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An extreme example concerning factorization products on the Schwartz space $\mathfrak{S}(\mathbb{R}^n)$

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Abstract. We construct linear operators S, T mapping the Schwartz space $\mathfrak{S}$ into its dual $\mathfrak{S}'$, such that any operator $R \in \mathcal{L}(\mathfrak{S}, \mathfrak{S}')$ may be obtained as factorization product $S \circ T$. More precisely, given $R \in \mathcal{L}(\mathfrak{S}, \mathfrak{S}')$, there exists a Hilbert space H_R such that $\mathfrak{S} \subset H_R \subset \mathfrak{S}'$, the embeddings $\mathfrak{S} \hookrightarrow H_R$ and $H_R \hookrightarrow \mathfrak{S}'$ are continuous, $\mathfrak{S}$ is dense in H_R, $T(\mathfrak{S}) \subset H_R$, and S has a continuous extension $\widetilde{S} : H_R \to \mathfrak{S}'$ such that $\widetilde{S}(T\varphi) = R\varphi$ for all $\varphi \in \mathfrak{S}$.

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1 Introduction

There exist several approaches to partial multiplications in spaces of linear operators connected with Gelfand triples $D \subset H \subset D^+$, e.g. products of operators on nested Hilbert spaces or on PIP-spaces [1, 5], partial *-algebras [3], quasi-algebras, quasi-*-algebras [10,11], and partial multiplications on $\mathcal{L}(D, D^+)$ [2,4,6,7,14]. It is known that such products may depend on the choice of factorizing spaces needed to define the products. Hence, in order to obtain a well-defined factorization product, it is necessary to be careful and restrictive in the choice of spaces which are allowed to serve as factorizing spaces. Since partially defined products are a useful tool as well in theoretical investigations as in applications, it is of interest to know to what extent the product may depend on the choice of the factorizing space. In [7–9] pairs of operators were constructed for which products with respect to different factorizing spaces differ just by a rank-one operator. Here we present a modified approach of [12] which shows that there exist two linear operators defined on the Schwartz space $\mathfrak{S}$ with the property that their factorization product can deliver *any* operator $R \in \mathcal{L}(\mathfrak{S}, \mathfrak{S}')$ depending on the factorizing space chosen.

2 Preliminaries

First of all we introduce some notations and definitions. For locally convex spaces E and F, E' denotes the strong dual, and $\mathcal{L}(E,F)$ denotes the space of all continuous linear operators from E into F. Given an index set $J \subset \mathbb{Z}^n$, a family $(x_\iota)_{\iota\in J}$ of complex numbers, and $s \in \mathbb{Z}$, we set

$$p_s\left((x_\iota)_{\iota\in J}\right) = \left(\sum_{\iota=(j_1,j_2,\ldots,j_n)\in J} \left(\prod_{l=1}^{n} (|j_l|+2)^{2s} \right) |x_\iota|^2 \right)^{1/2},$$

$$\mathfrak{s}(J) = \left\{ (x_\iota)_{\iota\in J} \in \mathbb{C}^J \mid p_s\left((x_\iota)_{\iota\in J}\right) < \infty \text{ for all } s \in \mathbb{N} \right\}.$$

The locally convex topology of $\mathfrak{s}(J)$ is generated by the seminorms p_s, $s \in \mathbb{N}_0$. There are natural embeddings $\mathfrak{s}(J) \subset l_2(J) \subset \mathfrak{s}'(J)$. By using the coordinate basis $(\varphi_\iota)_{\iota\in J}$, the element $(x_\iota)_{\iota\in J}$ of $\mathbb{C}^J$ may be written as $\sum_{\iota\in J} x_\iota\,\varphi_\iota$.

