Translation-invariant generalized topologies induced by probabilistic norms

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Abstract. One considers probabilistic normed spaces as defined by Alsina, Sklar, and Schweizer, but with non necessarily continuous triangle functions. Such spaces are endowed with a generalized topology that is Fréchet-separated, translation-invariant and countably generated by radial and circled 0-neighborhoods. Conversely, we show that such generalized topologies are probabilistically normable.

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1 Introduction

Probabilistic normed spaces (briefly, PN spaces) were first defined by Šerstnev in the early sixties [see [13]], thus originating a fruitful theory that extended the theory of ordinary normed spaces. Thirty years later, Alsina, Schweizer, and Sklar gave in [1] a quite general definition of PN space, based on the definition of Menger’s betweenness in probabilistic metric spaces; [see [14], p. 232].

We here consider PN spaces in which the involved triangle functions are non necessarily continuous. With regards to a generalized topology in the sense of Fréchet and for probabilistic metric spaces, the problem was treated by Höhle in [6] where he showed that all generalized topologies which are Fréchet-separated and first-numerable are induced by certain probabilistic metrics. The main result of this paper is a similar result for probabilistic norms, where the $t$-norm has certain restriction:

1 Theorem. Let $T$ be a $t$-norm such that $\sup_{0 \leq x < 1} T(x, x) < 1$. Suppose that $T(x, y) \leq xy$, whenever $x, y < \delta$, for some $\delta > 0$. A Fréchet-separated, translation-invariant, generalized topology $(U_p)_{p \in S}$ on a real vector space $S$ is
derivable from a Menger PN space \((S, \nu, \tau, \tau^*)\), if and only if \(\mathcal{U}_\theta\) admits a countable base of radial and circled subsets, where \(\theta\) is the origin of \(S\).

In fact, this result also holds if one assumes \(T\) to be Archimedean near the origin, i.e. there is a \(\delta > 0\) such that \(0 < T(x, x) < x\), for all \(0 < x < \delta\) [see Remark [8], after Theorem [1]].

We think that a similar result could be interesting for fuzzy normed spaces in the sense of Felbin [3], but allowing non-continuity of the \(t\)-norms, and \(t\)-conorms involved in the fuzzy structure.

In [10] the authors use this generalized topology to define bounded subsets in PN spaces (with non necessarily continuous triangle functions) and study its relationship with \(D\)-bounded subsets (a concept which is defined in probabilistic terms).

2 PM and PN spaces

Recall from [1] and [14] some definitions on probabilistic metric and probabilistic normed spaces.

As usual, \(\Delta^+\) denotes the set of distance distribution functions, i.e. distribution functions with \(F(0) = 0\), endowed with the metric topology given by the modified Lévy-Sybley metric \(d_L\) [see 4.2 in [14]]. Given a real number \(a\), \(\varepsilon_a\) denotes the distribution function defined as \(\varepsilon_a(x) = 0\) if \(x \leq a\) and \(\varepsilon_a(x) = 1\) if \(x > a\). Hence, the set of non-negative real numbers \(\mathbb{R}^+\) can be viewed as a subspace of \(\Delta^+\). A triangle function \(\tau\) is a map from \(\Delta^+ \times \Delta^+ \to \Delta^+\) which is commutative, associative, nondecreasing in each variable and has \(\varepsilon_0\) as the identity. Such functions give rise to all possible extensions of the sum of real numbers, so that (M3) below corresponds to the triangle inequality.

A probabilistic metric space (briefly, a PM space) is a triple \((S, F, \tau)\) where \(S\) is a non-empty set, \(F\) is a map from \(S \times S \to \Delta^+\), called the probabilistic metric, and \(\tau\) is a triangle function, such that:

(M1) \(F_{p,q} = \varepsilon_0\) if and only if \(p = q\).

(M2) \(F_{p,q} = F_{q,p}\).

(M3) \(F_{p,q} \geq \tau(F_{p,r}, F_{r,q})\).

When only (M1) and (M2) are required, it the pair \((S, F)\) is said to be a probabilistic semi-metric space (briefly, PSM space).

