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Nullity index of Bochner-Kähler manifolds

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Abstract. The nullity index of a Bochner-Kähler manifold is calculated. This also furnishes an alternative proof of the classification of Bochner-Kähler semi-symmetric manifolds.

Keywords: Bochner-Kähler manifolds, semi-symmetric spaces, nullity index.

MSC 2000 classification: 53C15, 53C25, 53C35

1 Introduction

Let (M^{2n}, g, J) be a Kähler manifold. M is said to be Bochner-Kähler if its *Bochner curvature tensor* vanishes (these spaces are also known as *Bochner-flat* manifolds). Bochner-Kähler manifolds are the Kähler analogous of conformally flat Riemannian manifolds, in the sense that a Kähler manifold has constant holomorphic sectional curvature if and only if it is Einstein and Bochner-Kähler. So, one might expect the theory of Bochner-Kähler manifolds to parallel the one of conformally flat Riemannian manifolds, but many interesting differences occur. Bochner-Kähler manifolds have non-trivial local geometry and they do not have a global isometric uniformization. The interest toward Bochner-Kähler manifolds was recently revived by the fundamental paper by Bryant [4], who provided an explicit local classification of Bochner-Kähler metrics and an in-depht study of their global geometry, also including a classification of the compact and complete Bochner-Kähler manifolds.

A locally symmetric Bochner-Kähler manifold is

- either a space of constant holomorphic sectional curvature, or
- locally isometric to the product of two spaces of constant holomorphic sectional curvature c and $-c$.

Indeed, a stronger classification result holds, since the spaces listed above are the only examples of Bochner-Kähler manifolds having constant scalar curvature [8].

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Semi-symmetric spaces are a well-known and natural generalization of locally symmetric spaces. A *semi-symmetric space* is a Riemannian manifold (M, g) such that its curvature tensor R satisfies the condition

$$R(X, Y) \cdot R = 0,$$

for all vector fields X, Y on M, where $R(X, Y)$ acts as a derivation on R [12]. Locally symmetric spaces are semi-symmetric, but in general the converse is not true. In fact, in any dimension greater than two there exist examples of semi-symmetric spaces which are not locally symmetric (we can refer to [13] for the first examples and to [3] for a survey). Nevertheless, semi-symmetry implies local symmetry in several cases and it is an interesting problem, given a class of Riemannian manifolds, to decide whether inside that class semi-symmetry implies local symmetry or not. Some examples can be found in [1], [2], [6]. In the framework of Kähler geometry, semi-symmetry and pseudo-symmetry were already investigated in [7], [9], [10], [11], [14].

In [5], the author classified conformally flat semi-symmetric spaces, calculating their nullity index at any point and using the local structure of a semi-symmetric space as described by Szabó [12]. It was shown in [5] that *real cones* are the only semi-symmetric conformally flat manifolds which are not locally symmetric. In this paper, we calculate the nullity index at any point of a Bochner-Kähler manifold. As an application, we get a straightforward proof of the following classification result, already obtained in [14]:

1 Theorem. *A semi-symmetric Bochner-Kähler manifold is locally symmetric.*

It is interesting to remark that differently from the conformally flat case, no proper semi-symmetric Bochner-Kähler manifolds occur, the difference being based exactly on the different possibilities about the nullity index.

The paper is organized in the following way. In Section 2, we recall some basic facts and results about Bochner-Kähler manifolds and semi-symmetric spaces. In Section 3 we calculate the nullity index of a Bochner-Kähler manifold and apply it to prove Theorem 1. We end the paper in Section 4 by relating Theorem 1 with some known results of Bochner-Kähler geometry.

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2 Preliminaries

Let (M^{2n}, g, J) be a Kähler manifold and R its curvature tensor, taken with the sign convention $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$, for all vector fields X, Y

on M, where ∇ denotes the Levi Civita connection of M. By ϱ, Q and τ we denote respectively the Ricci tensor, the Ricci operator associated to ϱ through g and the scalar curvature of M. The *Bochner curvature tensor* is defined as

$$\begin{aligned} B(X,Y)Z =& R(X,Y)Z - \frac{1}{2n+4}\,(g(Y,Z)QX - g(X,Z)QY \\ & + \varrho(Y,Z)X - \varrho(X,Z)Y + g(JY,Z)QJX \\ & - g(JX,Z)QJY + \varrho(JY,Z)JX - \varrho(JX,Z)JY \\ & -2\varrho(JX,Y)JZ - 2g(JX,Y)QJZ) \\ & + \frac{\tau}{(2n+2)(2n+4)}(g(Y,Z)X - g(X,Z)Y \\ & + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ). \end{aligned} \tag{1}$$

(M^{2n}, g, J) is said to be *Bochner-Kähler* if and only if $B = 0$.

