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Translation-invariant generalized topologies induced by probabilistic norms

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Abstract. One considers probabilistic normed spaces as defined by Alsina, Sklar, and Schweizer, but with non necessarily continuous triangle functions. Such spaces are endowed with a generalized topology that is Fréchet-separated, translation-invariant and countably generated by radial and circled 0-neighborhoods. Conversely, we show that such generalized topologies are probabilistically normable.

Keywords: Probabilistic norms, probabilistic metrics, triangle functions, generalized topologies, generalized uniformities.

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1 Introduction

Probabilistic normed spaces (briefly, PN spaces) were first defined by Šerstnev in the early sixties [see [13]], thus originating a fruitful theory that extended the theory of ordinary normed spaces. Thirty years later, Alsina, Schweizer, and Sklar gave in [1] a quite general definition of PN space, based on the definition of Menger's betweenness in probabilistic metric spaces; [see [14],p. 232].

We here consider PN spaces in which the involved triangle functions are non necessarily continuous. With regards to a generalized topology in the sense of Fréchet and for probabilistic metric spaces, the problem was treated by Höhle in [6] where he showed that all generalized topologies which are Fréchet-separated and first-numerable are induced by certain probabilistic metrics. The main result of this paper is a similar result for probabilistic norms, where the t-norm has certain restriction:

1 Theorem. *Let T be a t-norm such that $\sup_{0\leq x<1} T(x,x) < 1$. Suppose that $T(x,y) \leq xy$, whenever $x, y < \delta$, for some $\delta > 0$. A Fréchet-separated, translation-invariant, generalized topology $(\mathcal{U}_p)_{p\in S}$ on a real vector space S is* *derivable from a Menger PN space* $(S, \nu, \tau_T, \tau_{T^*})$*, if and only* $\mathcal{U}_\theta$ *admits a countable base of radial and circled subsets, where* θ *is the origin of* S*.*

In fact, this result also holds if one assumes T to be Archimedean near the origin, i.e. there is a $\delta > 0$ such that $0 < T(x, x) < x$, for all $0 < x < \delta$ [see Remark [8], after Theorem [1]].

We think that a similar result could be interesting for fuzzy normed spaces in the sense of Felbin [3], but allowing non-continuity of the t-norms, and t-conorms involved in the fuzzy structure.

In [10] the authors use this generalized topology to define bounded subsets in PN spaces (with non necessarily continuous triangle functions) and study its relationship with $\mathcal{D}$-bounded subsets (a concept which is defined in probabilistic terms).

2 PM and PN spaces

Recall from [1] and [14] some definitions on probabilistic metric and probabilistic normed spaces.

As usual, Δ^+ denotes the set of distance distribution functions, i.e. distribution functions with $F(0) = 0$, endowed with the metric topology given by the modified Lévy-Sybley metric d_L [see 4.2 in [14]]. Given a real number a, ε_a denotes the distribution function defined as $\varepsilon_a(x) = 0$ if $x \leq a$ and $\varepsilon_a(x) = 1$ if $x > a$. Hence, the set of non-negative real numbers $\mathbb{R}^+$ can be viewed as a subspace of Δ^+. A triangle function τ is a map from $\Delta^+ \times \Delta^+ \to \Delta^+$ which is commutative, associative, nondecreasing in each variable and has ε_0 as the identity. Such functions give rise to all possible extensions of the sum of real numbers, so that (M3) below corresponds to the triangle inequality.

A *probabilistic metric space* (briefly, a PM space) is a triple (S, F, τ) where S is a non-empty set, F is a map from $S \times S \to \Delta^+$, called the probabilistic metric, and τ is a triangle function, such that:

(M1) $F_{p,q} = \varepsilon_0$ if and only if $p = q$.

(M2) $F_{p,q} = F_{q,p}$.

(M3) $F_{p,q} \geq \tau(F_{p,r}, F_{r,q})$.

When only (M1) and (M2) are required, it the pair (S, F) is said to be a *probabilistic semi-metric space* (briefly, PSM space).