A PN space is a quadruple \((S, \nu, \tau, \tau^*)\) in which \(S\) is a vector space over \(\mathbb{R}\), the probabilistic norm \(\nu\) is a map \(S \to \Delta^+\), \(\tau\) and \(\tau^*\) are triangle functions\(^1\)

\(^1\)In the definition of PN space given in [1] the triangle functions are assumed to be continuous
such that the following conditions are satisfied for all \( p, q \) in \( S \):

(N1) \( \nu_p = \varepsilon_0 \) if and only if \( p = \theta \), where \( \theta \) is the origin of \( S \).

(N2) \( \nu_{-p} = \nu_p \).

(N3) \( \nu_{p+q} \geq \tau(\nu_p, \nu_q) \).

(N4) \( \nu_p \leq \tau^*(\nu_{\lambda p}, \nu_{(1-\lambda)p}) \) for every \( \lambda \in [0, 1] \).

Observe that every PN space \((S, \nu, \tau, \tau^*)\) is a PM space, where \( F_{p,q} := \nu_p - \nu_q \).

Recall that a \( \tau \)-norm is a binary operation on \([0,1]\) that is commutative, associative, nondecreasing in each variable, and has 1 as identity. Dually, a \( \tau \)-conorm is a binary operation on \([0,1]\) that is commutative, associative, non-decreasing in each variable, and has 0 as identity. If \( T \) is a \( \tau \)-norm, its associated \( \tau \)-conorm \( \tau^* \) is defined by

\[
\tau^*(x,y) := 1 - T(1-x,1-y).
\]

Given a \( \tau \)-norm \( T \) one defines the functions \( \tau_T \) and \( \tau_T^* \) by

\[
\tau_T(F,G)(x) := \sup \{ T(F(s),G(t)) : s + t = x \},
\]

and

\[
\tau_T^*(F,G)(x) := \inf \{ T^*(F(s),G(t)) : s + t = x \}.
\]

Recall that if \( T \) is left-continuous then \( \tau_T \) is a triangle function [14, p. 100], although this is not necessary; For example, if \( Z \) denotes the weakest \( \tau \)-norm, defined as \( Z(x,1) = Z(1,x) = x \) and \( Z(x,y) = 0 \) elsewhere, then \( \tau_Z \) is a triangle function which is not continuous.

A \u00c5Serstnev PN space is a PN space \((V, \nu, \tau, \tau^*)\) where \( \nu \) satisfies the following \u00e5Serstnev condition:

\[
(\bar{\Sigma}) \quad \nu_{\lambda p}(x) = \nu_p \left( \frac{x}{\lambda} \right), \text{ for all } x \in \mathbb{R}^+, p \in V \text{ and } \lambda \in \mathbb{R} \setminus \{0\}.
\]

which clearly implies (N2) and also [see [1]](N4) in the strengthened form

\[
\nu_p = \tau_M(\nu_{\lambda p}, \nu_{(1-\lambda)p}), \tag{1}
\]

for all \( p \in V \) and \( \lambda \in [0,1] \) [see [1, Theorem 1]], where \( M \) is the \( \tau \)-norm defined as \( M(x,y) = \min\{x,y\} \).

Let \( T \) be a \( \tau \)-norm. A Menger PM space under \( T \) is a PM space of the form \((S,F,\tau_T)\). Analogously, a Menger PN space under \( T \) is a PN space of the form \((S,\nu,\tau_T,\tau_T^*)\). Note that every metric space \((S,d)\) is a Menger space \((S,F,\tau_M)\) where \( F_{p,q} = \varepsilon_d(p,q) \). Analogously, every normed space \((S,\|\cdot\|)\) is a Menger and \u00e5Serstnev PN space \((S,\nu,\tau_M,\tau_M)\) where \( \nu_p = \varepsilon_{\|p\|} \).
3 Probabilistic metrization of generalized topologies

In [6] Höhle solved a problem posed by Thorp about the probabilistic metrization of generalized topologies. We recall some definitions and results that we shall use in the next section.

Let $S$ be a (non-empty) set. A generalized topology (of type $V_D$) on $S$ is a family of subsets $(U_p)_p \subseteq S$, where $U_p$ is a filter on $S$ such that $p \in U$ for all $U \in U_p$ [see e.g. [14, p. 38], [2, p. 22]]. Elements of $U_p$ are called neighborhoods at $p$. Such a generalized topology is called Fréchet-separated if $\bigcap_{U \in U_p} U = \{p\}$.