We now recall some basic facts about semi-symmetric spaces. Let (M, g) be a smooth, connected Riemannian manifold. As already mentioned in the Introduction, (M, g) is said to be *semi-symmetric* if its curvature tensor R satisfies $R(X,Y) \cdot R = 0$, for all vector fields X, Y, where $R(X,Y)$ acts as a derivation on R. The local classification of semi-symmetric spaces makes use of the following

2 Definition. The nullity vector space of the curvature tensor at a point p of a Riemannian manifold (M, g) is given by

$$E_{0p} = \{X \in T_pM \mid R(X,Y)Z = 0 \text{ for all } Y, Z \in T_pM\}.$$

The index of nullity at p is the number $\nu(p) = \dim E_{0p}$. The index of conullity at p is the number $u(p) = \dim M - v(p)$.

By means of the indeces of nullity and conullity, Szabó [12] classified locally irreducible semi-symmetric spaces, proving that such a space must be locally isometric to one of the following spaces:

(a) a symmetric space when $\nu(p) = 0$ at each point p, or

(b) a real cone when $\nu(p) = 1$ and $u(p) = n - 1 > 2$ at each point p, or

(c) a Kählerian cone when $\nu(p) = 2$ and $u(p) = n - 2 > 2$ at each point p,

or

(d) a Riemannian manifold foliated by Euclidean leaves of codimension two when $\nu(p) = n - 2$ and $u(p) = 2$ at each point p of a dense open subset U of M.

The following result describes the local structure of a semi-symmetric space (M, g).

3 Theorem. *[[12]] There exists an open dense subset U of M such that around every point of U the manifold is locally isometric to a Riemannian product of type*

$$\mathbb{R}^k \times M_1 \times \cdots \times M_r, \tag{2}$$

where $k \geq 0$, $r \geq 0$ and each M_i is either a symmetric space, a two-dimensional manifold, a real cone, a Kählerian cone or a Riemannian space foliated by Euclidean leaves of codimension two.

The decomposition (2) may vary in the different connected components U_α of U, while it is constant on each U_α. For more details and references we refer to [3]. It is worthwhile to emphasize that Theorem 3 was proved by Szabó [12] to hold *for an arbitrary Riemannian manifold*, not just for a semi-symmetric one, the only required adjustment being in the definition of primitive holonomy (Remark at p.15 of [3]).

3 Nullity index of a Bochner-Kähler manifold and the classification result

Theorem 3 makes very interesting to determine the possible values of the index of nullity for Riemannian manifolds inside a certain class. We now prove the following

4 Theorem. *Let (M^{2n}, g, J) be a Bochner-Kähler manifold. Then, at each point p of M, the index of nullity is either $\nu(p)$ is either 0 or $2n$.*

Proof. Consider a point $p \in M$. If the curvature tensor R_p at p vanishes, then $\nu(p) = 2n$ and the conclusion follows. Hence, we now assume $R_p \neq 0$ and we prove that $\nu(p) = 0$. For this purpose, it is enough to show that if $\nu(p) \neq 0$, then $R_p = 0$, against our assumption.

Suppose then $\nu(p) \neq 0$ and let $e_0 \in E_{0p}$ be a unit vector. Then, we also have $e_{\bar{0}} = Je_0 \in E_{0p}$. In fact, since M is a Kähler manifold, we have

$$R_p(Je_0, Y)Z = R_p(Je_0, -J^2Y)Z = R_p(e_0, -JY)Z = 0,$$

for all $Y, Z \in T_pM$. Next, since $e_0, e_{\bar{0}} \in E_{0p}$, we have at once

$$\varrho_p(e_0, Y) = \varrho_p(e_{\bar{0}}, Y) = 0, \tag{3}$$

for all $Y \in T_pM$. If $\dim M = 2$, then (3) is sufficient to conclude that $R_p = 0$. So, in the sequel we assume $\dim M \geq 4$ and consider an orthonormal basis