In the following we will use spaces defined for the index sets $\mathbb{N}$ and (the from now on fixed) $J = \mathbb{N}\times\mathbb{N}\times\{1,2,3,4\}\times\mathbb{Z}\times\mathbb{N}$. The space $\mathfrak{s} = \mathfrak{s}(\mathbb{N})$ is the Schwartz space of rapidly decreasing sequences. By constructing a suitable bijection of index sets it is easy to see that the spaces $\mathfrak{s}(J)$ and $\mathfrak{s}$ are isomorphic. It is also well-known that they are isomorphic to the Schwartz space $\mathfrak{S}(\mathbb{R}^n)$, which is also denoted by $\mathfrak{S}$. Using a basis consisting of products of Hermite functions, one can construct an isomorphism of $\mathfrak{S}(\mathbb{R}^n)$ onto $\mathfrak{s}({\mathbb{N}_0}^n)$ that may be extended to an isometric isomorphism of $L_2(\mathbb{R}^n)$ onto $l_2({\mathbb{N}_0}^n)$ (cf., e.g., [13]).

Hilbert spaces will often be defined as domains $D(A)$ of self-adjoint operators A endowed with their *graph norms* defined by $\|\varphi\|_A = \sqrt{\|\varphi\|^2 + \|A\varphi\|^2}$.

For locally convex spaces E, F such that $\mathfrak{S} \subset E$ and $F \subset \mathfrak{S}'$, where the embeddings are linear, continuous, and have dense ranges, we define

$$\mathcal{C}(E,F) = \{ T \in \mathcal{L}(\mathfrak{S},\mathfrak{S}') \mid \text{There exists } S \in \mathcal{L}(E,F) \text{ such that } S\varphi = T\varphi \text{ for all } \varphi \in \mathfrak{S} \}.$$

1 Definition. Suppose that $\mathfrak{K}$ is a set of locally convex spaces G which are linear subspaces of $\mathfrak{S}'$ containing $\mathfrak{S}$ such that the embeddings $\mathfrak{S} \hookrightarrow G$ and $G \hookrightarrow \mathfrak{S}'$ are continuous and such that for any pair (E,F) of elements of $\mathfrak{K}$, $\mathfrak{S}$ is dense in $E \cap F$ w.r.t. the topology generated by the union of the topologies induced by E and F. The product $T_n\circ\cdots\circ T_1$ of elements of $\mathcal{L}(\mathfrak{S},\mathfrak{S}')$ is said to be defined w.r.t. $\mathfrak{K}$ if there are spaces $E_0,\ldots,E_n \in \mathfrak{K}$ such that $T_j \in \mathcal{C}(E_{j-1},E_j)$ for $j \in \{1,\ldots,n\}$. If $S_j \in \mathcal{L}(E_{j-1},E_j)$ is the unique extension of T_j, the *factorization product* $T_n \circ\cdots\circ T_1$ is defined by

$$T_n \circ\cdots\circ T_1\,\varphi = S_n\left(\ldots(S_1\varphi)\ldots\right) \qquad (\varphi \in \mathfrak{S}).$$

2 Remark. The only difference between Definition 1 and the definition given in [6, 7] in a more general situation consists in the use of the dual space $\mathfrak{S}'$ here instead of the space $\mathfrak{S}^+$ of continuous conjugate linear functionals in the general case. Using the conjugate linear bijection between $\mathfrak{S}'$ and $\mathfrak{S}^+$ given by complex conjugation, one obtains easily the equivalence of both definitions. In particular, the operator $T_n \circ \cdots \circ T_1$ defined in Definition 1 does not depend on the special choice of the spaces E_j in $\mathfrak{K}$ such that $T_j \in \mathcal{C}(E_{j-1}, E_j)$.

The following construction is essentially taken from [6]. However some of the statements needed here were not formulated explicitely there. For the convenience of the reader we include this construction here.

3 Proposition. *There exist self-adjoint operators A_0, A_1 on l_2, an element η of $D(A_0) \cap D(A_1)$ and linear functionals f_0, f_1 on $D(A_0)$ and $D(A_1)$, resp., such that the following assertions are satisfied:*
i) $f_r(\eta) = r \quad (r \in \{0, 1\})$,
ii) $|f_r(\varphi)| \leq 2\,\|A_r\,\varphi\| \quad (r \in \{0, 1\}, \varphi \in D(A_r))$,
iii) $\|A_r\,\eta\| = r \quad (r \in \{0, 1\})$,
iv) $\mathfrak{s}$ is a dense linear subspace of $D(A_1)$ and of $D(A_2)$,
v) $f(\varphi) = f_0(\varphi) = f_1(\varphi)$ for all $\varphi \in \mathfrak{s}$.