A generalized uniformity $U$ on $S$ is a filter on $S \times S$ such that every $V \in U$ contains the diagonal $\{(p,p) : p \in S\}$, and for all $V \in U$, we have that $V^{-1} := \{(q,p) : (p,q) \in V\}$ also belongs to $U$. Elements of $U$ are called vicinities (or "entourages"). Every generalized uniformity $U$ induces a generalized topology as follows: for $p \in S$,

$$U_p := \{U \subseteq S \mid \exists V \in U : U \supseteq \{q \in S \mid (p,q) \in V\}\}. \quad (2)$$

A uniformity $U$ is called Hausdorff-separated if the intersection of all vicinities is the diagonal on $S$. Theorem 1 in [6] claims:

2 Theorem. [Höhle] Every Fréchet-separated generalized topology $(U_p)_p \subseteq S$ on a given set $S$ is derivable from a Hausdorff-separated generalized uniformity $U$ in the sense of (2). $\square$

Let $(S,F)$ be a PSM space. Consider the system $(N_p)_p \subseteq S$, where $N_p = \{N_p(t) : t > 0\}$ and

$$N_p(t) := \{q \in S : F_{p,q}(t) > 1 - t\}.$$ 

This is called the strong neighborhood system. If we define $\delta(p,q) := d_L(F_{p,q}, \varepsilon_0)$, then $\delta$ is a semi-metric on $S$ (i.e. it may not satisfy the triangle inequality of the standard metric axioms), and $N_p(t) = \{q : d_L(F_{p,q}, \varepsilon_0) < t\}$. Clearly $p \in N$ for every $N \in N_p$, and the intersection of two strong neighborhoods at $p$ is a strong neighborhood at $p$. Furthermore, $N_p$ admits a countable filter base given by $\{N_p(1/n) : n \in \mathbb{N}\}$, hence the strong neighborhood system is first-countable.

The above explanation yields the following fact [see more details in [14], p. 191]:

3 Theorem. Let $(S,F)$ be a PSM space, then the strong neighborhood system defines a generalized topology of type $V_D$ which is Fréchet-separated and first-countable. $\square$

This generalized topology is called the strong generalized topology of the PSM space $(S,F)$.

The main result in [6] is the following.
4 Theorem. [Höhle] Let $T$ be a $t$-norm such that $\sup_{0 \leq x < 1} T(x, x) < 1$. A Fréchet-separated generalized topology $(U_p)_{p \in S}$ on a set $S$ is derivable from a Menger PM space $(S, F, T)$ if and only if there exists a Hausdorff-separated, generalized uniform structure $U$ having a countable filter base, such that $U$ is compatible with $(U_p)_{p \in S}$.

5 Remark. If $(S, F, \tau)$ is a PM space with $\tau$ continuous, then the associated generalized topology is in fact a topology. This topology is called the strong topology. Because of (M1) this topology is Hausdorff. Since it is first-countable and uniformable, it is metrizable [see [14, Theorem 12.1.6]]. Conversely, if $\sup_{0 \leq x < 1} T(x, x) = 1$, then a Fréchet-separated, uniformable topology is derivable from a Menger space $(S, F, T)$ if and only if there exists a Hausdorff uniformity $U$ on $S$ having a countable filter base [6].

4 Translation-invariant generalized topologies

Assume now that $S$ is a vector space over $\mathbb{R}$. A generalized topology $(U_p)_{p \in S}$ on $S$ is said to be translation-invariant if for all $U \in U_p$ and $q \in S$, we have $q + U \in U_{p+q}$. Consequently, a translation-invariant generalized topology is uniquely determined by the neighborhood system $U_\theta$ at the origin $\theta$ of $S$. In this case, the generalized uniformity from which one can derive the generalized topology is:

$$U := \left\{ V \subseteq S \times S \mid \exists U \in U_\theta : V \supseteq \left\{ (p, q) \mid p - q \in U \right\} \right\}.$$ 

Recall that a subset $U$ of a vector space is called radial if $-U = U$; it is called circled (or balanced) if $\lambda U \subseteq U$ for all $|\lambda| \leq 1$.