$\{e_0, e_1, .., e_{n-1}, e_{\bar{0}}, e_{\bar{1}} = Je_1, .., e_{\overline{n-1}} = Je_{n-1}\}$ of T_pM. Since $B = 0$, from (1) it follows that the components R_{ABCD} of R_p with respect to this basis are completely determined in function of the Ricci components and the scalar curvature. Explicitly, by (1) we get

$$\begin{aligned} R_{ABCD} = &-\frac{1}{2n+4}(\delta_{BC}\varrho_{AD} - \delta_{BD}\varrho_{AC} + \delta_{AD}\varrho_{BC} - \delta_{AC}\varrho_{BD} + \delta_{\bar{B}C}\varrho_{\bar{A}D} \\ &-\delta_{\bar{B}D}\varrho_{\bar{A}C} + \delta_{\bar{A}D}\varrho_{\bar{B}C} - \delta_{\bar{A}C}\varrho_{\bar{B}D} - 2\delta_{\bar{C}D}\varrho_{\bar{A}B} - 2\delta_{\bar{A}B}\varrho_{\bar{C}D}) \\ &+ \frac{\tau_p}{(2n+2)(2n+4)}(\delta_{AD}\delta_{BC} - \delta_{AC}\delta_{BD} + \delta_{\bar{A}D}\delta_{\bar{B}C} - \delta_{\bar{A}C}\delta_{\bar{B}D} \\ &- 2\delta_{\bar{A}B}\delta_{\bar{C}D}), \end{aligned} \tag{4}$$

where indeces A, B, C, D run from 0 to $\overline{n-1}$. Now, choosing $A = 0$, $C = \bar{0}$ and $B, D \notin \{0, \bar{0}\}$, from (4) and $e_0 \in E_{0p}$ we get at once

$$0 = R_{0B\bar{0}D} = \frac{1}{2n+4}\varrho_{\bar{B}D} - \frac{\tau_p}{(2n+2)(2n+4)}\delta_{\bar{B}D},$$

that is,

$$\varrho_{\bar{B}D} = \frac{\tau_p}{2n+2}\delta_{\bar{B}D}, \tag{5}$$

for all indeces $B, D \notin \{0, \bar{0}\}$. Since B is an arbitrary index different from $0, \bar{0}$, so is $\bar{B}$. Hence, we can rewrite (5) as

$$\varrho_{BD} = \frac{\tau_p}{2n+2}\delta_{BD}, \tag{6}$$

for all $B, D \notin \{0, \bar{0}\}$. But then, because of (6) the scalar curvature τ_p at p satisfies

$$\tau_p = \sum_A \varrho_{AA} = \sum_{A \neq 0, \bar{0}} \varrho_{AA} = \frac{2n-2}{2n+2}\tau_p,$$

from which it follows at once that $\tau_p = 0$. Hence, (3) and (6) imply that all Ricci components at p vanish and by (4) we conclude $R_{ABCD} = 0$ for all A, B, C, D, that is, $R_p = 0$ QED

Applying Theorem 4, we can now proceed with the

Proof of Theorem 1. Let (M^{2n}, g, J) be a semi-symmetric Bochner-Kähler manifold. We first assume M is locally irreducible. Then, obviously M cannot be flat. According to Szabó's classification, M is locally isometric to one of the spaces (a)-(d) listed before Theorem 3. For these spaces, the index of nullity is the same at each point. On the other hand, because of Theorem 4, the nullity index at any point p of M is $\nu(p) = 0$. Therefore, M is locally isometric to an irreducible symmetric space and the conclusion follows.

We now assume that M is locally reducible. If M is flat, then in particular is locally symmetric. On the other hand, if M is not flat, then by Theorem 4 it follows that the nullity index at any point p of M is $\nu(p) = 0$. As the author already remarked in [5], if a Riemannian manifold (M, g) is locally isometric to a Riemannian product $M_1 \times M_2$, then, at any point $p = (p_1, p_2)$ of M, we have

$$\nu(p) = \nu(p_1) + \nu(p_2)$$

(and the result extends at once to the case when M is locally isometric a Riemannian product $M_1 \times .. \times M_k$). In our case, $\nu(p) = 0$ and so, $\nu(p_1) = \nu(p_2) = 0$. Therefore, each space in the local decomposition of M has (constant) nullity index equal to zero and so, is a locally symmetric space and this ends the proof. *QED*

Taking into account the classification of locally symmetric Bochner-Kähler spaces already mentioned in the Introduction, we obtain at once the following classification result:

5 Theorem. *A semi-symmetric Bochner-Kähler manifold is*

- *either a space of constant holomorphic sectional curvature, or*
- *locally isometric to the product of two spaces of constant holomorphic sectional curvature c and $-c$.*

4 Remarks and conclusions

Matsumoto and Tanno [8] proved that a Kähler manifold with *parallel* Bochner curvature tensor is either locally symmetric or Bochner-Kähler. Combining this result with Theorem 1, we have at once the following

6 Theorem. *A semi-symmetric Kähler manifold with parallel Bochner tensor is locally symmetric.*

A four-dimensional Kähler manifold (M^4, g, J) is Bochner-Kähler if and only if it is *self-dual*, that is, the self-dual part of its Weyl curvature tensor vanishes [10]. Then, in dimension four Theorem 5 can be rewritten in the following way:

7 Theorem. *A four-dimensional semi-symmetric self-dual Kähler manifold is either a space of constant holomorphic sectional curvature, or locally isometric to the product of two 2-dimensional spaces of constant holomorphic sectional curvature c and $-c$.*

Finally, we remark that Theorems 1, 5 and 6 above can be reformulated replacing semi-symmetry by pseudo-symmetry.