A *PN space* is a quadruple (S, ν, τ, τ^*) in which S is a vector space over $\mathbb{R}$, the *probabilistic norm* ν is a map $S \to \Delta^+$, τ and τ^* are triangle functions[1]

[1] In the definition of PN space given in [1] the triangle functions are assumed to be continuous

such that the following conditions are satisfied for all p, q in S:

(N1) $\nu_p = \varepsilon_0$ if and only if $p = \theta$, where θ is the origin of S.

(N2) $\nu_{-p} = \nu_p$.

(N3) $\nu_{p+q} \geq \tau(\nu_p, \nu_q)$.

(N4) $\nu_p \leq \tau^*(\nu_{\lambda p}, \nu_{(1-\lambda)p})$ for every $\lambda \in [0, 1]$.

Observe that every PN space (S, ν, τ, τ^*) is a PM space, where $F_{p,q} := \nu_{p-q}$.

Recall that a *t-norm* is a binary operation on $[0, 1]$ that is commutative, associative, nondecreasing in each variable, and has 1 as identity. Dually, a t-conorm is a binary operation on $[0, 1]$ that is commutative, associative, nondecreasing in each variable, and has 0 as identity. If T is a t-norm, its associated t-conorm T^* is defined by $T^*(x, y) := 1 - T(1 - x, 1 - y)$. Given a t-norm T one defines the functions τ_T and τ_{T^*} by

$$\tau_T(F, G)(x) := \sup\{T(F(s), G(t)) : s + t = x\},$$

and

$$\tau_{T^*}(F, G)(x) := \inf\{T^*(F(s), G(t)) : s + t = x\}.$$

Recall that if T is left-continuous then τ_T is a triangle function [14, p. 100], although this is not necessary; For example, if Z denotes the weakest t-norm, defined as $Z(x, 1) = Z(1, x) = x$ and $Z(x, y) = 0$ elsewhere, then τ_Z is a triangle function which is not continuous.

A *Šerstnev PN space* is a PN space (V, ν, τ, τ^*) where ν satisfies the following Šerstnev condition:

$$(\check{\text{S}}) \quad \nu_{\lambda p}(x) = \nu_p\left(\frac{x}{|\lambda|}\right), \text{ for all } x \in \mathbb{R}^+, p \in V \text{ and } \lambda \in \mathbb{R} \setminus \{0\}.$$

which clearly implies (N2) and also [see [1]](N4) in the strengthened form

$$\nu_p = \tau_M(\nu_{\lambda p}, \nu_{(1-\lambda)p}), \tag{1}$$

for all $p \in V$ and $\lambda \in [0, 1]$ [see [1, Theorem 1]], where M is the t-norm defined as $M(x, y) = \min\{x, y\}$.

Let T be a t-norm. A *Menger PM space under* T is a PM space of the form (S, F, τ_T). Analogously, a *Menger PN space under* T is a PN space of the form $(S, \nu, \tau_T, \tau_{T^*})$. Note that every metric space (S, d) is a Menger space (S, F, τ_M) where $F_{p,q} = \varepsilon_{d(p,q)}$. Analogously, every normed space $(S, \|\cdot\|)$ is a Menger and Šerstnev PN space (S, ν, τ_M, τ_M) where $\nu_p = \varepsilon_{\|p\|}$.

3 Probabilistic metrization of generalized topologies

In [6] Höhle solved a problem posed by Thorp about the probabilistic metrization of generalized topologies. We recall some definitions and results that we shall use in the next section.

Let S be a (non-empty) set. A *generalized topology (of type V_D)* on S is a family of subsets $(\mathcal{U}_p)_{p\in S}$, where $\mathcal{U}_p$ is a filter on S such that $p \in U$ for all $U \in \mathcal{U}_p$ [see e.g. [14, p. 38], [2, p. 22]]. Elements of $\mathcal{U}_p$ are called *neighborhoods* at p. Such a generalized topology is called *Fréchet-separated* if $\bigcap_{U\in\mathcal{U}_p} U = \{p\}$.

