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Warfield Invariants of $V(RG)/G$

Peter V. Danchev [i]
13, General Kutuzov Str., block 7, floor 2, flat 4
4003 Plovdiv, Bulgaria.
pvdanchev@yahoo.com

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Abstract. Let R be a commutative unitary ring of prime characteristic p and let G be an Abelian group. We calculate only in terms of R and G (and their sections) Warfield p-invariants of the quotient group $V(RG)/G$, that is, the group of all normalized units $V(RG)$ in the group ring RG modulo G. This supplies recent results of ours in (Extr. Math., 2005), (Collect. Math., 2008) and (J. Algebra Appl., 2008).

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1 Introduction

Throughout the present article, suppose that R is a commutative unitary ring of prime characteristic p, fixed for the duration, and G is an Abelian group, written multiplicatively as is customary when discussing group rings, with p-primary component G_p and torsion part G_t. As usual, RG denotes the group ring of G over R with group of normalized invertible elements $V(RG)$ and its p-component of torsion $V_p(RG)$. Moreover, let we define inductively, $G^{p^0} = G$, $G^{p^\alpha} = (G^{p^{\alpha-1}})^p$ when α is isolated and $G^{p^\alpha} = \cap_{\beta<\alpha} G^{p^\beta}$ when α is limit. By analogy $R^{p^0} = R$, $R^{p^\alpha} = (R^{p^{\alpha-1}})^p$ when α is isolated and $R^{p^\alpha} = \cap_{\beta<\alpha} R^{p^\beta}$ when α is limit. We shall say that the ring R is perfect if $R = R^p$. For any set M, we let $|M|$ designate its cardinality, and ζ_d designate the primitive d-th root of unity whenever d is a positive integer.

All other unexplained explicitly notations and notions are standard and follow essentially the classical ones stated in ([5], [6] and [8]).

The goal of this paper, that we pursue, is to calculate only in terms of R and G Warfield p-invariants of $V(RG)/G$, defined for an arbitrary multiplicative Abelian group A in the following way (compare with [9]):

$$W_{\alpha,p}(A) = rank(A^{p^\alpha}/(A^{p^{\alpha+1}} A_p^{p^\alpha})),$$

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where α is an ordinal.

It easily follows that $W_{\alpha,p}(A) = |A^{p^\alpha}/(A^{p^{\alpha+1}}A_p^{p^\alpha})|$ when $|A^{p^\alpha}/(A^{p^{\alpha+1}}A_p^{p^\alpha})| \geq \aleph_0$ or $W_{\alpha,p}(A) = \log_p|A^{p^\alpha}/(A^{p^{\alpha+1}}A_p^{p^\alpha})|$ otherwise.

Our calculations illustrated in the sequel naturally arise for applicable purposes and are helpful for the isomorphism description of the factor-group $V(RG)/G$. In fact, Warfield p-invariants, together with Ulm-Kaplansky invariants, determine, up to isomorphism, p-mixed Warfield groups (e.g., [7]).

It is worthwhile noticing that in [1]-[4] we have computed Warfield p-invariants of $V(RG)$ under various restrictions on R and G. These computations will be used here because as it will be proved below, we can restrict in some instances Warfield p-invariants of $V(RG)/G$ to the Warfield p-invariants of $V(RG)$.

2 Preliminaries

Before stating and proving our main result, we need some preparatory machineries.

1 Lemma. *For every ordinal number α, the following two identities hold:*

(a) $G \cap V^{p^\alpha}(RG) = G^{p^\alpha}$;

(b) $(V(RG)/G)^{p^\alpha} = V^{p^\alpha}(RG)G/G$.

PROOF. (a) Since it is straightforward that $V^{p^\alpha}(RG) = V(R^{p^\alpha}G^{p^\alpha})$, the equality now follows without any difficulty.

(b) It suffices to show that $\cap_{\beta<\alpha}(V^{p^\beta}(RG)G) = [\cap_{\beta<\alpha}(V^{p^\beta}(RG))]G = V^{p^\alpha}(RG)G$ for each limit α. In fact, take $x \in \cap_{\beta<\alpha}(V^{p^\beta}(RG)G) = \cap_{\beta<\alpha}(V(R^{p^\beta} G^{p^\beta})G)$, hence $x = (r_1a_1 + \cdots + r_sa_s)g = (f_1b_1 + \cdots + f_sb_s)h = \cdots$, where $r_i \in R^{p^\beta}, a_i \in G^{p^\beta}, f_i \in R^{p^\gamma}, b_i \in G^{p^\gamma}, i \in [1,s], \beta < \gamma < \alpha; g,h \in G$. Now, we obtain that $r_i = f_i$ and $ga_i = hb_i$, whence $a_ia_j^{-1} = b_ib_j^{-1} \in G^{p^\gamma}$. Writing $x = ga_1(r_1 + \cdots + r_sa_sa_1^{-1})$, we observe that $x \in GV(R^{p^\gamma}G^{p^\gamma}) = GV^{p^\gamma}(RG)$.

