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A new definition of a pseudo-rigid continuum

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Abstract. In the context of a purely mechanical development, the concept of a "globally constrained" continuum is employed here to construct a theory of pseudo-rigid bodies. The Cauchy stress tensor $\mathbf{T}$ for the pseudo-rigid body is assumed to be decomposed into an "active" part $\mathbf{T}^{(A)}$, which is specified by a constitutive equation, and a "reactive" part $\mathbf{T}^{(R)}$, which is called into play to maintain the global constraint. The theory generalizes one presented by the author in 2004, in which the active stress tensor was given by the same response function throughout the body, i.e., pseudo-rigid bodies were regarded there as *homogeneous* globally constrained continua. Material inhomogeneity is now admitted. A set of Lagrange's equations follows as before.

Keywords: pseudo-rigid continua, global constraints, Lagrange's equations

MSC 2000 classification: primary 74A99, secondary 74L99

In honour of Professor Alan Day

Introduction

A pseudo-rigid continuum is a mathematical model that occupies an intermediate position between a rigid continuum and an arbitrarily deformable one, being closer in spirit to the former. (Ideally) rigid bodies can undergo only rotations and translations (6 degrees of freedom) no matter what forces are applied to them. Similarly, pseudo-rigid bodies can undergo only homogeneous, or affine, motions, and have 12 degrees of freedom.[1] General deformable continua have infinitely many degrees of freedom, and require partial differential equations for their mechanical description. As in the case of rigid bodies, pseudo-rigid continua can be described by ordinary differential equations.

The field of research was initiated by Slawianowski [28, 29] in the 1970s and was elaborated in works by Cohen and Muncaster [9, 11, 12, 19, 20]. Another approach to the theory is through the concept of a Cosserat point [14, 26, 27].

[1]For pseudo-rigid continua, the deformation gradient $\mathbf{F}$ is a spatially uniform tensor and varies with time only: $\mathbf{F} = \mathbf{F}(t)$.

The theory of elastic pseudo-rigid continua has been successfully applied in a variety of problems—see e.g., [10, 15, 21–23, 27, 30, 33], where references to other work may be found.

In the publications cited above, the question was not raised as to how pseudo-rigid continua could maintain exactly the homogeneity of their deformation fields in the presence of arbitrary applied loading.[2] One might suspect that some sort of *material constraint* must be involved. If so, however, this constraint is not of the classical type, such as incompressibility or inextensibility, since for pseudo-rigid bodies, the deformation gradient $\mathbf{F}$ is not itself subjected to a finite constraint equation.[3] Instead, the mapping that describes the motion of the continuum is restricted, or equivalently, the spatial derivatives of $\mathbf{F}$ are restricted to be zero. Such constraints were first recognized by Antman & Marlow in 1991, who called them *global material constraints* [2].[4] In particular, when shells and rods are regarded as 3-dimensional continua that can experience only a restricted class of motions, such as in the Kirchhoff-Love, or Bernoulli-Euler, or Cosserat theories, global constraints appear (although they are usually not recognized as such). Antman & Marlow [2] proposed that a field of reactive stresses much accompany a global constraint.[5] In 2004, I proposed that a pseudo-rigid body may be regarded as a globally constrained homogeneous continuum [3].[6] A list of assumptions formalizing this idea were provided, and the equations for homogeneous pseudo-rigid continua were deduced. A set of Lagrange's equations were also derived, using a geometrical procedure that had been developed for particle systems in [4] and for a rigid body in [5].

In the present Note, the restriction made in [3] to homogeneous materials is removed and a less restrictive list of assumptions is presented. The theory of [3] is included as a special case of the present one.

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[2]This is obviously a very different question than "How well do the equations of the theory of pseudo-rigid bodies approximate the stress and strain fields of certain real bodies for some given range of loading conditions?" The latter question is adequately addressed in the cited papers that are devoted to applications.

[3]The classical material constraints all reduce to equations of the form $f(\mathbf{F}) = 0$, where f is a scalar-valued function. For treatments of these constraints, the reader is referred to [8, 31], where additional references may be found.