A *generalized uniformity* $\mathcal{U}$ on S is a filter on $S \times S$ such that every $V \in \mathcal{U}$ contains the diagonal $\{(p, p) : p \in S\}$, and for all $V \in \mathcal{U}$, we have that $V^{-1} := \{(q, p) : (p, q) \in V\}$ also belongs to $\mathcal{U}$. Elements of $\mathcal{U}$ are called *vicinities* (or "entourages"). Every generalized uniformity $\mathcal{U}$ induces a generalized topology as follows: for $p \in S$,

$$\mathcal{U}_p := \{U \subseteq S \mid \exists V \in \mathcal{U} : U \supseteq \{q \in S \mid (p, q) \in V\}\}. \tag{2}$$

A uniformity $\mathcal{U}$ is called *Hausdorff-separated* if the intersection of all vicinities is the diagonal on S. Theorem 1 in [6] claims:

2 Theorem. *[Höhle] Every Fréchet-separated generalized topology $(\mathcal{U}_p)_{p\in S}$ on a given set S is derivable from a Hausdorff-separated generalized uniformity $\mathcal{U}$ in the sense of (2).* QED

Let (S, F) be a PSM space. Consider the system $(\mathcal{N}_p)_{p\in S}$, where $\mathcal{N}_p = \{N_p(t) : t > 0\}$ and

$$N_p(t) := \{q \in S : F_{p,q}(t) > 1 - t\}.$$

This is called the *strong neighborhood system.* If we define $\delta(p, q) := d_L(F_{p,q}, \varepsilon_0)$, then δ is a semi-metric on S (i.e. it may not satisfy the triangle inequality of the standard metric axioms), and $N_p(t) = \{q : d_L(F_{p,q}, \varepsilon_0) < t\}$. Clearly $p \in N$ for every $N \in \mathcal{N}_p$, and the intersection of two strong neighborhoods at p is a strong neighborhood at p. Furthermore, $\mathcal{N}_p$ admits a countable filter base given by $\{N_p(1/n) : n \in \mathbb{N}\}$, hence the strong neighborhood system is first-countable. The above explanation yields the following fact [see more details in [14], p. 191]:

3 Theorem. *Let (S, F) be a PSM space, then the strong neighborhood system defines a generalized topology of type V_D which is Fréchet-separated and first-countable.* QED

This generalized topology is called the *strong generalized topology* of the PSM space (S, F).

The main result in [6] is the following.

4 Theorem. *[Höhle] Let T be a t-norm such that $\sup_{0\le x<1} T(x,x) < 1$. A Fréchet–separated generalized topology $(\mathcal{U}_p)_{p\in S}$ on a set S is derivable from a Menger PM space (S,F,T) if and only if there exists a Hausdorff–separated, generalized uniform structure $\mathcal{U}$ having a countable filter base, such that $\mathcal{U}$ is compatible with $(\mathcal{U}_p)_{p\in S}$.* QED

5 Remark. If (S,F,τ) is a PM space with τ continuous, then the associated generalized topology is in fact a topology. This topology is called *the strong topology*. Because of (M1) this topology is Hausdorff. Since it is first-countable and uniformable, it is metrizable [see [14, Theorem 12.1.6]].

Conversely, if $\sup_{0\le x<1} T(x,x) = 1$, then a Fréchet-separated, uniformable topology is derivable from a Menger space (S,F,T) if and only if there exists a Hausdorff uniformity U on S having a countable filter base [6].

4 Translation-invariant generalized topologies

Assume now that S is a vector space over $\mathbb{R}$. A generalized topology $(\mathcal{U}_p)_{p\in S}$ on S is said to be *translation-invariant* if for all $U \in \mathcal{U}_p$ and $q \in S$, we have $q + U \in \mathcal{U}_{p+q}$. Consequently, a translation-invariant generalized topology is uniquely determined by the neighborhood system $\mathcal{U}_\theta$ at the origin θ of S. In this case, the generalized uniformity from which one can derive the generalized topology is:

$$\mathcal{U} := \{V \subseteq S\times S \mid \exists U \in \mathcal{U}_\theta : V \supseteq \{(p,q) \mid p-q \in U\}\}.$$

Recall that a subset U of a vector space is called *radial* if $-U = U$; it is called *circled (or balanced)* if $\lambda U \subset U$ for all $|\lambda| \le 1$.