Since the support is finite whereas the number of equalities is not because α is infinite being limit, we may assume that all relations are of the above type. That is why, $x \in (\cap_{\gamma<\alpha}V^{p^\gamma}(RG))G = V^{p^\alpha}(RG)G$ as required. QED

The next assertion appeared in ([1], Lemma 2). Nevertheless, for the reader's convenience and for the completeness of the exposition we shall provide a proof.

2 Lemma. *For each ordinal number α the following equality holds:*

$$G^{p^\alpha} \cap (V^{p^{\alpha+1}}(RG)V_p^{p^\alpha}(RG)) = G^{p^{\alpha+1}}G_p^{p^\alpha}.$$

PROOF. Since it is routinely checked that $V^{p^\alpha}(RG) = V(R^{p^\alpha}G^{p^\alpha})$, we may write $g = uv$, where $g \in G^{p^\alpha}, u \in V^{p^{\alpha+1}}(RG) = V(R^{p^{\alpha+1}}G^{p^{\alpha+1}})$ and $v \in V_p^{p^\alpha}(RG) = V_p(R^{p^\alpha}G^{p^\alpha})$. Therefore, $g(r_1a_1 + \cdots + r_sa_s) = f_1b_1 + \cdots + f_sb_s$ and $r_i = f_i$ with $ga_i = b_i$, for each $i \in [1, s]$, where $r_i \in R^{p^{\alpha+1}}, a_i \in G^{p^{\alpha+1}}$ and $f_i \in R^{p^\alpha}, b_i \in G^{p^\alpha}$. Since $f_1b_1 + \cdots + f_sb_s \in V_p(R^{p^\alpha}G^{p^\alpha})$, there is an index, say j, such that $b_j \in G_p^{p^\alpha}$. Thus $ga_j = b_j$ secures that $g = b_ja_j^{-1} \in G_p^{p^\alpha}G^{p^{\alpha+1}}$. QED

The next statement may be found in ([5], p. 157, Exercise 14) as well.

3 Lemma. *[Dlab] Let A be an Abelian multiplicative group with finite rank and $B \leq A$. Then B is neat in A (i.e., $B \cap pA = pB$) if and only if $r(A) = r(B) + r(A/B)$.*

The following corresponding claim is also useful.

4 Corollary. *[[5], p. 105, Exercise 4] If A is a multiplicative Abelian group and $B \leq A$ is a direct factor of A, then $r(A) = r(B) + r(A/B)$.*

3 Main Results

We are now in a position to prove the following

5 Theorem. *Suppose G is an Abelian group and R is a commutative unitary ring of prime characteristic p without zero divisors. Then, for each ordinal α, the following holds:*

(1) $$W_{\alpha,p}(V(RG)/G) = W_{\alpha,p}(V(RG)) - W_{\alpha,p}(G)$$

when $W_{\alpha,p}(V(RG)/G) < \aleph_0$. Thus

(1') $$W_{\alpha,p}(V(RG)/G) = \sum\nolimits_{d/|G_t/G_p|} a(d).W_{\alpha,p}(G/\coprod\nolimits_{l\neq p} G_l) - W_{\alpha,p}(G)$$

where $a(d) = |\{g \in G_t/G_p : order(g) = d\}|/(R(\zeta_d) : R)$ provided that R is a perfect field.

(2) $$W_{\alpha,p}(V(RG)/G) = W_{\alpha,p}(V(RG))$$

when $W_{\alpha,p}(V(RG)/G) \geq \aleph_0$. Thus

(2') $$W_{\alpha,p}(V(RG)/G) = |G_t/G_p|W_{\alpha,p}(G)$$

provided that R is perfect.