[4]See also Marlow [17] and Antman [1].

[5]In 1989, Podio-Guidugli [24] made the important observation that the theory of linearly elastic plates could be rendered exact, rather than approximate, if a field of reactive stresses were to be permitted. See also [13, 16, 25].

[6]Also, by way of physical motivation, I suggested that an elastic pseudo-rigid body could be thought of as a combination of an elastic material together with an adjustable constraining system. In this connection, see Fig. 1 of [3]. Additional remarks may be found in [6].

1 Background material

In this section, let us consider an arbitrary 3-dimensional deformable continuum $\mathcal{B}$ undergoing an arbitrary motion. Let X be any particle belonging to $\mathcal{B}$. Choose a fixed occupiable reference configuration $\boldsymbol{\kappa}_0$ of $\mathcal{B}$ in a Newtonian frame of reference. Let $\mathbf{X}$ be the position vector of X in $\boldsymbol{\kappa}_0$, and let $\mathbf{x}$ be the position vector of X in the current configuration $\boldsymbol{\kappa}$ of $\mathcal{B}$ at time t. The motion of $\mathcal{B}$ is described by the function

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t) \ , \tag{1}$$

which may be taken to be as smooth as desired. Let $\mathbf{v}$ $(= \dot{\mathbf{x}})$ be the velocity of X and let $\mathbf{F}$ $(= \frac{\partial \boldsymbol{\chi}}{\partial \mathbf{X}})$ be the deformation gradient. Also,

$$J = \det \mathbf{F} > 0 \ . \tag{2}$$

The spatial velocity and acceleration gradients are

$$\mathbf{L} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} = \dot{\mathbf{F}}\mathbf{F}^{-1} \ , \quad \mathbf{A} = \frac{\partial \dot{\mathbf{v}}}{\partial \mathbf{x}} = \ddot{\mathbf{F}}\mathbf{F}^{-1} \ . \tag{3}$$

The rate of deformation tensor $\mathbf{D}$ is the symmetric part of $\mathbf{L}$ (i.e., $\frac{1}{2}(\mathbf{L} + \mathbf{L}^T)$).

For any nonempty subset $\mathcal{S} \subseteq \mathcal{B}$, let $\mathcal{P}$ be the region occupied by $\mathcal{S}$ in the current configuration and let $\partial\mathcal{P}$ be the boundary of $\mathcal{P}$; likewise, let $\mathcal{P}_0$ be the region occupied by $\mathcal{S}$ in the reference configuration $\boldsymbol{\kappa}_0$, and let $\partial\mathcal{P}_0$ be the boundary of $\mathcal{P}_0$; for $\mathcal{S} = \mathcal{B}$, we employ the notations $\mathcal{R} = \mathcal{P}$, $\partial\mathcal{R} = \partial\mathcal{P}$, $\mathcal{R}_0 = \mathcal{P}_0$, $\partial\mathcal{R}_0 = \partial\mathcal{P}_0$. Let ρ_0 (> 0) and ρ be the mass densities of $\mathcal{B}$ in the configurations $\boldsymbol{\kappa}_0$ and $\boldsymbol{\kappa}$, respectively, and let m be the mass of $\mathcal{B}$. Conservation of mass yields

$$m = \int_{\mathcal{R}} \rho \, \mathrm{d}v = \int_{\mathcal{R}_0} \rho_0 \, \mathrm{d}V = \text{const.} \ , \tag{4}$$

where $\mathrm{d}v$ and $\mathrm{d}V$ are the volume elements in $\mathcal{R}$ and $\mathcal{R}_0$, respectively $(\mathrm{d}v = J \, \mathrm{d}V)$. Since a statement of the form (4) holds for all $\mathcal{S} \subseteq \mathcal{B}$, we have

$$\rho_0 = \rho J \ , \quad \dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0 \ . \tag{5}$$

