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Constructions of Sperner k-Spreads of Dimension tk

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Abstract. In this article, a very general construction of Sperner spaces with given collineation group is given that produces a very large number of non-isomorphic subgeometry partitions of projective spaces as well as new focal-spreads. Also, a generalization of 'algebraic lifting' for Sperner 2-spreads is given.

Keywords: Sperner space, subgeometry partitions, focal-spread.

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1 Introduction

A major construction technique of finite translation planes with underlying vector space of dimension $2k$ over a field K isomorphic to $GF(q)$ involves choosing a basis in which two of the lines through the origin ('components') have coordinates $x = 0, y = 0$, where x and y are k-vectors over K. The remaining components are given by a set S of $q^k - 1$ non-singular $k \times k$ matrices such that, for any two matrices M and N, then $M - N$ is non-singular. We write the 'spread' for the translation plane as

$$x = 0,\ y = 0,\ y = xM;\ M \in S.$$

The lines of the translation plane are translates of the spread components. In this context we then focus on the set

$$S_{mat} : y = xM;\ M \in S,$$

which, if x is nonzero, we have vectors of the form (x, y), for x and y both nonzero.

Now to generalize this construction process for a tk-dimensional vector space, which has a spread of k-dimensional subspaces. Consider components $(x_1, x_2, \ldots, x_t)$, where x_i are k-vectors, to mirror the spread for a translation plane, we always take as 'components' the subspaces $(x_1, 0, 0, \ldots, 0)$, $(0, x_2, 0, 0, \ldots, 0)$, $\ldots$,$(0, 0, \ldots, 0, x_t)$. The remaining vectors will be partitioned in what are called '$j = 0$-sets', by which we shall mean the set of all vectors $(x_1, x_2, \ldots, x_t)$, where there are exactly j of the elements x_i equal to 0. Of course, there are $\binom{t}{j}$ such sets, for each $j = 0, 1, 2, \ldots, t-1$, and where $(x_1, 0, 0, \ldots, 0)$, $(0, x_2, 0, 0, \ldots, 0)$, $\ldots$,$(0, 0, \ldots, 0, x_t)$ for x_i non-zero, may be considered $t - 1 = 0$-sets. In this article, we shall construct spreads of k-subspaces in vector spaces of dimension tk, which correspond to partitions of the $j = 0$-sets, in the sense that each partition element when unioning the zero vector becomes a k-subspace. We call such structures "unions of $j = 0$-spreads'. Then taking the lines as translates of the spread elements, we obtain translation Sperner spaces.

We begin by giving a very general construction principle for the union of $j = 0$-spreads. Referring back to k-spreads of $2k$-dimensional vectors spaces equivalent to finite translation planes of order q^k with kernel containing $GF(q)$, we review and generalize the Hiramine-Oyama-Matsumoto 'algebraic lifting' construction that produces from a 2-spread of a 4-dimensional vector space over $GF(q)$, a 2-spread in a 4-dimensional vector space over $GF(q^2)$. Jha and Johnson [6] have used the process to identify certain so-called 'retraction' groups in the algebraically lifted translation plane, which, in turn, construct subgeometry partitions in an associated $PG(3, q^2)$ by subgeometries isomorphic to $PG(3, q)$ and $PG(1, q^2)$. We develop a method that we called 'lifting and twisting' by which the construction generalizes to produce from a 2-spread of a $4t$-dimensional vector space over $GF(q)$ a 2-spread of a $4t$-dimensional vector space over $GF(q^2)$, on which there is a natural retraction group and hence produces a new class of subgeometry partitions of $PG(4t-1, q^2)$ by subgeometries isomorphic to $PG(1, q^2)$ and $PG(3, q^2)$.

Recently the second author in Johnson [10] has constructed new classes of translation Sperner spaces that are called extended André spreads, in a manner analogous to the construction of the André translation planes from the associated Desarguesian affine plane. Furthermore, these general spaces produce a large variety of subgeometry partitions by Johnson [11]. In previous work, there were either one or two different types of subgeometries in the partition, whereas using extended André spreads, there is essentially no restriction on the type of projective space used as subgeometries.

The new constructions that are given in this article generalize the extended André constructions as well as the subgeometries that may be constructed.

In previous articles, [4], [5], [3], [2], the authors have constructed partitions called 'focal-spreads' of vector spaces of dimension $t+k$ by exactly one subspace of dimension t (the 'focus') and where the remaining vectors are covered by a partial k-spread of q^t k-subspaces. This article gives some generalizations of focal-spreads, by considering partitions of vector spaces of dimension $t+k$ also partitioned into exactly one t'-subspace and where the remaining vectors are covered by a partial k-spread. We call these partitions (t, t', k)-focal-spreads and show how these relate to our general constructions.

