Groups with Large Centralizer Subgroups

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Abstract. This article describes the structure of locally graded groups in which every (infinite) proper self-centralizing subgroup is abelian.

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1 Introduction

We shall say that a subgroup $X$ of a group $G$ is self-centralizing (in $G$) if $X$ contains its centralizer $C_G(X)$. Obvious examples of self-centralizing subgroups are provided by maximal abelian subgroups of arbitrary groups and by the Fitting subgroup of any soluble group. It follows immediately from Zorn’s Lemma that if a group $G$ does not contain proper self-centralizing subgroups, then $G$ is abelian. The aim of this paper is to study groups for which the set of self-centralizing subgroups is small in some sense.

In Section 2 a full description will be given of locally graded groups in which every proper self-centralizing subgroup is abelian; here a group $G$ is said to be locally graded if each finitely generated non-trivial subgroup of $G$ contains a proper subgroup of finite index. We work within the universe of locally graded groups in order to avoid Tarski groups (i.e. infinite simple groups whose proper non-trivial subgroups have prime order) and other similar pathological examples. The last section is devoted to the study of locally graded groups whose infinite proper self-centralizing subgroups are abelian.

Recall that a group is metahamiltonian if all its non-abelian subgroups
are normal. Groups with such property were introduced and investigated by G.M. Romalis and N.F. Sesekin ([6], [7], [8]), who proved in particular that (generalized) soluble metahamiltonian groups have finite commutator subgroup. Metahamiltonian groups are naturally involved in the study of groups with few self-centralizing subgroups; in fact, it is easy to show that groups whose proper self-centralizing subgroups are abelian must be metahamiltonian. Note also that in Section 3 a result of S.N. Černikov [1] concerning locally graded groups whose infinite non-abelian subgroups are normal will be used.

Most of our notation is standard and can be found in [5].

2 Centralizers of non-abelian subgroups

Let \( \mathcal{Q} \) be the class consisting of all groups whose proper self-centralizing subgroups are abelian (i.e. a group \( G \) has the property \( \mathcal{Q} \) if and only if \( C_G(X) \) is not contained in \( X \) for each proper non-abelian subgroup \( X \) of \( G \)). Of course, \( \mathcal{Q} \) contains all abelian groups, and also Tarski groups have the property \( \mathcal{Q} \). The main result of this section will characterize locally graded \( \mathcal{Q} \)-groups.

We begin with the following obvious property.

1 Lemma. Let \( G \) be a group and let \( X \) be a subgroup of \( G \). Then the normalizer \( N_G(X) \) is a self-centralizing subgroup of \( G \).

Proof. Clearly,

\[
C_G(N_G(X)) \leq C_G(X) \leq N_G(X),
\]

and hence the subgroup \( N_G(X) \) is self-centralizing in \( G \). \( \square \)

If \( \mathcal{X} \) is any class of groups, we will denote as usual by \( \mathcal{L}\mathcal{X} \) the class consisting of all groups with a local system by \( \mathcal{X} \)-subgroups (i.e. \( G \in \mathcal{L}\mathcal{X} \) if and only if every finite subset of \( G \) is contained in some \( \mathcal{X} \)-subgroup of \( G \)); the group class \( \mathcal{X} \) is local if \( \mathcal{L}\mathcal{X}=\mathcal{X} \). We shall say that a local group class \( \mathcal{X} \) is centrally stable if it satisfies the following conditions:

- \( \mathcal{X} \) is closed with respect to normal subgroups (i.e. every normal subgroup of an arbitrary \( \mathcal{X} \)-group belongs to \( \mathcal{X} \));

- if \( G \) is any group and \( X \) is an \( \mathcal{X} \)-subgroup of \( G \), then \( \langle g, X \rangle \in \mathcal{X} \) for each element \( g \in C_G(X) \).

Of course, for each non-negative integer \( c \) the class \( \mathfrak{N}_c \) of nilpotent groups with class at most \( c \) is centrally stable; in particular, the class \( \mathfrak{A} \) of abelian groups has such property.
2 Lemma. Let $\mathfrak{X}$ be a class of groups which is closed with respect to normal subgroups, and let $G$ be a group whose proper self-centralizing subgroups belong to $\mathfrak{X}$. Then every non-normal subgroup of $G$ is an $\mathfrak{X}$-group. Moreover, if the group class $\mathfrak{X}$ is centrally stable, then $G$ contains a maximal subgroup which is an $\mathfrak{X}$-group.

