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A Quest for an 'Extended' Continuum Mechanics

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Abstract. Earlier reflections on balance equations possibly appropriate for hyperfluids or pseudofluids (reflections broached originally so as to dispose of lack of strict objectivity in standard thermal quantities) are instanced again with a different slant. It is alleged that partially chaotic motions can be efficiently branded by assigning pertinent averages, moments *and* variances and these gauges can be all (not only the first two) considered mechanical; hence the qualifier 'extended'. A few immediate consequences of the approach are derived, open problems are listed, connections or dissensions with widely acknowledged notions are discussed.

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To a friend who needs not be told how murky the terrain
between continuum mechanics and thermodynamics still is, in places.

1 Preamble

An inquiry on the global behaviour of granular media is a vast, even prodigious, enterprise. Even a neat breakdown into separate stages of their flows (dense to rarefied, say, or quasistatic to fast) is still wanting to a large extent and, besides, often one phenomenon (a landslide, say) spans across any plausible, conceivable spectrum. Though the contours remain fuzzy, it is fair to expect that fast, sparse flows share some at least of the features of a molecular gas flow. Even then the order of magnitude of several quantities stays jarring: for instance, the peculiar speeds of grains cannot match those of molecules and, consequently, directly imported quasi-thermal measures end up by being grossly observer-dependent; an obnoxious event already evidenced even in the kinetic theory of gases, when that theory is pushed to high-order approximation via the Enskog procedure.

Hence the efforts expended to accommodate matters the least but enough to avoid awkward corollaries (e.g., by inserting explicitly apparent forces, as in many papers on extended thermodynamics). The heart of the matter appears to be the ascertainment of a field of local preferred frames with origin and orientation such that the densities of peculiar momentum and of peculiar moment of momentum both vanish ('peculiar' meaning here 'evaluated with respect to the local frame').

The inference is the following system of balance equations:

$$\frac{\partial \rho}{\partial \tau} + \operatorname{div}(\rho v) = 0, \tag{1}$$

$$\frac{\partial Y}{\partial \tau} + (\operatorname{grad} Y)\, v - BY - YB^T = 0, \tag{2}$$

$$\rho \left(\frac{\partial v}{\partial \tau} + Lv \right) = \rho f + \operatorname{div} T, \tag{3}$$

$$\rho \left(\frac{\partial K}{\partial \tau} + (\operatorname{grad} K)\, v - BK - H \right) = \rho M - A + \operatorname{div} \mathbf{m}, \tag{4}$$

$$\rho \left(\frac{\partial H}{\partial \tau} + (\operatorname{grad} H)\, v + BH + HB^T \right) = \rho S - Z + \operatorname{div} \mathbf{s}, \tag{5}$$

where notation is, in part, standard (ρ, density; v, velocity; $L = \operatorname{grad} v$; f, force per unit mass; T, Cauchy stress), partly easily interpreted (Y, intrinsic Euler inertia tensor per unit mass; K, intrinsic tensor moment of momentum always per unit mass; M, $-A$, $\mathbf{m}$ corresponding actions); the rest is, better, precisely defined below (the intrinsic 'whirling' B) or newly interpreted later (the Reynold's tensor H and the corresponding actions S, $-Z$, $\mathbf{s}$).

It is essential to recall that the route leading to (1)-(5) starts with the following epitome of reality. The region $\mathcal{B}$ occupied at time τ by the body (i.e., the set of all places where grains can be found) is fancied as split into loculi (i.e., representative volume elements) with a diameter indicative of the lower scale. At the gross scale each loculus $\mathfrak{e}$ is labelled by its place x in $\mathcal{B}$. However, each loculus is also observed at a magnification such that subplaces $z = x \oplus y$ can, in principle, be discerned in it; behold the notation: it reflects the crucial understanding that x and y belong to different Euclidean spaces. Indeed, we summon here a trivial vector bundle with the space of x as a base and the space of y as fibers. In our earlier analyses one grain was imagined posted in each subplace; here, instead, as we shall see, an entire family of grains is conjectured to belong to y with a spectrum of velocities w.

However, $v\,(x, \tau)$ and $B\,(x, \tau)$ are so chosen as to define a global fictitious affine motion of the grains in $\mathfrak{e}\,(x)$, a motion within which both momentum and tensor moment of momentum are exactly the same as in the actual motion of the grains in $\mathfrak{e}\,(x)$.