PROOF. We define orthogonal sequences $(\psi_{r,p})_{p\in\mathbb{N}}$, $r \in \{0, 1\}$, in l_2 by

$$\psi_{0,p} = (2p-1)\varphi_{2p-1} - 2p\,\varphi_{2p},$$
$$\psi_{1,1} = \varphi_1, \quad \psi_{1,p+1} = -2p\,\varphi_{2p} + (2p+1)\varphi_{2p+1}.$$

Then we set

$$\eta = \left(\frac{1}{p}\right)_{p\in\mathbb{N}},$$
$$D(A_r) = \left\{ \varphi \in l_2 \;\middle|\; \sum_{p\in\mathbb{N}} |\langle\varphi, \psi_{r,p}\rangle|^2\,\|\psi_{r,p}\|^2 < \infty \right\},$$
$$A_r\varphi = \sum_{p\in\mathbb{N}} \langle\varphi, \psi_{r,p}\rangle\,\psi_{r,p},$$
$$f_r : D(A_r) \ni \varphi \mapsto \sum_{p\in\mathbb{N}} \langle\varphi, \psi_{r,p}\rangle\,,$$
$$f : \mathfrak{s} \ni \varphi \mapsto \sum_{p\in\mathbb{N}} (-1)^{p-1} \cdot p \cdot \langle\varphi, \varphi_p\rangle\,.$$

Hence

$$|f_r(\varphi)| \leq \left(\sum_{p\in\mathbb{N}} |\langle\varphi, \psi_{r,p}\rangle|^2\,\|\psi_{r,p}\|^2\right)^{\frac{1}{2}} \left(\sum_{p\in\mathbb{N}} \frac{1}{\|\psi_{r,p}\|^2}\right)^{\frac{1}{2}} \leq \left(\sum_{p\in\mathbb{N}} \frac{1}{p^2}\right)^{\frac{1}{2}} \|A_r\varphi\|\,,$$

which implies ii).

It is easy to see that $\mathfrak{s} \subset D(A_r)$ and that already the linear span of $\{\varphi_p\}_{p\in\mathbb{N}}$ is dense in $D(A_r)$. Hence iv) is satisfied. Statements i), iii) and v) are immediate consequences of the construction of A_r, f_r, and f, which completes the proof.

QED

3 Statement of the result

4 Proposition. *There exist operators $S, T \in \mathcal{L}(\mathfrak{S}, \mathfrak{S}')$ satisfying the following assertion:*
Given $R \in \mathcal{L}(\mathfrak{S}, \mathfrak{S}')$, there exists a self-adjoint operator A_R on $L_2(\mathbb{R}^n)$ such that $T \in \mathcal{L}(\mathfrak{S}, D(A_R))$, S has a continuous extension $\widetilde{S} \in \mathcal{L}(D(A_R), \mathfrak{S}')$ and $\widetilde{S}(T\varphi) = R\varphi$ for all $\varphi \in \mathfrak{S}$.

5 Corollary. *The operators S, T of Proposition 4 satisfy also the following assertion:*
Given $R \in \mathcal{L}(\mathfrak{S}, \mathfrak{S}')$, there exists a Hilbert space H_R such that the factorization product w.r.t. $\mathfrak{K} = \{\, \mathfrak{S}, H_R, \mathfrak{S}' \,\}$ of S and T exists in the sense of Definition 1 and satisfies $S \circ T = R$.

4 Construction of operators

In this section we prove Proposition 4 by constructing the corresponding operators. Having in mind the isomorphisms mentioned before, the operators S and T will be constructed as elements of $\mathcal{L}(\mathfrak{s}(J), \mathfrak{s}')$ and $\mathcal{L}(\mathfrak{s}, l_2(J))$, respectively. The operator R will be assumed to be an element of $\mathcal{L}(\mathfrak{s}, \mathfrak{s}')$ and A_R will be a self-adjoint operator on $l_2(J)$.