Proof. Let $X$ be any subgroup of $G$ which is not in $\mathfrak{X}$. As $\mathfrak{X}$ is closed with respect to normal subgroups, the normalizer $N_G(X)$ cannot belong to $\mathfrak{X}$; moreover, $N_G(X)$ is self-centralizing in $G$, and so it follows that $N_G(X) = G$ and $X$ is normal in $G$. Suppose now that $\mathfrak{X}$ is also centrally stable, so that in particular by Zorn’s Lemma $G$ contains a maximal $\mathfrak{X}$-subgroup $M$ and $C_G(M) \leq M$. Let $H$ be any subgroup of $G$ which properly contains $M$. Then

$$C_G(H) \leq C_G(M) \leq M < H,$$

and hence $H$ is a self-centralizing subgroup of $G$ which is not in $\mathfrak{X}$, so that $H = G$ and $M$ is a maximal subgroup of $G$.

The above lemma provides information on the structure of groups whose proper self-centralizing subgroups belong to a given group class $\mathfrak{X}$, for several different choices of $\mathfrak{X}$. In particular, for $\mathfrak{X} = \mathfrak{A}$ we have the following consequence of Lemma 2.

3 Corollary. Let $G$ be a $\mathfrak{Q}$-group. Then $G$ is metahamiltonian and contains a maximal subgroup which is abelian.

Since it is well known that abelian-by-finite groups with finite commutator subgroup are central-by-finite, we also obtain the following result.

4 Corollary. Let $G$ be a locally graded $\mathfrak{Q}$-group. Then the factor group $G/Z(G)$ is finite.

Proof. The group $G$ is metahamiltonian by Corollary 3, so that in particular its commutator subgroup $G'$ is finite. Moreover, $G$ contains a maximal subgroup $M$ which is abelian, and of course the index $[G : M]$ is finite. Thus $G$ is abelian-by-finite and hence $G/Z(G)$ is finite.

5 Lemma. A locally graded group $G$ belongs to the class $\mathfrak{Q}$ if and only if $G = XZ(G)$ for each non-abelian subgroup $X$ of $G$.

Proof. Suppose first that $G$ is a $\mathfrak{Q}$-group, and assume for a contradiction that $G$ contains a non-abelian subgroup $X$ such that $XZ(G) \neq G$. As $G/Z(G)$ is finite by Corollary 4, there exists a maximal subgroup $M$ of $G$ containing $XZ(G)$. By hypothesis, $M$ is not self-centralizing and so we may consider an element $g$ of $C_G(M) \setminus M$; then $G = \langle g, M \rangle$ and hence $g$ belongs to $Z(G)$. This contradiction proves that $G = XZ(G)$ for every non-abelian subgroup $X$ of $G$.

Conversely, suppose that $G$ satisfies the condition of the statement, and let $X$ be any proper non-abelian subgroup of $G$. Then $G = XZ(G)$, so that the
centre $Z(G)$ is not contained in $X$ and in particular $X$ is not self-centralizing. Therefore $G$ is a $Q$-group.

6 Corollary. A locally graded group $G$ belongs to the class $Q$ if and only if all proper subgroups of $G$ containing $Z(G)$ are abelian.

It is known that a locally graded group $G$ is metahamiltonian if and only every non-abelian subgroup of $G$ contains the commutator subgroup $G'$ of $G$ (see [3]). Thus the above corollary provides further evidence of the fact that the centre and the commutator subgroup of a group have dual behaviours. In fact, since any $Q$-group is metahamiltonian, we obtain the following information.

7 Corollary. Let $G$ be a locally graded group. If all proper subgroups of $G$ containing the centre $Z(G)$ are abelian, then every non-abelian subgroup of $G$ contains the commutator subgroup $G'$ and in particular all proper subgroups of $G'$ are abelian.

We can now describe locally graded $Q$-groups, starting with the nilpotent case.

8 Theorem. Let $G$ be a nilpotent group. Then $G$ belongs to the class $Q$ if and only if it is abelian or the factor group $G/Z(G)$ has order $p^2$ for some prime number $p$.

Proof. Suppose first that $G$ is a non-abelian $Q$-group. By Corollary 4 we have that $G/Z(G)$ is a finite (non-cyclic) group, and so it contains two distinct maximal subgroups $M_1/Z(G)$ and $M_2/Z(G)$. Moreover, it follows from Lemma 5 that $M_1$ and $M_2$ are abelian, so that $M_1 ∩ M_2 = Z(G)$ and $G/Z(G)$ has order $p^2$ for some prime number $p$.

Conversely, assume that $G/Z(G)$ has order $p^2$ for some prime number $p$. If $X$ is any non-abelian subgroup of $G$, the group $XZ(G)/Z(G)$ cannot be cyclic and hence $XZ(G) = G$. Therefore $G$ belongs to $Q$ by Lemma 5.