6 Theorem. Every PN space $(S, \nu, \tau, \tau^*)$ admits a generalized topology $(U_p)_{p \in S}$ of type $V_D$ which is Fréchet-separated, translation-invariant, and countably-generated by radial and circled $\theta$-neighborhoods.

Proof. Let $(S, \nu, \tau, \tau^*)$ be a PN space with $\tau$ non-necessarily continuous. Let $(S, F)$ be its associated PSM space, where $F_{p,q} = \nu_{p-q}$. The strong neighborhoods at $p$ are given by $N_p(t) = \{ q \in S : \nu_{p-q}(t) > 1-t \} = p + N_\theta(t)$. In particular, the generalized topology is translation-invariant. By (N1) we have that this generalized topology is Fréchet-separated (as in the case of PSM spaces). The countable base of $\theta$-neighborhoods is $\{ N_{\theta}(1/n) : n \in \mathbb{N} \}$, whose elements are clearly radial and circled, by axioms (N2) and (N4), respectively.

Note that the generalized topology induced by a PN space $(S, \nu, \tau, \tau^*)$ is derivable from the following generalized uniformity:

$$U := \left\{ V \subseteq S \times S \mid \exists n \in \mathbb{N} : V \supseteq \left\{ (p, q) \mid \nu_{p-q} \left( \frac{1}{n} \right) \geq 1 - \frac{1}{n} \right\} \right\},$$
which is translation-invariant and has a countable filter base of radial and circled vicinities.

Adapting the methods in [6], we next show that a converse result holds for such generalized topologies (or generalized uniformities).

Let $S$ be a vector space and $(U_p)_{p \in S}$ be a Fréchet-separated, translation-invariant, generalized topology of type $V_D$ on $S$. Then, there is a unique translation-invariant, Hausdorff-separated generalized uniformity, which is defined as follows

\[ U := \{ V \subseteq S \times S \mid \exists U \in \mathcal{U}_0 : V \supseteq \{(p, q) : p - q \in U\} \} . \]

The analogous result of Theorem 4 for PN spaces is the following. (Note that there is an extra assumption on $T$):

7 Theorem. Let $T$ be a t-norm such that $\sup_{0 \leq x < 1} T(x, x) < 1$. Suppose that $T(x, y) \leq xy$, whenever $x, y < \delta$, for some $\delta > 0$. A Fréchet-separated, translation-invariant, generalized topology $(U_p)_{p \in S}$ on a real vector space $S$ is derivable from a Menger PN space $(S, \nu, \tau_T, \tau_T^*)$, if and only $U_0$ admits a countable base of radial and circled subsets.

Proof. The direct implication has been shown above. For the converse, let $B = \{V_n \mid n \in \mathbb{N}\}$ be a countable filter base for $U_0$ consisting on radial and circled $\theta$-neighborhoods.

Let $N_0 \in \mathbb{N}$ such that $1 - \frac{1}{N_0} \geq \sup_{0 \leq x < 1} T(x, x)$. We can assume that $\frac{1}{N_0} < \delta$, so that $T(x, y) \leq xy$, for all $x, y \leq \frac{1}{N_0}$, where $\delta$ is given by hypothesis.

Before defining $\nu$, recall from [6, Theorem 2] the distribution functions $F_n$ (used to define the probabilistic metric $F$):

\[ F_n(x) := \begin{cases} 0 & : x \leq 0 \\ 1 - 1/(N_0(n + 1)) & : 0 < x \leq \frac{1}{n+1}, \\ 1 - 1/(2N_0(n + 1)) & : \frac{1}{n+1} < x \leq 1, \\ 1 - 1/(2^{m+1}N_0(n + 1)) & : m < x \leq m + 1 \quad \text{for} \quad m \in \mathbb{N}. \end{cases} \]

By putting “$\nu_p = F_{p, \theta}$” in [6, Theorem 2]) we define:

\[ \nu_p(x) := \begin{cases} F_0, & p \notin V_1 \\ F_n, & p \in V_n \setminus V_{n+1}, \text{for} \quad n \in \mathbb{N} \\ \varepsilon_0, & p \in \cap_n V_n. \end{cases} \]