Besides the index set $J = \mathbb{N} \times \mathbb{N} \times \{\,1, 2, 3, 4\,\} \times \mathbb{Z} \times \mathbb{N}$ we also use the set $I = \mathbb{N} \times \mathbb{N} \times \{\,1, 2, 3, 4\,\} \times \mathbb{Z}$. If $\iota = (j, k, l, m) \in I$ or $\iota = (j, k, l, m, p) \in J$ we will also write $j = j_\iota, k = k_\iota, \ldots$. The following maps are defined on $\mathbb{C}^J$ or $\mathbb{C}^{\mathbb{N}}$, but they can be interpreted also as linear operators between subspaces:

$$
\begin{aligned}
M &: \mathbb{C}^{\mathbb{N}} \ni (x_n)_{n\in\mathbb{N}} &&\mapsto ((n+2)\cdot x_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}, \\
P_{(j,k,l,m)} &: \mathbb{C}^{J} \ni (x_\iota)_{\iota\in J} &&\mapsto \sum_{n\in\mathbb{N}} x_{(j,k,l,m,n)}\, \varphi_n \in \mathbb{C}^{\mathbb{N}} \quad ((j,k,l,m) \in I), \\
Q_{(j,k,l,m)} &: \mathbb{C}^{\mathbb{N}} \ni (x_n)_{n\in\mathbb{N}} &&\mapsto \sum_{n\in\mathbb{N}} x_n\, \varphi_{(j,k,l,m,n)} \in \mathbb{C}^{J} \quad ((j,k,l,m) \in I).
\end{aligned}
$$

The operators $S \in \mathcal{L}(\mathfrak{s}(J), \mathfrak{s}')$ and $T \in \mathcal{L}(\mathfrak{s}, l_2(J))$ are defined by

$$S(\psi) = \sum_{(j,k,l,m)\in I} \mathrm{i}^l \, (j+2)(|m|+2) \, f\big(P_{(j,k,l,m)}\,\psi\big)\,\varphi_j,$$

$$T((x_k)_{k\in\mathbb{N}}) = \sum_{(j,k,l,m)\in I} \frac{x_k}{(j+2)(|m|+2)}\, 2^{-m_+}\, Q_{(j,k,l,m)}\eta,$$

where η is the vector constructed in Proposition 3 and $m_+ = \max\{\,0, m\,\}$ is the positive part of m.

Let us now fix an operator $R \in L(\mathfrak{s}, \mathfrak{s}')$ and consider its matrix elements $(a_{j,k})_{j,k\in\mathbb{N}}$. This means that $R((x_k)_{k\in\mathbb{N}}) = \big(\sum_{k\in\mathbb{N}} a_{j,k}x_k\big)\,\varphi_j$. Since the bilinear form $\mathfrak{s}\times\mathfrak{s} \ni (\varphi_1, \varphi_2) \to (R\varphi_1)(\varphi_2)$ is jointly continuous, there exists $s \in \mathbb{N}$ such that $|a_{j,k}| \le (j+2)^s \cdot (k+2)^s$ for all $j, k \in \mathbb{N}$. Writing the real and imaginary parts of $a_{j,k}$ as differences of their positive and negative parts, resp., and representing these nonnegative reals as sums of values 1 (as often as integer part requires) and an element of $[0,1)$, that we decomposit into its binary digits, we assign to R a fixed number $s \in \mathbb{N}$ and a family $(r_\iota)_{\iota\in I} \in \{\,0,1\,\}^I$ such that $r_{(j,k,l,m)} = 0$ whenever $m < -(j+2)^s \cdot (k+2)^s$ and that

$$a_{j,k} = \sum_{l=1}^{4} \sum_{m=-\infty}^{\infty} \mathrm{i}^l\, 2^{-m_+}\, r_{(j,k,l,m)}.$$

Now using the operators A_0 and A_1 from Proposition 3 the formulas

$$D(A_R) = \left\{\, \psi \in l_2(J) \;\middle|\; \sum_{\iota\in I} (k_\iota+2)^2 (j_\iota+2)^{-8s} (|m_\iota|+2)^4 \|A_{r_\iota} P_\iota \psi\|^2 < \infty \,\right\},$$

$$A_R\psi = \sum_{\iota\in I} (k_\iota+2)(j_\iota+2)^{-4s}(|m_\iota|+2)^2\, Q_\iota A_{r_\iota} P_\iota\psi$$

define a self-adjoint operator A_R on $l_2(J)$.