9 Theorem. Let $G$ be a locally graded non-nilpotent group. Then $G$ belongs to the class $Q$ if and only if $G = A ⋉ P$, where $P$ is a finite abelian group of prime exponent $p = A$ and $A = \langle a, Z(G) \rangle$ for some element $a$ acting irreducibly on $P$, and $\langle a \rangle ∩ Z(G) = \langle a^q \rangle$ for some prime number $q > 1$ which is prime to $p$; moreover, $a^k$ acts irreducibly on $P$ for each positive integer $k < q$. Assume for a contradiction that $q$ is not a prime number, so that there exists a positive divisor $r$ of $q$ such that $\langle a^r \rangle < \langle a^r \rangle < \langle a \rangle$. Thus the subgroup $\langle a^r, P \rangle$ is not abelian and hence
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\[ G = \langle a^r, P \rangle Z(G), \text{ a contradiction since } \langle a^r, Z(G) \rangle \text{ is properly contained in } A. \] Therefore \( q \) is a prime number.

Conversely, suppose that \( G = A \rtimes P \) has the structure described in the statement, so that in particular \( G \) is metahamiltonian (see [3], Theorem 2). Let \( X \) be any proper non-abelian subgroup of \( G \). Since \( P \) is a minimal normal subgroup of \( G \), it follows that \( P = G' \) is contained in \( X \) (see [3], Theorem 3). Assume for a contradiction that \( X \) contains also \( Z(G) \); then \( PZ(G) \leq X \) and \( |G : PZ(G)| = q \), so that \( X = PZ(G) \) is abelian. This contradiction shows that \( Z(G) \) is not contained in \( X \), so that in particular \( X \) is not self-centralizing. Therefore \( G \) belongs to the class \( Q \).

Finally, we note that Corollary 4 can be extended to the case of groups with finitely many self-centralizing non-abelian subgroups. Since every self-centralizing subgroup contains the centre, it is clear that if \( G \) is a central-by-finite group, then the set of all self-centralizing subgroups of \( G \) is finite.

10 Theorem. Let \( G \) be a locally graded group with finitely many self-centralizing non-abelian subgroups. Then the factor group \( G/Z(G) \) is finite.

Proof. By Lemma 1 the group \( G \) has finitely many normalizers of non-abelian subgroups, so that its commutator subgroup \( G' \) is finite (see [2]). Let \( A \) be any maximal abelian subgroup of \( G \); then \( C_G(A) = A \) and hence \( A \) is finite. In particular, \( G \) is periodic. Moreover, it follows from the Hall-Kulatilaka-Kargapolov theorem (see [5] Part 1, Theorem 3.43) that \( G \) is not locally finite, so that it contains an infinite finitely generated subgroup \( E \). If \( X \) is any subgroup of finite index of \( E \), the normalizer \( N_G(X) \) is an infinite self-centralizing subgroup, so

3 Centralizers of infinite non-abelian subgroups

The consideration of Tarski groups shows that the condition that the group is locally graded is necessary in our next result.

11 Lemma. Let \( G \) be an infinite locally graded group whose proper self-centralizing subgroups are finite. Then \( G \) is abelian.

Proof. Assume for a contradiction that \( G \) is not abelian. Let \( A \) be any maximal abelian subgroup of \( G \); then \( C_G(A) = A \) and hence \( A \) is finite. In particular, \( G \) is periodic. Moreover, it follows from the Hall-Kulatilaka-Kargapolov theorem (see [5] Part 1, Theorem 3.43) that \( G \) is not locally finite, so that it contains an infinite finitely generated subgroup \( E \). If \( X \) is any subgroup of finite index of \( E \), the normalizer \( N_G(X) \) is an infinite self-centralizing subgroup, so
that $N_G(X) = G$ and $X$ is normal in $G$; in particular, all subgroups of finite index of $E$ are normal. Let $J$ be the finite residual of $E$; then $E/J$ is nilpotent and so finite. Since $G$ is locally graded, it follows that $E$ itself is finite. This contradiction proves the statement.

Let $\mathfrak{Q}_\infty$ be the class consisting of all groups in which all infinite proper self-centralizing subgroups are abelian. Applying the argument used in the proof of the first part of Lemma 2 to the class $\mathfrak{A}_\infty$ of all infinite abelian groups, the following result can be proved.

**12 Lemma.** Let $G$ be an infinite $\mathfrak{Q}_\infty$-group. Then all infinite non-abelian subgroups of $G$ are normal.

Groups in which every infinite non-abelian subgroup is normal have been described by S.N. Černikov [1]. We state here his main result as a lemma; it will be used in order to describe (generalized soluble) $\mathfrak{Q}_\infty$-groups.

**13 Lemma.** Let $G$ be a locally graded group in which every infinite non-abelian subgroup is normal. Then either the commutator subgroup $G'$ of $G$ is finite or $G$ is a Černikov group whose divisible part contains no infinite proper $G$-invariant subgroups.

