambrosio, N. Fusco, D. Pallara: Functions of bounded variation and free discontinuity problems, Oxford Science Publications, Oxford 2000.

[2] C. Banfi, M. Fabrizio: *Sul concetto di sottocorpo nella meccanica dei continui*, Rend. Accad. Naz. Lincei, **66** (1979), 136–142.

[3] G. Del Piero: *A class of fit regions and a universe of shapes for continuum mechanics*, J. of Elasticity, **70** (2003), 175–195.

[4] M. E. Gurtin, W. O. Williams, W. P. Ziemer: *Geometric measure theory and the axioms of continuum thermodynamics*, Arch. Rational Mech. Anal., **92** (1986), 1–22.

[5] A. N. Kolmogorov, S. V. Fomin: Introductory Real Analysis, Dover Publications, New York 1970.

[6] W. Noll: *Lectures on the foundations of Continuum Mechanics and Thermodynamics*, Arch. Rational Mech. Anal., **52** (1973), 62–92.

[7] W. Noll: *Continuum Mechanics and geometric integration theory*, in: F.W. Lawvere and S.H. Schnauel (Eds.), *Categories in Continuum Physics*, Buffalo 1982, Springer Lecture Notes in Mathematics, **1174**, Berlin (1986), 17–29.

[8] W. Noll, E. G. Virga: *Fit regions and functions of bounded variation*, Arch. Rational Mech. Anal., **102** (1988), 1–21.

[9] C. A. Truesdell: A First Course in Rational Continuum Mechanics, Vol. 1, 2nd Edition, Academic Press, Boston 1991.

[10] Z. Buczolich: *Density points and bi-Lipschitz functions in* $\mathbf{R}^m$, Proceedings AMS, **116** (1992), 53–59.