6 Theorem. *Every PN space (S,ν,τ,τ^*) admits a generalized topology $(\mathcal{U}_p)_{p\in S}$ of type V_D which is Fréchet-separated, translation-invariant, and countably-generated by radial and circled θ-neighborhoods.*

Proof. Let (S,ν,τ,τ^*) be a PN space with τ non-necessarily continuous. Let (S,F) be its associated PSM space, where $F_{p,q} = \nu_{p-q}$. The strong neighborhoods at p are given by $N_p(t) = \{q \in S : \nu_{p-q}(t) > 1-t\} = p+N_\theta(t)$. In particular, the generalized topology is translation-invariant. By (N1) we have that this generalized topology is Fréchet-separated (as in the case of PSM spaces). The countable base of θ-neighborhoods is $\{N_\theta(\frac{1}{n}) : n \in \mathbb{N}\}$, whose elements are clearly radial and circled, by axioms (N2) and (N4), respectively. QED

Note that the generalized topology induced by a PN space (S,ν,τ,τ^*) is derivable from the following generalized uniformity:

$$\mathcal{U} := \left\{V \subset S\times S \mid \exists n \in \mathbb{N} : V \supseteq \left\{(p,q) \mid \nu_{p-q}\left(\frac{1}{n}\right) \ge 1-\frac{1}{n}\right\}\right\},$$

which is translation-invariant and has a countable filter base of radial and circled vicinities.

Adapting the methods in [6], we next show that a converse result holds for such generalized topologies (or generalized uniformities).

Let S be a vector space and $(\mathcal{U}_p)_{p\in S}$ be a Fréchet-separated, translation-invariant, generalized topology of type V_D on S. Then, there is a unique translation-invariant, Hausdorff-separated generalized uniformity, which is defined as follows

$$\mathcal{U} := \{V \subseteq S \times S \mid \exists U \in \mathcal{U}_\theta : V \supseteq \{(p,q) : p - q \in U\}\}.$$

The analogous result of Theorem 4 for PN spaces is the following. (Note that there is an extra assumption on T):

7 Theorem. *Let T be a t-norm such that $\sup_{0\le x<1} T(x,x) < 1$. Suppose that $T(x,y) \le xy$, whenever $x, y < \delta$, for some $\delta > 0$. A Fréchet-separated, translation-invariant, generalized topology $(\mathcal{U}_p)_{p\in S}$ on a real vector space S is derivable from a Menger PN space $(S,\nu,\tau_T,\tau_{T^*})$, if and only $\mathcal{U}_\theta$ admits a countable base of radial and circled subsets.*

Proof. The direct implication has been shown above. For the converse, let $\mathcal{B} = \{V_n \mid n \in \mathbb{N}\}$ be a countable filter base for $\mathcal{U}_\theta$ consisting on radial and circled θ-neighborhoods.

Let $N_0 \in \mathbb{N}$ such that $1 - \frac{1}{N_0} \ge \sup_{0\le x<1} T(x,x)$. We can assume that $\frac{1}{N_0} < \delta$, so that $T(x,y) \le xy$, for all $x, y \le \frac{1}{N_0}$, where δ is given by hypothesis.

Before defining ν, recall from [6, Theorem 2] the distribution functions F_n (used to define the probabilistic metric F):

$$F_n(x) := \begin{cases} 0 & : x \le 0 \\ 1 - 1/(N_0(n+1)), & 0 < x \le \frac{1}{n+1}, \\ 1 - 1/(2N_0(n+1)), & \frac{1}{n+1} < x \le 1, \\ 1 - 1/(2^{m+1}N_0(n+1)), & m < x \le m+1 \quad \text{for } m \in \mathbb{N}. \end{cases}$$

By putting "$\nu_p = F_{p,\theta}$" in [6, Theorem 2]) we define:

$$\nu_p(x) := \begin{cases} F_0, & p \notin V_1 \\ F_n, & p \in V_n \setminus V_{n+1}, \text{for } n \in \mathbb{N} \\ \varepsilon_0, & p \in \cap_n V_n. \end{cases}$$