In the, perhaps wiser, approach mentioned at the end of the previous sentence, the model of reality above is complemented by the assignment of a distribution function $\theta\,(\tau, x; y, w)$ of grain velocities w (velocities belonging to the vector space $\mathcal{V}$) at y within $\mathfrak{e}$, with the origin of y chosen so that

$$\int_{\mathfrak{e}}\int_{\mathcal{V}} \theta y = 0. \tag{6}$$

Then an ancillary, though crucial, rôle was found to be played by the 'reduced' distributions $\hat{\theta}$ and $\tilde{\theta}$:

$$\hat{\theta} = \int_{\mathfrak{e}} \theta, \quad \tilde{\theta} = \int_{\mathcal{V}} \theta. \tag{7}$$

Indeed it was observed that the main fields could be expressed in terms of $\tilde{\theta}$ alone. In fact, some with either $\hat{\theta}$ or $\tilde{\theta}$, but some with $\tilde{\theta}$ only; in particular condition (6) may involve $\tilde{\theta}$ only

$$0 = \int_{\mathfrak{e}}\int_{\mathcal{V}} \theta y = \int_{\mathfrak{e}} \tilde{\theta} y. \tag{8}$$

Besides

$$v = \omega^{-1} \int_{\mathfrak{e}}\int_{\mathcal{V}} \theta w = \omega^{-1} \int_{\mathcal{V}} \hat{\theta} w = \omega^{-1} \int_{\mathfrak{e}} \tilde{\theta} \tilde{w}, \tag{9}$$

$$\omega = \int_{\mathfrak{e}}\int_{\mathcal{V}} \theta = \int_{\mathfrak{e}} \tilde{\theta} = \int_{\mathcal{V}} \hat{\theta}, \tag{10}$$

$$\tilde{w} = \tilde{\theta}^{-1} \int_{\mathcal{V}} \theta w; \tag{11}$$

$$Y = \omega^{-1} \int_{\mathfrak{e}}\int_{\mathcal{V}} \theta y \otimes y = \omega^{-1} \int_{\mathfrak{e}} \tilde{\theta} y \otimes y; \tag{12}$$

$$K = \omega^{-1} \int_{\mathfrak{e}}\int_{\mathcal{V}} \theta y \otimes (w - v) = \omega^{-1} \int_{\mathfrak{e}}\int_{\mathcal{V}} \theta y \otimes w = \omega^{-1} \int_{\mathfrak{e}} \tilde{\theta} y \otimes \tilde{w}; \tag{13}$$

$$B = K^T Y^{-1}; \tag{14}$$

$$H = \omega^{-1} \int_{\mathfrak{e}} \tilde{\theta} w^r \otimes w^r, \tag{15}$$

where w^r is the 'relative' velocity

$$w^r = \tilde{w} - v - By. \tag{16}$$

The ensuing view is that the grains instantaneously (at time τ) occupying the loculus $\mathfrak{e}$ make up a microscopic subbody $\mathfrak{s}$ at x characterized by the fields above. One is induced to hope that the characterization be exhaustive and that no further reference need, in practice, be made to the distributions; actually circumstances are not so clear-cut as the remarks below show.

True, from the kinetic viewpoint the process extracts from the chaotic granular motion an average momentum (and, consequently, a primary velocity v), an average tensor moment of momentum (and, consequently, a primary whirling B), and also a variance H of peculiar velocities; and even that further detail emerging from the chaotic maelstrom is disciplined mechanically.

However, to achieve balance equations we still have to confront the arduous challenge of conceiving and recommending expressions for the totals of the internal and external actions on the element.

Here we meet severe, and deplorable, handicaps in our earlier essays. There are intimate actions among grains within the element, actions internal to the body exchanged from element to element and totally external actions, external to the body, such as those due to gravity and those due to contacts through the boundary. All these actions may be either caused by molecular collisions or owed to cohesion, attraction or repulsion, at short or long distance. A detailed scrutiny as to how all those actions sum up in totals went beyond our writ.

One could expect that the want could be allayed (as in kinetic theory) via corollaries of an appropriate version of the Boltzmann equation. However, perplexities about the consequences on it of the boundedness of the domain $\mathfrak{e}$ apart, one would simply skirt the stumbling block to face the subtlest quandary as the appropriate form of the collision-coherence operator.

Therefore, in analogy with suggestions successful in classical cases, though perhaps less justifiably, we presumed the existence of fields of stress T, hyperstress $\mathbf{m}$ and agitation flux $\mathbf{s}$. Besides, in view of the fact that moments and virials of internal actions do not necessarily balance and thus lead to non-null resultants, we also presumed the existence of fields of tensor moments $-A$ and tensor virials $-Z$ of the actions; plus of those actions, of course, the representatives of actions at distance ρf, ρM, ρS.

Evidently, the sombre, furtive assumptions behind the choice of the right-hand sides above are very restrictive of types of interactions allowed. On the other hand, they leave still a vast scope for fantasy and adjustment to a wealth of circumstances via the choice of constitutive laws.