The position vectors of the mass center of $\mathcal{B}$ in the configurations $\boldsymbol{\kappa}_0$ and $\boldsymbol{\kappa}$ are given by

$$\bar{\mathbf{X}} = \frac{1}{m} \int_{\mathcal{R}_0} \rho_0 \mathbf{X} \, \mathrm{d}V \ , \quad \bar{\mathbf{x}} = \frac{1}{m} \int_{\mathcal{R}} \rho \mathbf{x} \, \mathrm{d}v \ . \tag{6}$$

Set

$$\boldsymbol{\Pi} = \mathbf{X} - \bar{\mathbf{X}} \ , \quad \boldsymbol{\pi} = \mathbf{x} - \bar{\mathbf{x}} \ . \tag{7}$$

Clearly,

$$\int_{\mathcal{R}} \rho \boldsymbol{\pi} \, \mathrm{d}v = \mathbf{0} \; . \tag{8}$$

Hence, employing the transport theorem and $(5)_2$, we see that

$$\int_{\mathcal{R}} \rho \dot{\boldsymbol{\pi}} \, \mathrm{d}v = \mathbf{0} \; , \quad \int_{\mathcal{R}} \rho \ddot{\boldsymbol{\pi}} \, \mathrm{d}v = \mathbf{0} \; . \tag{9}$$

The Euler tensors of $\mathcal{B}$, defined with respect to its mass center in the configurations $\boldsymbol{\kappa}_0$ and $\boldsymbol{\kappa}$ are

$$\mathbf{E}_0 = \int_{\mathcal{R}_0} \rho_0 \boldsymbol{\Pi} \otimes \boldsymbol{\Pi} \, \mathrm{d}V \; , \quad \mathbf{E} = \int_{\mathcal{R}} \rho \boldsymbol{\pi} \otimes \boldsymbol{\pi} \, \mathrm{d}v \; , \tag{10}$$

respectively, where $\otimes$ denotes the tensor product operation $[(\mathbf{a} \otimes \mathbf{b})\mathbf{c} = \mathbf{a}\,\mathbf{b} \cdot \mathbf{c}$, for any vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}]$. Note that $\mathbf{E}_0$ is a constant tensor.

A motion $\boldsymbol{\chi}^+$ for $\mathcal{B}$ is said to differ from $\boldsymbol{\chi}$ by a superposed rigid motion if and only if

$$\boldsymbol{\chi}^+(\mathbf{X}, t^+) = \mathbf{Q}(t)\boldsymbol{\chi}(\mathbf{X}, t) + \mathbf{a}(t) \; , \quad t^+ = t + a \; , \tag{11}$$

where $\mathbf{Q}(t)$ is any proper orthogonal tensor-valued function of t, $\mathbf{a}(t)$ is any vector-valued function of t, and a, which may be interpreted as a chronometer change, is any real number. Under all such superposed rigid motions, the quantities $\mathbf{F}$, J, ρ, $\bar{\mathbf{x}}$, $\boldsymbol{\pi}$, and $\mathbf{E}$ transform as

$$\mathbf{F}^+ = \mathbf{QF}, \; J^+ = J, \; \rho^+ = \rho, \; \bar{\mathbf{x}}^+ = \mathbf{Q}\bar{\mathbf{x}} + \mathbf{a}, \; \boldsymbol{\pi}^+ = \mathbf{Q}\boldsymbol{\pi}, \; \mathbf{E}^+ = \mathbf{QEQ}^T. \tag{12}$$