Our next theorem deals with the case of finite-by-abelian groups, and in particular it applies to metahamiltonian $\mathfrak{Q}_\infty$-groups which are locally graded.

**14 Theorem.** Let $G$ be an infinite $\mathfrak{Q}_\infty$-group with finite commutator subgroup. Then $G$ belongs to $\mathfrak{Q}$.

**Proof.** Assume for a contradiction that $G$ is not a $\mathfrak{Q}$-group, so that it contains a proper non-abelian subgroup $X$ such that $C_G(X) \leq X$. Thus $X$ is finite, so that in particular the centre $Z(G)$ of $G$ is finite and hence $Z_2(G)$ has finite exponent (see [5] Part 1, Theorem 2.23). On the other hand, as $G'$ is finite, the index $[G : Z_2(G)]$ is likewise finite (see [5] Part 1, p.113). It follows that $G$ has finite exponent and so the infinite abelian group $G/G'$ contains a subgroup $H/G'$ of finite index such that $|G/H| > |X|$. Then $XH$ is an infinite proper non-abelian subgroup and

$$C_G(XH) \leq C_G(X) \leq X < XH,$$

and this contradiction proves the theorem.

We can now complete the description of locally graded $\mathfrak{Q}_\infty$-groups.

**15 Theorem.** Let $G$ be a locally graded group with infinite commutator subgroup. Then $G$ has the property $\mathfrak{Q}_\infty$ if and only if $G$ is a Černikov group whose divisible part $J$ contains no infinite proper $G$-invariant subgroups and the factor group $G/JZ(G)$ has prime order.
Proof. Suppose first that \( G \) is a \( \mathcal{Q}_\infty \)-group. As \( G' \) is infinite, it follows from Lemma 12 and Lemma 13 that \( G \) is a Černikov group and its divisible part \( J \) has no infinite proper \( G \)-invariant subgroups. Moreover, \( J \) cannot be contained in \( Z(G) \), and hence the centralizer \( C_G(J) \) is an infinite proper subgroup of \( G \). On the other hand, 
\[
C_G(C_G(J)) \leq C_G(J)
\]
and so \( C_G(J) \) must be abelian. Let \( X \) be any subgroup of \( G \) properly containing \( C_G(J) \). Then \( X \) is not abelian and 
\[
C_G(X) \leq C_G(J) \leq X,
\]
so that \( X = G \). It follows that \( C_G(J) \) is a maximal subgroup of \( G \) and the index \( [G : C_G(J)] \) is a prime number. Let \( x \) be an element of \( G \) such that \( G = \langle x, C_G(J) \rangle \) and consider the infinite non-abelian subgroup \( \langle x, JZ(G) \rangle \). Then 
\[
C_G(\langle x, JZ(G) \rangle) \leq C_G(J) \cap C_G(x) = Z(G) \leq \langle x, JZ(G) \rangle,
\]
and hence \( G = \langle x, JZ(G) \rangle \) by the property \( \mathcal{Q}_\infty \). Therefore 
\[
C_G(J) = \langle x, JZ(G) \rangle \cap C_G(J) = JZ(G)(\langle x \rangle \cap C_G(J)) = JZ(G),
\]
so that the group \( G/JZ(G) \) has prime order.

Assume conversely that \( G \) is a Černikov group satisfying the conditions of the statement, and let \( X \) be any infinite non-abelian subgroup of \( G \) such that \( C_G(X) \leq X \). Then \( Z(G) \) lies in \( X \) and \( X \) is not contained in \( JZ(G) \), so that 
\[
G = JZ(G)X = JX.
\]
As \( X \) is infinite, its divisible part \( Y \) is likewise infinite and of course \( Y \) is a normal subgroup of \( G \). It follows that \( J = Y \leq X \) and hence \( X = G \). Therefore all infinite proper self-centralizing subgroups of \( G \) are abelian and \( G \) has the property \( \mathcal{Q}_\infty \).

Our last result provides further information on the structure of locally graded \( \mathcal{Q}_\infty \)-groups.

16 Corollary. Let \( G \) be a locally graded \( \mathcal{Q}_\infty \)-group with infinite commutator subgroup. Then \( G \) is a Černikov group and \( G' \) is the divisible part of \( G \).

Proof. By Theorem 15, \( G \) is a Černikov group and its divisible part \( J \) has no infinite proper \( G \)-invariant subgroups. Thus every infinite normal subgroup of \( G \) contains \( J \) and in particular \( J \leq G' \). On the other hand, Theorem 15 also yields that the factor group \( G/JZ(G) \) has prime order, so that \( G/J \) is central-by-cyclic and hence abelian. Therefore \( G' = J \).
References