However, before we could proceed with some special, even if appealing, suggestions, we had to derive the appropriate kinetic energy theorem. Unluckily an expression in terms of $\tilde{\theta}$ only is found to be impossible for the kinetic energy tensor W (even if possible in terms of $\hat{\theta}$)

$$W = \frac{1}{2}\omega^{-1}\int_{\mathfrak{e}}\int_{\mathcal{V}} \theta w \otimes w = \frac{1}{2}\omega^{-1}\int_{\mathcal{V}} \hat{\theta} w \otimes w. \tag{17}$$

W can be split into the sum $W = \tilde{W} + \hat{H}$, where

$$\tilde{W} = v \otimes v + BYB^T + H \tag{18}$$

can be obtained with recourse to $\tilde{\theta}$ only, whereas

$$\hat{H} = \omega^{-1} \int_{\mathfrak{e}} \int_{\mathcal{V}} \theta w^s \otimes w^s \tag{19}$$

(where $w^s = w - \tilde{w} = w - w^r - v - By$ is the ultimate 'stray' velocity of each grain at y) involves the fundamental distribution θ, irremediably. Thus, though, as mentioned, H can be ruled mechanically what is left out of that discipline is the contribution $\hat{H}$ to W and only $\hat{H}$ keeps thermal peculiarities.

A terse rendition of H (but, finally, also of $\hat{H}$) is based on its being symmetric and definite, so that it can be written in terms of eigenvalues $\chi^2_{(s)}$ and unit vectors $h_{(s)}$ $(s = 1, 2, 3)$

$$H = \sum_{s=1}^{3} \chi^2_{(s)} h_{(s)} \otimes h_{(s)}. \tag{20}$$

Thus H could be naively regarded as generated by six countervailing and equally populous tribes of granules with speeds $\chi_{(s)} h_{(s)}$, $-\chi_{(s)} h_{(s)}$ $(s = 1, 2, 3)$. This interpretation ensues: H provides a measure of cross-over rate and, accordingly, should be accepted as a citizen of mechanics just as, say, L and B are. Remark that the tensorial character of H is essential in view of the possible anisotropy of the chaotic motions, near boundaries in particular.

As hinted above, a similar additive decomposition could, in principle, be adopted for $\hat{H}$. However, to enter in such details for the stray motion seems futile; one may rest context with the kinetic energy per unit mass

$$\hat{\kappa} = \frac{1}{2} tr \hat{H} \tag{21}$$

associated with it.

2 Outline of kinetics

A vital prerequisite to progress is to list again and comment on the stipulations which justify the definitions in the preamble.

The continuum is viewed at two scales and the distributions θ, $\hat{\theta}$, $\tilde{\theta}$ belong to the microscale. For instance, if, as always presumed for simplicity, all grains have the same mass μ, then $\mu\tilde{\theta}$ measures the mass of grains per unit volume at

y within the loculus $\mathfrak{e}$ and only the ratio of integrals $\mu\omega\,(\mathrm{meas}\,\mathfrak{e})^{-1}$ leads to the gross density ρ at x.

$\hat{\theta}$ is the distribution summoned in the prolegomena of the kinetic theory of monoatomic gases. Contrariwise, by its definition, $\tilde{\theta}$ relinquishes statistical attributes; indeed it should not be categorized as a distribution at all as, instead, θ and $\hat{\theta}$ properly are.

However, only by mustering the more specific distribution θ (i.e., by sharing out grains also according to microplace not only to velocity) and, consequently, inferring $\tilde{\theta}$, could we have access to Y, K and the local preferred frame, independent of that assigned by the macroplacement gradient F and to a local metric distinct from that associated with the macrostrain $C = F^T F$.

The cost of the extension is peremptory: because of the explicit dependence on location y within the bounded domain $\mathfrak{e}$, the rate of change of θ is ruled not only by collisions (as in the kinetic theory) and coherence (possible in semifluids), but also by transit of grains from and to neighbouring loculi. The route to recommend a consequent variant of the Boltzmann equation is eased by some remarks excerpted from an earlier paper.

As we have already mentioned, availability of the fields v and $\tilde{w}$ grants us licence to invent a shadow speck of matter $\mathfrak{s}$ which, in imagination, simply translates with the velocity v, within which the shadow subspeck at y flies with velocity $\tilde{w}$ and therefore the moment of momentum is K. Then one can take a further step and devise a congruent affine field (congruent in the sense that it leads to the same value of moment of momentum) with rate of deformation B and, through integration, a fictitious affine displacement with gradient G such that $B = \dot{G}G^{-1}$. Thus G is determined a multiplicative right-factor apart and, in principle, there might not be any obvious preferred choice. However, one may always decide on one, conventionally, as happens in standard gas dynamics (typically that occurring 'at normal atmospheric pressure and room temperature'). Similarly, if, in imagination, only the field L where known for the macromotion, the actual placement gradient F could be sought as the solution of $\dot{F} = LF$, again a multiplicative factor apart. Of course, one thinks rather of integrating $\dot{x} = v\,(x, \tau)$ and, if the case may be, of determining F, for instance, as the derivative of x with respect to its initial value; in this case the 'initial' value of F would be the identity.