We next check that $(S, \nu, \tau_T, \tau_T^*)$ is a PN space. Axiom (N1) holds because the generalized topology is Fréchet-separable. (N2) holds because all $V_n$’s are radial. (N3) holds as in [6]:

\[ \tau_T(\nu_p, \nu_q)(x) = \sup_{r + s = x} T(\nu_p(r), \nu_q(s)) \leq 1 - 1/N_0 \leq \nu_{p+q}(r + s) = \nu_{p+q}(x). \]
Finally, for (N4): Let \( p \in V_n \) and \( \lambda \in [0, 1] \). Then, \( \lambda p \) and \( (1 - \lambda)p \) are also in \( V_n \), because \( V_n \) is circled. For \( x = r + s \), we have to show that

\[
\nu_p(x) \leq T^\ast(\nu_{\lambda p}(r), \nu_{(1-\lambda)p}(s)).
\]

Suppose first that \( r \) and \( s \) are strictly greater than 1, \( r, s > 1 \). Let \( a, b, c \in \mathbb{N} \) such that \( a < r \leq a + 1, b < s \leq b + 1 \), and \( c < r + s \leq c + 1 \). Then,

\[
\nu_{\lambda p}(r) = 1 - 1/(2a+1N_0(n+1)),
\]
\[
\nu_{(1-\lambda)p}(s) = 1 - 1/(2b+1N_0(n+1)),
\]
\[
\nu_p(r + s) = 1 - 1/(2c+1N_0(n+1)).
\]

By the properties of \( T \) it follows that

\[
T^\ast(\nu_{\lambda p}(r), \nu_{(1-\lambda)p}(s)) = 1 - T(1 - \nu_{\lambda p}(r), 1 - \nu_{(1-\lambda)p}(s))
\]
\[
= 1 - T(1/(2a+1N_0(n+1)), 1/(2b+1N_0(n+1)))
\]
\[
\geq 1 - (1/(2a+1N_0(n+1))) \cdot (1/(2b+1N_0(n+1)))
\]
\[
= \nu_p(r + s) = \nu_p(x).
\]

In the third line we have used that the arguments of \( T \) are smaller than \( 1/N_0 \), thus we can apply \( T(x, y) \leq xy \). Then, we obtain \( \nu_p \leq \tau_{T^\ast}(\nu_{\lambda p}, \nu_{(1-\lambda)p}) \) as desired. The inequality for the other possible values of \( r \) and \( s \), is checked in a similar way. We conclude that \( (S, \nu, \tau_T, \tau_{T^\ast}) \) is a Menger PN space under \( T \).

It only remains to show that the generalized topology induced by \( \nu \) is the same as the one given at the beginning. As in [6], we have by construction that

\[
V_n = \left\{ p \in S \mid \nu_p \left( \frac{1}{n+1} \right) \geq 1 - \frac{1}{N_0(n+1)} \right\}.
\]

Thus, the filter base \( \{ p \in S \mid \nu_p \left( \frac{1}{n+1} \right) \geq 1 - \frac{1}{n+1} \} \) induced by \( \nu \) is equivalent to \( \mathcal{B} \), hence the proof is finished.

**Remark.** Theorem 6 also holds if instead of assuming \( T(x, y) \leq xy \) near the origin, one assumes that \( T \) is Archimedean near the origin (i.e. there is a \( \delta > 0 \) such that \( 0 < T(x, x) < x \), for all \( 0 < x < \delta \)). In that case, the distribution function \( F_n \) can be chosen as:

\[
F_n(x) := \begin{cases} 
0 & : x \leq 0 \\
1 - z & : 0 < x \leq \frac{1}{n+1} \\
1 - T(z, z) & : \frac{1}{n+1} < x \leq 1 \\
1 - T^{m+1}(z, z) & : m < x \leq m + 1 \quad \text{for } m \in \mathbb{N},
\end{cases}
\]

where \( z = 1/(N_0(n+1)) \), \( T^1(x, y) = T(x, y) \) and recursively

\[
T^r(x, y) = T(T^{r-1}(x, y), T^{r-1}(x, y)).
\]
References