2 The General Construction

First a special case of the construction will be given, so the reader can better understand the general case.

Choose five k-spreads of a $2k$-dimensional vector space by mutually disjoint k-spaces $\mathcal{M}_i$, $i = 1, 2, 3, 4, 5$. These spreads are chosen independently so some can be the same or all can be distinct.

So, we would have spreads of the form:

$$x = 0, y = 0, \ y = xM_i, \text{ for } M_i \in \mathcal{M}_i$$

where the x's and $y^{'}s$ are k-vectors, and where the $\mathcal{M}_i$ are $k \times k$ matrices over the same field $GF(q)$, and there are $(q^{2k} - 1)/(q^k - 1) - 2 = q^k - 1$ matrices in each set.

Now form the $3k$-space with vectors (x_1, x_2, x_3), where the x_i are k-vectors. Consider the following set:

$$(x_1, x_1M_1, x_1M_2); y = (x_1M_1, x_1M_2),$$

where $M_i \in \mathcal{M}_i$. There are $(q^k - 1)^2$ such sets and, when unioning the zero vector, each is a $GF(q)$-space. For x_1 non-zero, this is the $0 = 0$-set.

There are three $1 = 0$-sets, which we cover in the following manner:

$$\begin{gathered}(0, x_1, x_1M_3),\\ (x_1, 0, x_1M_4),\\ (x_1, x_1M_5, 0)\end{gathered}$$

to obtain $3(q^k - 1)$ $GF(q)$-spaces, when unioning each with the zero vector.

Finally, there are three $2 = 0$-sets:

$$(x_1, 0, 0), (0, x_2, 0), (0, 0, x_3),$$

which are k-subspaces when unioning the zero vector to each.

We obtain a set of

$$(q^k-1)^2+3(q^k-1)+3=\frac{q^{3k}-1}{q^k-1},$$

k-spaces.

We now claim that these are mutually disjoint as subspaces; that we have constructed a k-spread as a union of $j=0$-spreads.

It is clear that we need only check the $0=0$-set type. Suppose $y=(x_1M_1,x_1M_2)$ and $y=(x_1M_1^*,x_1M_2^*)$ share a vector (x_1,x_1M_1,x_1M_2) and $(x_1,x_1M_1^*,x_1M_2^*)$ (note the first k-vector must be the same for the shared vector). Then

$$x_1M_1=x_1M_1^*, \text{ and } x_1M_2=x_1M_2^*,$$

which immediately shows for x_1 non-zero that $M_1=M_1^*$ and $M_2=M_2^*$.

Therefore, from 5 k-spreads (not necessarily distinct), we may construct a $(3,k)$-spread.

We now generalize this construction to construct k-spreads of tk-dimensional vector spaces by finding partitions of the $j=0$-sets, which we also refer to as '(t,k)-spreads'.

Suppose we wish to construct a (t,k)-spread over $GF(q)$, of a tk-dimensional vector space, with vectors $(x_1,x_2,\ldots,x_t)$, where all x_i are k-vectors. Note that

$$(q^{tk}-1)/(q^k-1)=\sum_{j=0}^{t-1}\tbinom{t}{j}(q^k-1)^{t-j-1}\,.$$

So, we would need

$$\sum_{j=0}^{t-2}\tbinom{t}{j}(t-j-1)$$

k-spreads arising from translation planes. (Note we don't require a spread for the t, $(t-1)=0$-sets.)

Therefore take $\sum_{j=0}^{t-2}\binom{t}{j}(t-j-1)=N_t$ $(2,k)$-spreads $\mathcal{M}_i$, $i=1,2,\ldots,N_t$.

Note that when $t=2$, $N_t=1$ and when $t=3$, $N_t=5$. Just repeat the previous construction to obtain a partition of a tk-vector space by k-spaces; a (t,k)-spread and note that the placement of the individual pieces within the various $j=0$-sets may change the (t,k)-spread.

Hence, we obtain our main construction theorem.