Pseudo-symmetry ($R \cdot R = fQ(R)$) implies semi-symmetry for Kähler manifolds of dimension $n \geq 6$ [7], while Z. Olszak [11] proved that there exist four-dimensional pseudo-symmetric Kähler manifolds which are not semi-symmetric.

However, also in dimension four, a pseudo-symmetric Bochner-Kähler manifold is semi-symmetric. In fact, any pseudo-symmetric space is Ricci-pseudo-symmetric, that is, satisfies $R \cdot \varrho = fQ(\varrho)$. Moreover, a Ricci-pseudo-symmetric Kähler manifold is Ricci-semi-symmetric (Theorem 1 of [10]), i.e., $R \cdot \varrho = 0$ holds. Finally, the Ricci-semi-symmetry, together with the vanishing of the Bochner curvature tensor, implies the semi-symmetry. Hence, Theorems 1, 5 and 6 yield at once the following

8 Theorem. *A pseudo-symmetric Bochner-Kähler manifold is locally symmetric.*

9 Theorem. *A pseudo-symmetric Bochner-Kähler manifold is*

- *either a space of constant holomorphic sectional curvature, or*
- *locally isometric to the product of two spaces of constant holomorphic sectional curvature c and $-c$.*

10 Theorem. *A pseudo-symmetric Kähler manifold with parallel Bochner tensor is locally symmetric.*

References

[1] E. Boeckx: *Einstein-like semi-symmetric spaces*, Arch. Math. (Brno) **29** (1993), 235–240.

[2] E. Boeckx and G. Calvaruso: *When is the unit tangent sphere bundle semi-symmetric?*, Tôhoku Math. J. (2) **56** (2004), 357–366.

[3] E. Boeckx, O. Kowalski and L. Vanhecke: Riemannian manifolds of conullity two, World Scientific, (1996)

[4] R.L. Bryant: *Bochner-flat Kähler metrics*, J. Amer. Math. Soc. **14** (2001), 623–715.

[5] G. Calvaruso: *Conformally flat semi-symmetric spaces*, Arch. Math. (Brno) **41** (2005), 27–36.

[6] G. Calvaruso and L. Vanhecke: *Semi-symmetric ball-homogeneous spaces and a volume conjecture*, Bull. Austral. Math. Soc. **57** (1998), 109–115.

[7] J. Deprez, R. Deszcz and L. Verstraelen: *Pseudo-symmetry curvature conditions on hypersurfaces of Euclidean spaces and on Kahlerian manifolds*, Ann. Fac. Sci. Toulouse (2) **9** (1988), 183–192.

[8] M. Matsumoto and S. Tanno: *Kählerian spaces with parallel or vanishing Bochner curvature tensor*, Tensor N.S. **27** (1973), 291–294.

[9] Y. Ogawa: *A condition for a compact Kaehlerian space to be locally symmetric*, Nat. Sci. Rep. Ochanomizu Univ. **28** (1977), 21–23.

[10] Z. Olszak: *Bochner flat Kählerian manifolds with a certain condition on the Ricci tensor*, Simon Stevin **63** (1989), 295–303.

[11] Z. Olszak: *On the existence of pseudosymmetric Kähler manifolds*, Colloquium Math. (2) **95** (2003), 185–189.

[12] Z. I. Szabó: *Structure theorems on Riemannian manifolds satisfying* $R(X,Y)\cdot R = 0$*, I, the local version*, J. Diff. Geom. **17** (1982), 531–582.

[13] H. Takagi: *An example of Riemannian manifold satisfying* $R(X,Y)\cdot R$ *but not* $\nabla R = 0$, Tôhoku Math. J. **24** (1972), 105–108.

[14] H. Takagi, Y. Watanabe: *Kaehlerian manifolds with vanishing Bochner curvature tensor satisfying* $R(X,Y)\cdot R_1 = 0$, Hokaido Math. J. **1** (1974), 129–132.