Formally the velocity of each grain can now be split into the sum

$$w = v + By + w^r + w^s, \tag{22}$$

where w^r and w^s (relative and stray velocity) have been already introduced. Of course, by definition, w^s has null average at y. Recall that $\tilde{w}$ is the velocity attributed to $\mathfrak{s}$ at y as a weighted average of velocities of grains there and that

v and B are chosen so that the field of w^r leads to vanishing relative momentum and tensor moment of momentum.

Finally the flux of grains through the boundary into the loculus, due to the disorderly motion, can be evaluated by

$$-\int_{\mathfrak{e}} \tilde{\theta}\,(\tilde{w}-v)\cdot n = -\int_{\mathfrak{e}} \mathrm{div}_y\left(\tilde{\theta}\tilde{w}\right), \tag{23}$$

n, exterior normal.

Having assembled all the ingredients we venture to suggest the structure at least, if not all specifics of the surrogate, appropriate here, for the Boltzmann equation; a surrogate which reflects our modelling of $\mathfrak{s}$ as a subbody occupying the bounded domain $\mathfrak{e}$ subject to grain suffusion and which is inspired by the standard equation of conservation of mass. Thus we express the rate of change of grain number by $\frac{d\theta}{d\tau}+\theta\,(\mathrm{div}_x v+\mathrm{div}_y\tilde{w})$ and we propose the balance equation in the following disguise:

$$\frac{\partial\theta}{\partial\tau}+\frac{\partial\theta}{\partial x}\cdot v+\frac{\partial\theta}{\partial y}\cdot\tilde{w}+\frac{\partial\theta}{\partial w}\cdot h+\theta\,(\mathrm{div}_x v+\mathrm{div}_y\tilde{w})=\Gamma, \tag{24}$$

where h is related to force acting on $\mathfrak{s}$ and Γ is the collision/coherence operator.

As already remarked, equations (1)-(5) should follow from (24) as corollaries. In fact, a persistent discrepancy recurs due to the fact (ignored in the rough derivation of (1)-(5)) that, rigorously, $\mathfrak{s}$ is a subbody with variable mass. Warnings on the matter, which is mostly of marginal relevance, was already given repeatedly.

To evidence the discrepancy one needs only to integrate (24) over $\mathcal{V}\times\mathfrak{e}$ and exploit basic properties of h and Γ

$$\frac{\mu}{(\mathrm{meas}\,\mathfrak{e})}\int_{\mathcal{V}}\int_{\mathfrak{e}}\left(\frac{\partial\theta}{\partial\tau}+\mathrm{div}_x\,(\theta v)+\mathrm{div}_y\,(\theta\tilde{w})\right)=0, \tag{25}$$

to obtain the 'correct' version of (1)

$$\frac{\partial\rho}{\partial\tau}+\mathrm{div}_x\,(\rho v)=-\frac{\mu}{(\mathrm{meas}\,\mathfrak{e})}\int_{\partial\mathfrak{e}}\tilde{\theta}\tilde{w}\cdot n \tag{26}$$

(n, exterior normal to $\partial\mathfrak{e}$).

As mentioned above the right-hand side measures the mass per unit volume of the flux of grains through the boundary of $\mathfrak{e}$.

Equation (2) should suffer of a similar correction, with the flux of moment of inertia through the boundary of $\mathfrak{e}$ brought in. However, we will not pursue here the corollaries of (24) in the standard manner followed in treatises on the kinetic theory to obtain versions of all balances (1)-(5) corrected from possible grain flux through $\partial\mathfrak{c}$. Rather, worried by the complexities already inherent in (1)-(5), we accept them below within their declared uncertainties and dedicate the following paragraphs to important implications even as they stand and also explore some special consequences, contrasting the latter with old developments and results.

3 Strain

The likely inexistence of a natural reference, from which to count F and G unequivocally, does not deny necessarily value, or at least service, for allied concepts of strain. Indeed, a word such as 'volume' enters even in the fundamental law of elementary gas dynamics and, in general, if it is only agreed that the reference, as already suggested (and often feasible), be isotropic, a common bare scalar factor in F and G would again attend a change.