Let $\mathbf{n}$ be the outward unit normal vector to $\partial\mathcal{P}$, and let $\mathbf{t}$ be the Cauchy stress vector. Also, let $\mathbf{b}$ be the body force field per unit mass acting on $\mathcal{B}$ at time t. Euler's laws for the balance of momentum and angular momentum of $\mathcal{S} \subseteq \mathcal{B}$ are equivalent to the pair of statements

$$\int_{\partial\mathcal{P}} \mathbf{t} \, \mathrm{d}a + \int_{\mathcal{P}} \rho \mathbf{b} \, \mathrm{d}v = \int_{\mathcal{P}} \rho \dot{\mathbf{v}} \, \mathrm{d}v = m\ddot{\bar{\mathbf{x}}} \; , \tag{13a}$$

$$\int_{\partial\mathcal{P}} \boldsymbol{\pi} \times \mathbf{t} \, \mathrm{d}a + \int_{\mathcal{P}} \rho \boldsymbol{\pi} \times \mathbf{b} \, \mathrm{d}v = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{P}} \rho \boldsymbol{\pi} \times \dot{\boldsymbol{\pi}} \, \mathrm{d}v \; , \tag{13b}$$

where $\mathrm{d}a$ is the area element of $\partial\mathcal{P}$, and the moments in (13b) are taken about the mass center.

By Cauchy's tetrahedron argument, (13a) leads to the existence of the Cauchy stress tensor $\mathbf{T}$ such that

$$\mathbf{t} = \mathbf{T}\mathbf{n} \, . \tag{14}$$

Equations (13a,b) and (14) imply Cauchy's laws

$$\operatorname{div} \mathbf{T} + \rho \mathbf{b} = \rho \ddot{\mathbf{x}} \; , \quad \mathbf{T}^T = \mathbf{T} \; , \tag{15}$$

where the superscript T denotes the transpose of a tensor.

A technique associated with the name of Signorini affords a means of extracting partial global information for the body $\mathcal{B}$ from Cauchy's laws,[7] which is especially useful for globally constrained continua. Denote the mean Cauchy stress in $\mathcal{B}$ by

$$\bar{\mathbf{T}} = \frac{1}{V} \int_{\mathcal{R}} \mathbf{T} \, \mathrm{d}v \; , \tag{16}$$

where $V = \int_{\mathcal{R}} \mathrm{d}v$, and note that $\bar{\mathbf{T}}^T = \bar{\mathbf{T}}$. Also, define a tensor $\mathbf{M}$ by

$$\mathbf{M} = \int_{\partial\mathcal{R}} \mathbf{t} \otimes \boldsymbol{\pi} \, \mathrm{d}a + \int_{\mathcal{R}} \rho \mathbf{b} \otimes \boldsymbol{\pi} \, \mathrm{d}v \; ; \tag{17}$$

$\mathbf{M}$ is referred to as the *Möbius tensor* about the mass center of $\mathcal{B}$ in the current configuration.[8] If we take the tensor product of both sides of $(15)_1$ with $\boldsymbol{\pi}$, integrate over $\mathcal{R}$, employ $(7)_2$, (8), (16), (17), and note that $\frac{\partial \boldsymbol{\pi}}{\partial \mathbf{x}} = \mathbf{I}$, where $\mathbf{I}$ is the identity tensor, we will obtain

$$\mathbf{M} - V\bar{\mathbf{T}} = \int_{\mathcal{R}} \rho \ddot{\boldsymbol{\pi}} \otimes \boldsymbol{\pi} \, \mathrm{d}v \, . \tag{18}$$

The skew-symmetric part of (18) is equivalent to Euler's second law (13b). The symmetric part of (18) supplies new global information stemming from $(15)_{1,2}$. It is worth emphasizing that (18) is a necessary condition of Euler's laws, and holds in every motion of any 3-dimensional continuum $\mathcal{B}$.

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[7] For a further discussion, see Truesdell & Toupin [32, Sects. 216–220], where references to Signorini's papers can be found.

[8] In connection with his pioneering studies on astatics, A.F. Möbius (1790–1868) was led to consider all linear moments of forces, and not just torques [18].

2 Definition of a pseudo-rigid continuum

We now provide a definition of pseudo-rigid bodies based on the concept of a globally constrained continuum.

1 Definition. A 3-dimensional deformable continuum $\mathcal{B}$ is *pseudo-rigid* if and only if the following three conditions are satisfied.