1 Theorem. *Let $\mathcal{S}_t$ be an ordered sequence of $\sum_{j=0}^{t-2}\binom{t}{j}(t-j-1)=N_t$ $(2,k)$-spreads over $GF(q)$. We may represent each of the $(2,k)$-spreads by a matrix spread set, where we identify two common components $x=0,y=0$, and where*

x and y are k-vectors. Then if $\mathcal{M}_i$ is a $(2,k)$-spread, there is a set of q^k-1 nonsingular matrices $M_{i,z}$, for $z=1,2,\ldots,q^k-1$, $i=1,2,\ldots,N_t$, whose differences are also non-singular. Hence, we represent the $(2,k)$-spreads by $y=xM_{iz}$, for $z=1,2,\ldots,q^k-1$. Consider the $j=0$-sets for $j=0,1,2,\ldots,t-2$, and assume an ordering for the N_t $(2,k)$-spreads. For each such $j=0$-set, for $j>2$, we eliminate the zero elements and write vectors in the form $(x_1,x_2,\ldots,x_{t-j})$, where x_w for $w=1,2,\ldots,t-j$ are all non-zero vectors. We partition this set by

$$(x_1, x_1M_{1,z_1}, x_1M_{2,z_2}, x_1M_{3,z_3}, \ldots, x_1M_{t-j,z_{t-j}})$$

where M_{i,z_i} varies over $\mathcal{M}_i$, and where the indices z_i are independent of each other. By adjoining the zero vector, we then may consider

$$y = (x_1M_{1,z_1}, x_1M_{2,z_2}, x_1M_{3,z_3}, \ldots, x_1M_{t-j,z_{t-j}})$$

as a k-subspace for fixed M_{i,z_i}, for each $i=1,2,\ldots,t-j$. Then we obtain a spread of k-spaces of a tk-dimensional vector space over $GF(q)$ as a union of $j=0$-spreads.

Now consider a (t,k)-spread, that is a k-spread of a tk-dimensional vector space, where it is assumed that the spread consists of the $t-1=0$-sets and the remaining k-subspaces partition each $j=0$-set. That is, assume that the k-spread is a union of $j=0$-spreads. Consider for example, a k-subspace within a $j=0$-set of the form (x_1,x_2,x_3), where all k-vectors x_i, are non-zero. Then, since (x_1,x_2,x_3), (x_1',x_2,x_3) in the same set means that the difference is $(x_1-x_1',0,0)$ which is in another $j=0$-set unless $x_1=x_1'$ (and we have adjoined the zero vector to this set). Hence, $(x_2,x_3)=f(x_1)$ and clearly f is a linear function over $GF(q)$. But, similarly $(x_2,x_3)=f(x_1)$ and $(x_2,x_3')=f(x_1')$ would imply that $(x_1-x_1',0,x_3-x_3')$ is a vector in another $j=0$-set, unless $x_3=x_3'$. This means that $(x_2,x_3)=(g(x_1),h(x_1))$,where g and h are linear functions over $GF(q)$. Considered in matrix form, in this case, we have (x_1,x_1M_1,x_1M_2), or rather that one of the k-spaces has the form $y=(xM_1,xM_1')$ in the notation established previously. In order to have a partition of the $j=0$-set by k-spaces, in this case, have $(q^k-1)^3$ vectors so we would require $(q^k-1)^2$ elements $y=(xM_i,xM_i')$. Since now the differences of the associated matrices in each entry are non-singular, this implies that we have k-spreads

$$\begin{gathered}
x=0, y=0, y=xM_i \text{ for } i=1,2,\ldots,q^k-1,\\
x=0, y=0, y=xM_i' \text{ for } i=1,2,\ldots,q^k-1,
\end{gathered}$$

there the notation now indicates that y is a k-vector. Hence, we obtain two k-spreads $\{M_i; i=1,2,\ldots,q^k-1\}$, and $\{M_i'; i=1,2,\ldots,q^k-1\}$. The same argument for $j=0$-sets of other sizes with a corresponding count on how many $j=0$-sets there are shows that the converse to the previous result is also valid.

2 Theorem. *Let $\mathcal{S}$ be a spread of k-subspaces of a tk-dimensional vector space over $GF(q)$. Assume that the spread is a union of $j=0$-spreads (k-spreads). Then there are $\sum_{j=0}^{t-2}\binom{t}{j}(t-j-1)=N_t$ k-spreads of a $2k$-dimensional vector space over $GF(q)$ such the the spread $\mathcal{S}$ may be reconstructed using the construction method of Theorem 1.*

3 Generalization to Vector Space Partitions with Many Spread types

In the previous section, we gave a method to construct a k-spread of a vector space of dimension tk, using spreads from translation planes. In this section, we show that we may generalize the construction to produce partitions of a vector space of dimension tk by subspaces of various dimensions n, for any divisor of k.