We next check that $(S,\nu,\tau_T,\tau_{T^*})$ is a PN space. Axiom (N1) holds because the generalized topology is Fréchet-separable. (N2) holds because all V_n's are radial. (N3) holds as in [6]:

$$\tau_T(\nu_p,\nu_q)(x) = \sup_{r+s=x} T(\nu_p(r),\nu_q(s)) \le 1 - 1/N_0 \le \nu_{p+q}(r+s) = \nu_{p+q}(x).$$

Finally, for (N4): Let $p \in V_n$ and $\lambda \in [0,1]$. Then, λp and $(1-\lambda)p$ are also in V_n, because V_n is circled. For $x = r+s$, we have to show that

$$\nu_p(x) \leq T^*(\nu_{\lambda p}(r), \nu_{(1-\lambda)p}(s)).$$

Suppose first that r and s are strictly greater than 1, $r, s > 1$. Let $a, b, c \in \mathbb{N}$ such that $a < r \leq a+1$, $b < s \leq b+1$, and $c < r+s \leq c+1$. Then,

$$\begin{aligned}
\nu_{\lambda p}(r) &= 1 - 1/(2^{a+1}N_0(n+1)),\\
\nu_{(1-\lambda)p}(s) &= 1 - 1/(2^{b+1}N_0(n+1)),\\
\nu_p(r+s) &= 1 - 1/(2^{c+1}N_0(n+1)).
\end{aligned}$$

By the properties of T it follows that

$$\begin{aligned}
T^*(\nu_{\lambda p}(r), \nu_{(1-\lambda)p}(s)) =& 1 - T(1-\nu_{\lambda p}(r), 1-\nu_{(1-\lambda)p}(s))\\
=& 1 - T(1/(2^{a+1}N_0(n+1)), 1/(2^{b+1}N_0(n+1)))\\
\geq& 1 - (1/(2^{a+1}N_0(n+1))) \cdot (1/(2^{b+1}N_0(n+1)))\\
\geq& 1 - 1/(2^{c+1}N_0(n+1))\\
=& \nu_p(r+s) = \nu_p(x).
\end{aligned}$$

In the third line we have used that the arguments of T are smaller than $1/N_0$, thus we can apply $T(x,y) \leq xy$. Then, we obtain $\nu_p \leq \tau_{T^*}(\nu_{\lambda p}, \nu_{(1-\lambda p)})$ as desired. The inequality for the other possible values of r and s, is checked in a similar way. We conclude that $(S, \nu, \tau_T, \tau_{T^*})$ is a Menger PN space under T.

It only remains to show that the generalized topology induced by ν is the same as the one given at the beginning. As in [6], we have by construction that

$$V_n = \left\{ p \in S \mid \nu_p\left(\frac{1}{n+1}\right) \geq 1 - \frac{1}{N_0(n+1)} \right\}.$$

Thus, the filter base $\{p \in S \mid \nu_p(\frac{1}{n+1}) \geq 1 - \frac{1}{n+1}\}$ induced by ν is equivalent to $\mathcal{B}$, hence the proof is finished. QED

8 Remark. Theorem 6 also holds if instead of assuming $T(x,y) \leq xy$ near the origin, one assumes that T is Archimedean near the origin (i.e. there is a $\delta > 0$ such that $0 < T(x,x) < x$, for all $0 < x < \delta$). In that case, the distribution function F_n can be chosen as:

$$F_n(x) := \begin{cases} 0 & : x \leq 0\\ 1 - z & : 0 < x \leq \frac{1}{n+1}\\ 1 - T(z,z) & : \frac{1}{n+1} < x \leq 1\\ 1 - T^{m+1}(z,z) & : m < x \leq m+1 \quad \text{for } m \in \mathbb{N}, \end{cases}$$

where $z = 1/(N_0(n+1))$, $T^1(x,y) = T(x,y)$ and recursively

$$T^r(x,y) = T(T^{r-1}(x,y), T^{r-1}(x,y)).$$

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