(a) Every motion $\boldsymbol{\chi}$ of $\mathcal{B}$ is of the form

$$\boldsymbol{\chi}(\mathbf{X}, t) = \mathbf{F}\mathbf{X} + \mathbf{c} \ , \tag{19}$$

where the deformation gradient $\mathbf{F}$ is a function of t only and $\mathbf{c}$ is a vector-valued function of t only. Thus, $\mathcal{B}$ can experience only *homogeneous* motions.

(b) The Cauchy stress tensor at each $X \in \mathcal{B}$, and at each t, can be expressed as

$$\mathbf{T}(X, t) = \mathbf{T}^{(A)}(X, t) + \mathbf{T}^{(R)}(X, t) \ . \tag{20}$$

Here, the *active stress tensor* $\mathbf{T}^{(A)} = \left(\mathbf{T}^{(A)}\right)^T$ is specified by a constitutive equation, which we take to be of the form

$$\mathbf{T}^{(A)} = \boldsymbol{\mathcal{T}}(\mathbf{F}, X) \ , \tag{21}$$

where $\boldsymbol{\mathcal{T}}$ is a functional of the deformation gradient and an ordinary function of X, i.e., we are considering "simple" materials that are inhomogeneous, in general [31]. The *reactive stress tensor* $\mathbf{T}^{(R)} = \left(\mathbf{T}^{(R)}\right)^T$ is completely indeterminate, and we assume that it is uninfluenced by the rate of deformation tensor $\mathbf{D}$. The active stress tensor is assumed to satisfy the invariance requirement

$$(\mathbf{T}^{(A)})^+ = \mathbf{Q}\mathbf{T}^{(A)}\mathbf{Q}^T \tag{22}$$

under superposed rigid motions $(11)_{1,2}$ of $\mathcal{B}$.[9]

(c) The integral of the stress power of the reactive stresses is zero for all (homogeneous) motions (19) of $\mathcal{B}$:

$$\int_{\mathcal{R}} \mathbf{T}^{(R)} \cdot \mathbf{D} \, \mathrm{d}v = 0 \ . \tag{23}$$

[9]Standard objectivity arguments lead to reduced forms of the constitutive equation (21)—see e.g., [31]. The reactive stress, being indeterminate, cannot be expected to be objective [7,8].

Equation (19) is a *global material constraint* in the sense of Antman & Marlow [2] and Marlow [17]. The reactive stresses are required to enforce the constraint in the face of arbitrary applied loading.

For every motion (19), $\mathbf{D}$ depends on t only (see $(3)_1$), and hence for every pseudo-rigid body $\mathcal{B}$,

$$\int_{\mathcal{R}} \mathbf{T}^{(R)}\, \mathrm{d}v = \mathbf{0}\ . \tag{24}$$

Thus, the mean value of $\mathbf{T}^{(R)}$ is necessarily zero. It follows immediately from (20), (16), and (24) that

$$V\bar{\mathbf{T}} = \int_{\mathcal{R}} \mathbf{T}^{(A)}\, \mathrm{d}v\ . \tag{25}$$

In the definition previously given in [3], the response function for the active stress was assumed to be the same for all $X \in \mathcal{B}$, and equal in value to $\bar{\mathbf{T}}$. For such homogeneous pseudo-rigid bodies, (24) follows without assuming condition (c). For inhomogeneous continua, some global condition such as (c) is needed.

For every pseudo-rigid body, it follows from (19), $(6)_{1,2}$, $(7)_{1,2}$, and $(3)_{1,2}$ that

$$\bar{\mathbf{x}} = \mathbf{F}\bar{\mathbf{X}} + \mathbf{c}\ , \quad \boldsymbol{\pi} = \mathbf{F}\boldsymbol{\Pi}\ , \quad \dot{\boldsymbol{\pi}} = \mathbf{L}\boldsymbol{\pi}\ , \quad \ddot{\boldsymbol{\pi}} = \mathbf{A}\boldsymbol{\pi}\ . \tag{26}$$