Hence the present, brief section to recall a few definitions and reflections from elsewhere, beginning with the primary notation

$$C = F^T F, \quad N = G^T G, \quad X = G^{-1} F, \quad \mathbf{n} = (\operatorname{grad} N)\, F, \tag{27}$$

leading to the multiplicative decompositions

$$F = RC^{\frac{1}{2}}, \quad G = R' N^{\frac{1}{2}}, \quad X = N^{\frac{1}{2}} R'^T R C^{\frac{1}{2}}, \tag{28}$$

with R and R' appropriate orthogonal tensors.

The last of relations (28) shows that X is independent of C and N only to within the orthogonal tensor Q

$$Q = R'^T R. \tag{29}$$

Conclusively, C, N and Q provide adequate first-order strain measures; $\mathbf{n}$ and a similar tensor involving $\operatorname{grad} C$ could offer higher order strains. Notice that C and N are symmetric, whereas $\mathbf{n}$ is symmetric with respect to the first two indices.

Time rates are also relevant later in the paper and we recall that

$$\dot{C} = 2F^T (\operatorname{sym} L)\, F, \quad \dot{N} = 2G^T (\operatorname{sym} B)\, G, \quad \dot{X} = G^{-1} (L - B)\, F. \tag{30}$$

More pertinent is a last consequence of definitions above

$$\dot{Q} = R'^T [\operatorname{skw}(L - B)]\, R = R'^T \operatorname{skw}\left(G\dot{X}F^{-1}\right) R. \tag{31}$$

The connection between the time-derivative $\dot{\mathbf{n}}$ of $\mathbf{n}$ and the third-order tensor $\mathbf{b}_{ijk}$ which express the gradient of B, $\mathbf{b}_{ijk} = B_{ij,k}$, is more remote. A slightly simpler expression obtains for the left-symmetric part

$$\mathbf{b}_{ijk} + \mathbf{b}_{jik} = G^{-1}_{Bi}\dot{\mathbf{n}}_{ABC}G^{-1}_{Aj}F^{-1}_{Ck} - \dot{N}_{AB}G^{-1}_{Bl}G_{lC,k}\left(G^{-1}_{Ai}G^{-1}_{Cj} + G^{-1}_{Aj}G^{-1}_{Ci}\right). \quad (32)$$

We have already remarked that a rôle similar to that of L and B could be imputed to H, so that it is not outlandish to pull it back also onto the phantom reference. In fact, a tensor $H_* = G^{-1}HG^{-T}$ was already introduced elsewhere and the rate $\dot{H}_*$ plays, on occasion, a rôle similar to that of $\dot{C}$, $\dot{N}$ etc.

In contrast to all said above and with the intent (in the search for response functions, say) to turn attention only and ever to current events, one could call upon, again as hinted elsewhere, to local metrics

$$\tilde{C} = FF^T \quad \text{and} \quad \tilde{N} = GG^T, \quad (33)$$

to the rotation

$$\tilde{Q} = R'R^T \quad (34)$$

and the wryness of underlying reference

$$\mathbf{w}_{ijk} = F^{-1}_{Aj}G_{iA,k}; \quad (35)$$

the latter leading to torsion (antisymmetric in the last two indices)

$$\mathbf{h} = \frac{1}{2}\left(\mathbf{w} - \mathbf{w}^t\right), \quad (36)$$

to the second order tensor measuring dislocation density

$$\mathbf{e}_{iab}\mathbf{h}_{jab} \quad (37)$$

and the corresponding Burgers vector b relative to any plane of normal n

$$\mathsf{b} = \left(\mathbf{e}\mathbf{h}^T\right)n. \quad (38)$$

Clearly one's interest may be only in the common invariants of all these tensors; the corresponding choices may be delicate.

In such a vein also for H an alternative decomposition comes to mind:

$$H = \tilde{N}^{\frac{1}{2}}\tilde{H}\tilde{N}^{\frac{1}{2}}, \quad (39)$$

and

$$\tilde{H} = \tilde{R}'^T H_* \tilde{R}' \quad (40)$$

gathers together the full local effect of, possibly internecine, cross-over rate, leading to $\tilde{N}$ the job of representing cooperative change of shape (in particular contraction/expansion).

Ultimately and in an incongruous temper here, H could also be dressed up in the disguise of a quasi-temperature Θ

$$H = \frac{k}{(\operatorname{meas} \mathfrak{e})\, \rho} \Theta, \tag{41}$$

where k is Boltzmann's constant. Say, a recurring, very simple, constitutive law for the stress relates it to H: $T = -\rho H$, as we shall see later. Then, in terms of Θ one gets

$$T = -k\Theta \left(\operatorname{meas} \mathfrak{e}\right)^{-1}, \tag{42}$$

a law that is the most direct adaptation of the elementary one stipulating that the pressure be proportional to the ratio of absolute temperature over volume.