PROOF. By definition we write

$$W_{\alpha,p} = \text{rank}((V(RG)/G)^{p^\alpha}/((V(RG)/G)^{p^{\alpha+1}}(V(RG)/G)_p^{p^\alpha})).$$

But according to Lemma 1 we may write

$$(V(RG)/G)^{p^\alpha} = (V^{p^\alpha}(RG)G)/G,$$

$$(V(RG)/G)^{p^{\alpha+1}} = (V^{p^{\alpha+1}}(RG)G)/G$$

and $(V(RG)/G)_p^{p^\alpha} = (V_p^{p^\alpha}(RG)G)/G$. Therefore, using the modular law from [5], we obtain

$$\begin{aligned}(V(RG)/G)^{p^\alpha}&/((V(RG)/G)^{p^{\alpha+1}}(V(RG)/G)_p^{p^\alpha})\\ &= (V^{p^\alpha}(RG)G)/G/(V^{p^{\alpha+1}}(RG)V_p^{p^\alpha}(RG)G)/G\\ &\cong (V^{p^\alpha}(RG)G)/(V^{p^{\alpha+1}}(RG)V_p^{p^\alpha}(RG)G)\\ &\cong V^{p^\alpha}(RG)/(V^{p^\alpha}(RG)\cap[GV^{p^{\alpha+1}}(RG)V_p^{p^\alpha}(RG)])\\ &= V^{p^\alpha}(RG)/[V^{p^{\alpha+1}}(RG)V_p^{p^\alpha}(RG)(G\cap V^{p^\alpha}(RG))]\\ &= V^{p^\alpha}(RG)/(V^{p^{\alpha+1}}(RG)V_p^{p^\alpha}(RG)G^{p^\alpha})\\ &\cong V^{p^\alpha}(RG)/(V^{p^{\alpha+1}}(RG)V_p^{p^\alpha}(RG))\\ &\quad /(V^{p^{\alpha+1}}(RG)V_p^{p^\alpha}(RG)G^{p^\alpha})/(V^{p^{\alpha+1}}(RG)V_p^{p^\alpha}(RG)).\end{aligned}$$

But

$$\begin{aligned}(V^{p^{\alpha+1}}(RG)V_p^{p^\alpha}(RG)G^{p^\alpha})&/(V^{p^{\alpha+1}}(RG)V_p^{p^\alpha}(RG))\\ &\cong G^{p^\alpha}/[G^{p^\alpha}\cap(V^{p^{\alpha+1}}(RG)V_p^{p^\alpha}(RG))] = G^{p^\alpha}/(G^{p^{\alpha+1}}G_p^{p^\alpha})\end{aligned}$$

by using Lemma 2.

Furthermore, since $V^{p^\alpha}(RG)/(V^{p^{\alpha+1}}(RG)V_p^{p^\alpha}(RG)G^{p^\alpha})$ is an epimorphic image of the quotient group $V^{p^\alpha}(RG)/(V^{p^{\alpha+1}}(RG)V_p^{p^\alpha}(RG))$, we observe that $W_{\alpha,p}(V(RG)/G) \leq W_{\alpha,p}(V(RG))$.

Next, we shall show that $W_{\alpha,p}(V(RG)/G) \geq W_{\alpha,p}(G)$ whenever $G_t \neq G_p$. In fact, we consider the element $e = (1/|C|)\sum_{c\in C} r_c c \in RC \leq RG_q \subseteq RG^{p^{\alpha+t}}$, for any $t \in \mathbb{N}$, where $|C| < \aleph_0$; clearly $|C|$ inverts in R since $\mathrm{char}(R) = p$. It is not hard to verify that e is an idempotent, i.e., $e^2 = e$. Let $g, h \in G^{p^\alpha}$ with $gG^{p^{\alpha+1}}G_p^{p^\alpha} \neq hG^{p^{\alpha+1}}G_p^{p^\alpha}$. Construct the elements $x_g = eg + (1-e)$ and $x_h = eh + (1-e)$. Apparently, $x_g, x_h \in V(RG)$. We claim that $x_g G^{p^\alpha} V^{p^{\alpha+1}}(RG)V_p^{p^\alpha}$ $(RG) \neq x_h G^{p^\alpha} V^{p^{\alpha+1}}(RG)V_p^{p^\alpha}(RG)$. If not, $x_g x_h^{-1} = x_g x_{h^{-1}} = (eg + (1-e))(eh^{-1} + (1-e)) = egh^{-1} + (1-e) = ea + (1-e) \in G^{p^\alpha}V^{p^{\alpha+1}}(RG)V_p^{p^\alpha}(RG)$, where we denote $a = gh^{-1} \notin G^{p^{\alpha+1}}G_p^{p^\alpha}$. By our assumption there exists a natural k such that $(ea + (1-e))^{p^k} = ea^{p^k} + (1-e) \in G^{p^{\alpha+k}}V^{p^{\alpha+k+1}}(RG) = G^{p^{\alpha+k}}V(R^{p^{\alpha+k+1}}G^{p^{\alpha+k+1}})$. Writing $e = \sum_{c\in C} f_c c$, we obtain that $\sum_{c\in C} f_c c a^{p^k} +$