By virtue of $(26)_2$ and $(5)_1$, the Euler tensors in $(10)_{1,2}$ are now related to one another by

$$\mathbf{E} = \mathbf{F}\mathbf{E}_0\mathbf{F}^T\ . \tag{27}$$

In view of $(26)_4$, $(10)_2$, and (27), the dynamical equation (18) becomes

$$\mathbf{M} - V\bar{\mathbf{T}} = \mathbf{A}\mathbf{E} = \ddot{\mathbf{F}}\mathbf{E}_0\mathbf{F}^T\ , \tag{28}$$

which is a differential equation associated with $\mathbf{F}$. The differential equation associated with the motion of the mass center of $\mathcal{B}$ is given by (13a), with $\mathcal{S} = \mathcal{B}$, namely

$$\mathbf{f} = m\ddot{\bar{\mathbf{x}}}\ , \tag{29}$$

where we have set $\mathbf{f}$ equal to the resultant external force acting on $\mathcal{B}$ at time t.

The kinetic energy of the pseudo-rigid body $\mathcal{B}$ is

$$T = \frac{1}{2}\int_{\mathcal{R}} \rho\, \mathbf{v}\cdot\mathbf{v}\, \mathrm{d}v\ . \tag{30}$$

Taking the material derivative of both sides of $(7)_2$, substituting for $\mathbf{v}$ in (30), and employing (4), $(9)_1$, and $(26)_3$ we may write T as

$$T = \bar{T} + T^{\star}\ , \tag{31}$$

where

$$\bar{T} = \frac{1}{2} m \dot{\bar{\mathbf{x}}} \cdot \dot{\bar{\mathbf{x}}} \ , \quad T^\star = \frac{1}{2} \int_{\mathcal{R}} \rho \, \dot{\boldsymbol{\pi}} \cdot \dot{\boldsymbol{\pi}} \, \mathrm{d}v = \frac{1}{2} \int_{\mathcal{R}} \rho \, \mathrm{tr}\big((\mathbf{L}\boldsymbol{\pi}) \otimes (\mathbf{L}\boldsymbol{\pi}) \big) \, \mathrm{d}v \ . \tag{32}$$

Further, with the aid of $(10)_2$, $(3)_1$, and (27), $T^\star$ may be expressed as

$$T^\star = \frac{1}{2} \, \mathrm{tr}\big(\mathbf{L}\mathbf{E}\mathbf{L}^T \big) = \frac{1}{2} \, \mathrm{tr}\big(\dot{\mathbf{F}} \mathbf{E}_0 \dot{\mathbf{F}}^T \big) \ . \tag{33}$$

The following power relations hold:

$$\mathbf{f} \cdot \dot{\bar{\mathbf{x}}} = \dot{\bar{T}} \ , \quad (\mathbf{M} - V\bar{\mathbf{T}}) \cdot \mathbf{L} = \dot{T}^\star \ . \tag{34}$$

Equation $(34)_1$ follows from $(32)_1$, (4), and (29), while $(34)_2$ may be obtained from the second expression in (33), the constancy and symmetry of $\mathbf{E}_0$ in $(10)_1$, $(3)_1$, and (28).[10]

3 Lagrange's equations for a pseudo-rigid body

The configuration of a pseudo-rigid body at time t is determined by the pair $(\bar{\mathbf{x}}, \mathbf{F})$, which may be regarded as the position vector of an abstract particle $\mathcal{P}$ in a 12-dimensional vector space.

Lagrange's equations for a pseudo-rigid body can be derived from equations (29) and (28), using the same geometrical procedure as described in [3]. We will not repeat the details of the derivation here, and will only outline the results.