Given $\psi \in D(A_R)$, we set furthermore

$$B_R\psi = \sum_{\iota\in I} \mathrm{i}^l\, (j_\iota+2)^{-4s}\, (|m_\iota|+2)\, f_{r_\iota}(P_\iota\psi)\, \varphi_{j_\iota},$$

$$S_R\psi = M^{4s+1}\, B_R\psi.$$

The next proposition collects the properties of these operators, needed to prove Proposition 4.

6 Proposition. *The operators S, T, A_R, B_R, and S_R constructed obove satisfy the following assertions:*

i) $T \in \mathcal{L}(\mathfrak{s}, D(A_R))$,
ii) $B_R \in \mathcal{L}(D(A_R), l_2)$,
iii) $S_R \in \mathcal{L}(D(A_R), \mathfrak{s}')$,
iv) $S_R\psi = S\psi$ *for all* $\psi \in \mathfrak{s}(J)$,
v) $S_R T\varphi = R\varphi$ *for all* $\varphi \in \mathfrak{s}$.

PROOF.

i) Note that $(m+2)^4\, 2^{-2m_+} \le (m+2)^4 \mathrm{e}^{-m} \le 2^6$ for $m \ge 0$ and $(|m|+2)^4 \le ((j+2)^{s+1} \cdot (k+1)^{s+1})^4 \le (j+2)^{8s}(k+2)^{8s}$ for $-(j+2)^s(k+2)^s \le m < 0$. Consequently, given $(x_k)_{k\in\mathbb{N}} \in \mathfrak{s}$, we obtain the estimate

$$\sum_{\iota\in I} (k_\iota+2)^2 (j_\iota+2)^{-8s} (|m_\iota|+2)^4 \|A_{r_\iota} P_\iota T((x_k)_{k\in\mathbb{N}})\|^2$$
$$= \sum_{\iota\in I} (k_\iota+2)^2 (j_\iota+2)^{-8s} (|m_\iota|+2)^4 \frac{|x_{k_\iota}|^2}{(j_\iota+2)^2(|m_\iota|+2)^2} 2^{-2m_{\iota+}} \|A_{r_\iota}\eta\|^2$$
$$\le \sum_{j,k=1}^{\infty} \sum_{l=1}^{4} \sum_{m\ge -(j+2)^s(k+2)^s} (k+2)^2 (j+2)^{-8s-2} (|m|+2)^{-2} (|m|+2)^4\, 2^{-2m_+} |x_k|^2$$
$$\le \sum_{\iota\in I} \frac{1}{(j_\iota+2)^2(|m_\iota|+2)^2} (k_\iota+2)^{8s+2} |x_{k_\iota}|^2$$
$$\le \left(\sum_{j=1}^{\infty} \sum_{l=1}^{4} \sum_{m=-\infty}^{\infty} \frac{1}{(j+2)^2(|m|+2)^2} \right) (p_{4s+1}((x_k)_{k\in\mathbb{N}}))^2 .$$

Together with $T \in \mathcal{L}(\mathfrak{s}, l_2(J))$ this yields $T(\mathfrak{s}) \subset D(A_R)$ and $T \in \mathcal{L}(\mathfrak{s}, D(A_R))$.

ii) We estimate

$$\|B_R\psi\|^2 \le \sum_{j=1}^{\infty} \Bigg(\sum_{k=1}^{\infty} \sum_{l=1}^{4} \sum_{m=-\infty}^{\infty} (j+2)^{-4s} (|m|+2)^{-1} (k+2)^{-1} \cdot$$
$$\cdot (k+2)(|m|+2)^2\, 2 \left\| A_{r_{(j,k,l,m)}} P_{(j,k,l,m)}(\psi) \right\| \Bigg)^2$$
$$\le \left(\sum_{k=1}^{\infty} \sum_{l=1}^{4} \sum_{m=-\infty}^{\infty} (|m|+2)^{-2} (k+2)^{-2} \right) \cdot$$
$$\cdot \sum_{(j,k,l,m)\in I} (j+2)^{-8s} (k+2)^2 (|m|+2)^4 \left\| A_{r_{(j,k,l,m)}} P_{(j,k,l,m)}(\psi) \right\|^2 ,$$

the last sum being $\|A_R\psi\|^2$. So the series defining $B_R\psi$ converges in l_2 and we have $B_R \in \mathcal{L}(D(A_R), l_2)$.