Thus, as in a suggestion of decades ago, one may imagine to evidence two temperatures: a macroscopic quasi-temperature recorded at x and a, more appropriately named, microtemperature measured at y in $\mathfrak{e}$. The conception should not surprise in a complex continuum. It is well known that in mixtures of diffusing components one may associate a different microtemperature with each component and a mesoscopic quasi-temperature with the mixture, based on the peculiar velocities.

4 Energies

To take a first step in the direction pointed above at the end of Section 2 we recall a 'reduced' kinetic energy theorem, reduced in the sense that it involves $\tilde{W}$ rather than W; a theorem already quoted repeatedly as exhaustive in earlier papers and which is an immediate consequence of (1)-(5).

The theorem is derived by adding term by term the equations (3)-(5) after multiplication of (3) tensorially by v, (4) on the left by B and (5) by $\frac{1}{2}$, taking symmetric parts and integrating over any subbody $\mathfrak{b}$ of $\mathcal{B}$

$$\int_{\mathfrak{b}} \rho \dot{\tilde{W}} = \int_{\mathfrak{b}} \rho \left[\operatorname{sym}\left(v \otimes f + BM\right) + \frac{1}{2} S \right] + \int_{\partial \mathfrak{b}} \left\{ \operatorname{sym}\left[v \otimes Tn + B\left(\mathbf{m}n\right)\right] + \frac{1}{2} \mathbf{s}n \right\} \\ - \int_{\mathfrak{b}} \operatorname{sym}\left(\frac{1}{2} Z + LT^T + BA + \mathbf{b}\mathbf{m}^t \right); \tag{43}$$

here the compact notation for third-order tensors entails ambiguities; we mean $(\mathbf{b})_{ikl} = (\operatorname{grad} B)_{ikl} = B_{ik,l}$ and $\left(\mathbf{b}\mathbf{m}^t\right)_{ij} = \mathbf{b}_{ikl}\mathbf{m}_{kjl}$.

Thus the last integral in (43) must be interpreted as the tensor power of internal (mechanical) actions; its density per unit volume amounts to

$$-\operatorname{sym}\left(\frac{1}{2}Z + LT^T + BA + \mathbf{b}\mathbf{m}^t\right). \tag{44}$$

Of course, the standard scalar version of the theorem is obtained by taking the trace of both sides of (43). The density of scalar power of internal actions is given by

$$-\left(\frac{1}{2}trZ + L\cdot T + B\cdot A + \mathbf{b}\cdot{}^t\mathbf{m}\right) \tag{45}$$

and the consequent requirement of its objectivity (i.e., indifference from an observer rotation) is satisfied if and only if

$$\operatorname{skw} T = \operatorname{skw} A. \tag{46}$$

The stricter condition

$$T = -A \tag{47}$$

would render even the corresponding tensor version (44) objective.

Note that, contrary to the usual paradigm, here balance of moment of momentum does not imply 'per se' independence of observer for the quantity (45) (and 'a fortiori' of (44)); though the two conditions could be crammed together by substituting A in (4) with $\operatorname{skw} T + A^*$, entailing the symmetry of A^* (or, more radically, substituting A with $-T^T$).

In view of the rôle of H within $\tilde{W}$, one insinuation, dismissed above but mooted elsewhere, is that H be classified as a quasi-thermal quantity and, perhaps even, that (5) takes the place of a principle of balance of energy. Indeed, in our first approach where the body element was imagined as a discrete collection of grains, later 'homogenized', and totals were sums, later expressed via densities, the insinuation seemed warranted as then $\frac{1}{2}trH$ exhausted the kinetic energy (per unit mass) beyond $\frac{1}{2}v^2 + \frac{1}{2}trBK$.

To contrast and elucidate further the approach here it is expedient to dwell again on and promote some of its significant features. The chaotic distribution of grain velocities in $\mathfrak{e}$ is drilled also here within these artificial ranks: (i) average velocity v and average momentum K; (ii) variance H of relative velocities w^r. However, in addition to the uncertainty in the details of the field $w^r(y)$ within the constraint placed by H, there remains that in the distribution of stray velocities w^s. Hence, at least an evolution equation for $\hat{H}$ is still missing; it must be provided (fittingly in view of the declared properly thermal character of $\hat{H}$) by an appropriate, separate stipulation of the first principle of thermodynamics.