It is useful to define two inner products

$$\langle \mathbf{A}, \mathbf{B} \rangle = \frac{1}{m} \, \mathrm{tr}\big(\mathbf{A}\mathbf{E}\mathbf{B}^T \big) \ , \quad \langle \mathbf{A}, \mathbf{B} \rangle_0 = \frac{1}{m} \, \mathrm{tr}\big(\mathbf{A}\mathbf{E}_0\mathbf{B}^T \big) \tag{35}$$

for any second-order tensors $\mathbf{A}$, $\mathbf{B}$. The expressions in $(32)_2$ may then be written as

$$T^\star = \frac{1}{2} m \langle \mathbf{L}, \mathbf{L} \rangle = \frac{1}{2} m \langle \dot{\mathbf{F}}, \dot{\mathbf{F}} \rangle_0 \ . \tag{36}$$

We may represent $\bar{\mathbf{x}}$ by three generalized coordinates (η^1, η^2, η^3), and $\mathbf{F}$ by nine generalized coordinates $(\xi^1, \xi^2, \ldots, \xi^9)$. Also, we introduce the tangent vectors

$$\mathbf{a}_\gamma = \frac{\partial \bar{\mathbf{x}}}{\partial \eta^\gamma} \quad (\gamma = 1, 2, 3) \ , \quad \mathbf{A}_\gamma = \frac{\partial \mathbf{F}}{\partial \xi^\gamma} \quad (\gamma = 1, 2, \ldots, 9) \ . \tag{37}$$

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[10] The inner product between tensors in $(34)_2$ is the standard one, i.e., $\mathbf{A} \cdot \mathbf{B} = \mathrm{tr}(\mathbf{A}\mathbf{B}^T)$, for any second-order tensors $\mathbf{A}$, $\mathbf{B}$.

The following kinematical formulae hold:[11]

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial \bar{T}}{\partial \dot{\eta}^\gamma}\Big) - \frac{\partial \bar{T}}{\partial \eta^\gamma} = m\ddot{\bar{\mathbf{x}}}\cdot \mathbf{a}_\gamma \quad (\gamma = 1,2,3)\ ,\\
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial T^\star}{\partial \dot{\xi}^\gamma}\Big) - \frac{\partial T^\star}{\partial \xi^\gamma} = m\langle \ddot{\mathbf{F}}, \mathbf{A}_\gamma\rangle_0 \quad (\gamma = 1,2,\dots,9)\ .
\end{aligned}
\tag{38}
$$

With the help of $(38)_1$, we may readily deduce Lagrange's equations for $\bar{\mathbf{x}}$ from (29):

$$
\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial \bar{T}}{\partial \dot{\eta}^\gamma}\Big) - \frac{\partial \bar{T}}{\partial \eta^\gamma} = \mathbf{f}\cdot \mathbf{a}_\gamma \quad (\gamma = 1,2,3)\ . \tag{39}
$$

Likewise, from (28) and $(38)_2$, we obtain Lagrange's equations for $\mathbf{F}$:

$$
\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial T^\star}{\partial \dot{\xi}^\gamma}\Big) - \frac{\partial T^\star}{\partial \xi^\gamma} = m\langle (\mathbf{M} - V\bar{\mathbf{T}})\mathbf{F}^{-T}\mathbf{E}_0^{-1}, \mathbf{A}_\gamma\rangle_0 \quad (\gamma = 1,2,\dots,9)\ . \tag{40}
$$

Other expressions for the generalized forces on the right-hand side of (40) may be found in [3].

4 Homogeneous materials

If the pseudo-rigid body $\mathcal{B}$ defined in Section 3 is composed of homogeneous material, then the stress response $\boldsymbol{\mathcal{T}}$ in (21) is independent of X. Further, for all pseudo-rigid bodies, $\mathbf{F}$ is independent of X. Hence, for homogeneous pseudo-rigid bodies, (25) becomes

$$
V\bar{\mathbf{T}} = \mathbf{T}^{(A)} \int_{\mathcal{R}} \mathrm{d}v \tag{41}
$$

and hence

$$
\mathbf{T}^{(A)} = \bar{\mathbf{T}}\ . \tag{42}
$$

It follows from (42) that for homogeneous pseudo-rigid bodies

$$
\operatorname{div} \mathbf{T}^{(A)} = \mathbf{0}\ . \tag{43}
$$

Additional results for this case may be found in [3].

[11] For details on the derivation of these key formulae, see Theorem 4.1 of [3].

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