iii) is an immediate consequence of ii) and the definition of S_R.
v) We evaluate $S_R T$ on $(x_k)_{k\in\mathbb{N}}$:

$$\begin{aligned} S_R(T((x_k)_{k\in\mathbb{N}})) &= \sum_{(j,k,l,m)\in I} \mathrm{i}^l (j+2)^{4s+1}(j+2)^{-4s}(|m|+2)\cdot \\ &\qquad \cdot f_{r_{(j,k,l,m)}}\left(\frac{x_k}{(j+2)(|m|+2)}\,2^{-m_+}\eta\right)\varphi_j \\ &= \sum_{(j,k,l,m)\in I} \mathrm{i}^l\, 2^{-m_+}\, r_{(j,k,l,m)} x_k \varphi_j \\ &= \sum_{k,l=1}^{\infty} a_{j,k} x_k \varphi_j = R((x_k)). \end{aligned}$$

Since iv) is an immediate consequence of the definitions of S and S_R the proof is complete. QED

As noted in Section 2, it is possible to define explicit isomorphisms

$$\begin{aligned} U_0 &\colon \mathfrak{S}(\mathbb{R}^n) \to \mathfrak{s}({\mathbb{N}_0}^n), \\ V_0 &\colon \mathfrak{s}({\mathbb{N}_0}^n) \to \mathfrak{s}, \\ W_0 &\colon \mathfrak{s} \to \mathfrak{s}(J) \end{aligned}$$

which admit unitary extensions $U \in \mathcal{L}(L_2(\mathbb{R}^n), l_2({\mathbb{N}_0}^n))$, $V \in \mathcal{L}(l_2({\mathbb{N}_0}^n), l_2)$, and $W \in \mathcal{L}(l_2, l_2(J))$ as well as continuous extensions $U_1 \in \mathcal{L}(\mathfrak{S}'(\mathbb{R}^n), \mathfrak{s}'({\mathbb{N}_0}^n))$, $V_1 \in \mathcal{L}(\mathfrak{s}'({\mathbb{N}_0}^n), \mathfrak{s}')$, and $W_1 \in \mathcal{L}(\mathfrak{s}', \mathfrak{s}'(J))$ which are isomorpisms. Using these isomorphisms, one can construct explicitly the operators needed to prove Proposition 4. To do so, we define $S', T' \in \mathcal{L}(\mathfrak{S}, \mathfrak{S}')$ by $S'\varphi = U_1^{-1}V_1^{-1}SW_0V_0U_0\varphi$ and $T'\varphi = U^{-1}V^{-1}W^{-1}TV_0U_0\varphi$, where S, T are the operators constructed above. Given $R' \in \mathcal{L}(\mathfrak{S}, \mathfrak{S}')$, we apply the construction described in the present section to the operator $R \in \mathcal{L}(\mathfrak{s}, \mathfrak{s}')$ defined by $R\varphi = V_1U_1R'U_0^{-1}V_0^{-1}\varphi$. In particular, we obtain a self-adjoint operator A_R and operators B_R and S_R such that the assertions of Proposition 6 are satisfied. Setting finally $A'_R = (WVU)^{-1}A_RWVU$ and $\widetilde{S}'\varphi = U_1^{-1}V_1^{-1}S_RWVU\varphi \quad (\varphi \in D(A'_R) = (WVU)^{-1}D(A_R))$, all assertions of Proposition 4 are satisfied for S', T', R', A'_R, $\widetilde{S}'$ in place of S, T, R, A_R, $\widetilde{S}$.

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