Having entrusted H with the task of highlighting any dominant anisotropy afflicting the chaotic motion, it seems reasonable to presume, as we have already hinted, that the main effects due to remaining stray motion be bound to the evolution of $tr\hat{H}$ and hence governed by a standard scalar version of the first principle. The question emerges then as to the appropriate notion of temperature; it may appear ancillary here, but for some purposes at least it is in the end crucial. Reference to the kinetic theory may be of help: there the temperature (a dimensional factor apart already quoted in Section 3) coincides with the peculiar molecular kinetic energy. Hence the plausible suggestion here that, such factor apart, $\frac{1}{2}tr\hat{H}$ take the rôle. However, always within the kinetic theory, another task is assigned to temperature: it is the essential parameter in the stipulation of the distribution $\hat{\theta}$ valid near equilibrium. So much so that many scholars deny legitimacy to the notion of temperature outside that strict bond. Now, Maxwell's distribution, which forecasts the existence of some, if few, molecules with speed beyond any limit, seems poorly adaptable to the granular flows. A distribution with compact support in the space of speeds seems more appropriate; consequences of that choice are deferred to a later stage.

Here we conclude this section recalling a more explicit version of the quantity (45) where L, B and $\mathbf{b}$ are substituted by their expressions in terms of $\dot{C}$, $\dot{N}$, $\dot{Q}$, $\dot{\mathbf{n}}$. The manipulations are elementary if modestly tricky; the result being

$$\begin{aligned}
-\left(\frac{1}{2}trZ + L\cdot T + B\cdot A^T + \mathbf{b}\cdot{}^t\mathbf{m}\right) = \\
= -\frac{1}{2}trZ - \frac{1}{2}\left(F^{-1}\left(\operatorname{sym}T\right)F^{-T}\right)\cdot\dot{C} - \left(R'^T\left(\operatorname{skw}T\right)R\right)\cdot\dot{Q} - \\
-\frac{1}{2}\left[M^* + G^{-1}\left(\operatorname{sym}A\right)G^{-T}\right]\cdot\dot{N} + \frac{1}{2}\left({}^t\mathbf{b}-\mathbf{b}\right)\cdot\mathbf{m} - \frac{1}{2}\mathbf{m}^*\cdot\dot{\mathbf{n}}, \quad (48)
\end{aligned}$$

where

$$M^*_{AB} = \left(G^{-1}_{Ai}G^{-1}_{Bj}\right)_{,k}\left(\mathbf{m}_{ijk} + \mathbf{m}_{jik}\right), \quad (49)$$

$$\mathbf{m}^*_{ABC} = G^{-1}_{Ai}G^{-1}_{Bj}\mathbf{m}_{ijk}F^{-1}_{Ck}. \quad (50)$$

The term containing ${}^t\mathbf{b}-\mathbf{b}$ (skew in the first two indices) is left undeveloped because of its complexity and because, in an interesting case quoted below, $\mathbf{m}$ turns out to be symmetric in the first two indices and, as a consequence, that term vanishes.

5 Some constitutive options

Ordinarily, in granular flows, the rôle of dissipation is paramount. However, the disposition, in this paper, is to essay new ideas inspired originally by those flows, but with an open mind and with attention also for allied incidents. In such an inquisitive mood, it does not seem impertinent to study a 'perfect' medium ruled by the balances (1)-(5); perfect in the sense that, for it, a potential function φ of the strains can be devised such that along any stream the power (45) (or, better, (48)) coincides with the time derivative of φ multiplied by ρ. The strength of the hypothesis of perfection depends ultimately on the primary variables entering φ. If those variables are C, N, Q and $\mathbf{n}$, then consequent choices for T, A, $\mathbf{m}$ are

$$\operatorname{sym} T = 2\rho F \frac{\partial \varphi}{\partial C} F^T, \tag{51}$$

$$\operatorname{skew} T = \operatorname{skw} A = \rho R' \frac{\partial \varphi}{\partial Q} R^T, \tag{52}$$

$$\mathbf{m}_{ijk} = 2\rho G_{iA} G_{jB} F_{kC} \frac{\partial \varphi}{\partial \mathbf{n}_{ABC}} \tag{53}$$

(a tensor symmetric in the first two indices),

$$(\operatorname{sym} A)_{ij} = 2\rho \left(G \frac{\partial \varphi}{\partial N} G^T \right)_{ij} - 4\rho G_{iA} \frac{\partial \varphi}{\partial \mathbf{n}_{ABC}} G_{jB,k} F_{kC}. \tag{54}$$

At the same time

$$trZ = 0. \tag{55}$$

No constitutive condition ensues for dev Z and $\mathbf{s}$, of course; the former could, possibly, descend from speculations regarding the tensor quantity (44).