$1-\sum_{c\in C} f_c c \in G^{p^{\alpha+k}}V(R^{p^{\alpha+k+1}}G^{p^{\alpha+k+1}})$; $f_c \in R$. Furthermore, $\sum_{c\in C} f_c c a^{p^k} + 1-\sum_{c\in C} f_c c = d^{p^k}\sum_{v\in G^{p^\alpha}} f_v v^{p^{k+1}} = \sum_{v\in G^{p^\alpha}} f_v d^{p^k} v^{p^{k+1}}$, where $f_v \in R$ and $d \in G^{p^\alpha}$. Thus, $d^{p^k}v^{p^{k+1}} \in C \subseteq G^{p^{\alpha+k+1}}$ for some $v \in G^{p^\alpha}$, and hence $d^{p^k} \in G^{p^{\alpha+k+1}}$. Therefore, $ca^{p^k} \in G^{p^{\alpha+k+1}}$ and so $a^{p^k} \in G^{p^{\alpha+k+1}}$ because $c \in G^{p^{\alpha+k+1}}$. Now, $a^{p^k} = b^{p^{k+1}}$ with $b \in G^{p^\alpha}$, i.e., $(ab^{-p})^{p^k} = 1$ and $ab^{-p} \in G_p^{p^\alpha}$. Consequently, $a \in G^{p^{\alpha+1}}G_p^{p^\alpha}$ which is the desired contradiction.

Since $V^{p^\alpha}(RG)/(V^{p^{\alpha+1}}(RG)V_p^{p^\alpha}(RG))$ is a group bounded by p, all its subgroups are pure and so they are direct factors (see, for example, [5], Theorem 27.5). That is why, by what we have just shown above, we may write $V^{p^\alpha}(RG)/(V^{p^{\alpha+1}}(RG)V_p^{p^\alpha}(RG)) \cong (G^{p^\alpha}/(G^{p^{\alpha+1}}G_p^{p^\alpha})) \times ((V(RG)/G)^{p^\alpha}/ /((V(RG)/G)^{p^{\alpha+1}}(V(RG)/G)_p^{p^\alpha}))$. Consequently, employing Lemma 3 and Corollary 4 (see also [5], p. 157, Exercise 14 and p. 105, Exercise 4), we deduce that

$$\mathrm{rank}(V^{p^\alpha}(RG)/(V^{p^{\alpha+1}}(RG)V_p^{p^\alpha}(RG))) = \mathrm{rank}(G^{p^\alpha}/G^{p^{\alpha+1}}G_p^{p^\alpha}) +$$

$$\mathrm{rank}((V(RG)/G)^{p^\alpha}/((V(RG)/G)^{p^{\alpha+1}}(V(RG)/G)_p^{p^\alpha})),$$

i.e., $W_{\alpha,p}(V(RG)) = W_{\alpha,p}(G) + W_{\alpha,p}(V(RG)/G)$. By what we have already shown above when $G_t \neq G_p$, if $W_{\alpha,p}(V(RG)/G)$ is finite, then $W_{\alpha,p}(G)$ is finite, whence $W_{\alpha,p}(V(RG))$ is finite and thus $W_{\alpha,p}(V(RG)/G) = W_{\alpha,p}(V(RG)) - W_{\alpha,p}(G)$ whenever $G_t \neq G_p$. Note that when $G_t = G_p$ we know via [1] that $W_{\alpha,p}(V(RG)) = W_{\alpha,p}(G)$ and that $W_{\alpha,p}(V(RG)/G) = 0$. So, the same formula is true even in this case. Further, we apply [3] and [4] to complete point (1').

Let us now $W_{\alpha,p}(V(RG)/G)$ be infinite; thus $G_t \neq G_p$. By virtue of the inequality $W_{\alpha,p}(V(RG)/G) \geq W_{\alpha,p}(G)$ established above we obtain that $W_{\alpha,p}(V(RG)/G) = W_{\alpha,p}(V(RG))$. Finally, we can apply [2] and [3] to conclude that point (2') is valid. QED

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