To compare and contrast some special implications of (1)-(5) with notions issuing from different lines of research we must betray again the original motivations and assume that quantities related to moments (precisely Y, K, M, $\mathbf{m}$) be negligible. Then the equation (2) is trivially satisfied, B is left unspecified but becomes insignificant, the equation (4) requires A to coincide with $-\rho H$ and, consequently, T to be symmetric. The remaining equations make up a decamoment system,formally reminiscent of one repeatedly proposed in papers on gas dynamics.

Actually if the deeper condition (47) is also accepted, then the (constitutive) equation

$$T = -\rho H, \tag{56}$$

already quoted toward the end of Section 3, must apply. Further, again formally, one could rewrite and reinterpret our equations so that they become akin to those appearing in papers on extended thermodynamics.

The fact that then, if indirectly, the last equation involves the time-rate of Cauchy stress recalls suggestions in hypoelasticity. True, both in the first paper of Truesdell and in the successive elaborations of Noll the law additional to Cauchy's is posited as a queer constitutive property. However, already in the corresponding thermomechanical formulation of Olsen and Bernstein, it appears more akin to a balance law. Such an admittedly controversial interpretation would, at least, give a more natural status to initial and (eventual) boundary condition (now for H, and only indirectly for T).

An alternative option to obtain from (1)-(5) a set of equations closer to the standard format is by the imposition of some perfect constraint; the simpler such constraint enforces the coincidence of micro and macro strain

$$G = F \quad \text{and} \quad B = L, \tag{57}$$

when the only kinetic fields left to be determined are v and H.

One is thus led to a continuum with latent complexity, now kinetic rather than Lagrangian as was one proposed long ago.

Absence of friction requires the vanishing of the power of reactive components (labelled below by the superscript r) for all virtual kinetic fields allowed by the constraint

$$\frac{1}{2} tr \overset{r}{Z} + L \cdot \left(\overset{r}{T} + \overset{r}{A}{}^{T} \right) + (\operatorname{grad} L) \cdot^{t} \overset{r}{\mathbf{m}} = 0, \qquad \forall L, \operatorname{grad} L. \tag{58}$$

Hence

$$\frac{1}{2} tr \overset{r}{Z} = 0, \quad \overset{r}{T} = -\overset{r}{A}{}^{T}, \quad \overset{r}{\mathbf{m}} = 0. \tag{59}$$

The pure equations of evolution (involving exclusively the active components $\overset{a}{T}$, $\overset{a}{A}$, $\overset{a}{\mathbf{m}}$, $\overset{a}{Z}$) are (1), (2) with B substituted by L, the scalar equation obtained from (5) by taking the trace of each side (again writing L for B) and finally the equation derived by substracting term by term the divergence of the transpose of (4) from (3)

$$\begin{aligned} \rho \left(\frac{\partial v}{\partial \tau} + L v \right) - \operatorname{div} \left[\rho \left(\frac{\partial L}{\partial \tau} + (\operatorname{grad} L) \, v + L^2 \right) Y \right] = \\ = \rho f - \operatorname{div} \left(\rho M^T \right) + \operatorname{div} \left[\operatorname{sym} \left(\overset{a}{T} + \overset{a}{A} \right) - \rho H - \left(\operatorname{div} \overset{a}{\mathbf{m}} \right)^T \right]. \end{aligned} \tag{60}$$

This last comes close to some equations proposed for the study of flows with fully developed turbulence. The interest and residual complexity, a relic of the original tangle, is in the intrusion of moments. The arduous task is the search of constitutive laws for A and $\mathbf{m}$, which would connect, by a physically motivated recipe, B and L and H, evidencing the ties between macromotion, turbulence and quasi-thermal quantities.

Again, when Y, M and $\mathbf{m}$ vanish, equation (60) collapses into the familiar form

$$\rho\left(\frac{\partial v}{\partial \tau}+Lv\right)=\rho f+\operatorname{div}\left[\operatorname{sym}\left(\overset{a}{T}+\overset{a}{A}\right)-\rho H\right], \tag{61}$$

where the active stress is augmented by the quasi-thermal stress $-\rho H$. Still, in all such simplified circumstances, the main thrust of the initial proposal is sidetracked.

However, a vestigial concurrence of effects remains: $-\rho H$ renders quasi-thermal dispersive actions which prevail in gases (where $\overset{a}{T}\sim 0$, $\overset{a}{A}\sim 0$); $\overset{a}{A}$ portrays intimate cohesive actions, which in solids balance the former (when $\overset{a}{A}-\rho H\sim 0$); $\overset{a}{T}$ standard interloculus actions.

6 Concluding remark

The speculations signified in this paper originated from a haphazard perusal of the literature which induced at times agreement or objection, even surprise. Casual citation is avoided: to list references appropriately would have been a major task, disproportionate to the modest goal of the paper.

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