In our construction method, we covered $j=0$-sets using k-spreads. However, for any particular $j=0$-set, we may instead choose to cover this set using n-spreads, where n properly divides k. For this part, we think of the vector space of dimension tmn, where $mn=k$ and we wish to use n-spreads arising from translation planes instead of k-spreads. To give an illustration, assume that $k=6$ and we have a vector space of dimension $t6$. Our previous construction uses N_t, $6-$spreads arising from translation planes. Take a particular $j=0$-set (x_1,x_2,x_3) (zeros are suppressed), where x_i is non-zero (for $t>2$). There are $(q^k-1)^3$ vectors and we would require three k-spreads to cover this set. Let $n=3$ and $m=2$, write $x_i=(x_{1i},x_{2i})$, where the x_{1i} and x_{2i} are 3-vectors. Now further decompose this set into $j=0$-sets. For example, it is possible to have $(x_{11},0,x_{12},x_{22},0,x_{22})$. Note that we could never have $(x_{11},0,0,0,0,0)$, since the x_i are non-zero. There is one $0=0$-set, exactly six $1=0$-set (subsets), twelve $2=0$-sets (choose two of x_1,x_2,x_3 then one of two 3-vectors), and eight $3=0$-sets. Hence, there are in total $1+6+12+8=27$ $j=0$-subsets. We have 6 sets with $(q^n-1)^5$ vectors, 12 sets with $(q^n-1)^4$ vectors and 8 sets with $(q^n-1)^3$ vectors.

Note that $(q^6-1)^3=(q^3-1)^6+6(q^3-1)^5+12(q^3-1)^4+8(q^3-1)^3$. To see this, just note that $(q^6-1)^3=(((q^3-1)+1)^2-1)^3=((q^3-1)^2+2(q^3-1))^3$. Thus, we have a $j=0$-set partition of n-vectors of the $j=0$-set (x_1,x_2,x_3) of k-vectors. Now we may apply our construction using n-spreads from translation planes. That is, we would require $5+6\cdot 4+12\cdot 3+8\cdot 2$ n-spreads.

In this way, beginning from a tk-dimensional vector space over $GF(q)$, then we may obtain a partition of the vector space by subspaces of dimensions n_i for any divisor n_i of k from n_i-spreads arising from translation planes. Of course, given any k-spread, choose any particular k-subspace and find a n_i-spread on that k-subspace. This simple idea will also produce partitions of vector spaces with subspaces of many dimensions. However, the construction method that we give here does not involve such refinements of the individual k-spaces.

Suppose that we would wish to find a finite vector space partition by subspaces of dimensions n_i, for $i = 1, 2, \ldots, z$. Then take the product $\prod_{i=1}^{z} n_i = k$. Then take any vector space V_{tk} of dimension tk over $GF(q)$. Our construction then gives a partition of of V_{tk} by subspaces not only of dimension n_i, but also of dimension m for any divisor of k.

3.1 The Formal Algorithmic Construction

Let V be a vector space of dimension tk over $GF(q)$. Initially consider the vectors of V as $(x_1, x_2, \ldots, x_t)$, where the x_i are k-vectors over $GF(q)$. Let $\mathcal{P}$ denote the partition of the non-zero vectors into the $j = 0$-sets. Let $\mathcal{S}_0$ denote the set of $t - 1 = 0$-sets and choose an arbitrary partition $\mathcal{Z}$ of the set of all $j = 0$-sets of $\mathcal{P} - \mathcal{S}_0$. For each element W of $\mathcal{Z}$, choose a divisor k_W of k. Suppressing the 0's, assume that in W, we have a $j = 0$-set J, say with $j = \widehat{j}$, with vectors represented by $(x_1, x_2, \ldots, x_{t-j})$. Let $k_W n_W = k$, and let $x_i = (x_{i1}, x_{i2}, \ldots, x_{in_W})$, where the x_{ij} are k_W-vectors. Noting that $n_W - 1$ of the x_{ij} could be zero without x_i being zero, we further partition J into $j = 0$-sets of k_W-vectors. So, in $(x_{11}, \ldots, x_{1n_W}, x_{21}, \ldots, x_{2n_W}, \ldots, x_{t-\widehat{j}1}, \ldots, x_{t-\widehat{j}n_W})$, then per $(x_{i1}, x_{i2}, \ldots, x_{in_W})$, it is possible to have the zero k_W-vector in any of 0 through $n_W - 1$ times. The number of ways of choosing zero k-vectors any of $s = 1, 2, \ldots, (t - \widehat{j})$ times is $\sum_{s=1}^{(t-\widehat{j})} \binom{(t-\widehat{j})}{s}$, whereas the number of ways of choosing zero k_W-vectors any of $r = 0, 1, \ldots, n_W(t - \widehat{j})$ times is $\sum_{r=0}^{n_W(t-\widehat{j})} \binom{n_W(t-\widehat{j})}{r}$, so the number of total $j = 0$-sets is

$$\sum_{r=0}^{n_W(t-\widehat{j})} \binom{n_W(t-\widehat{j})}{r} - \sum_{s=1}^{(t-\widehat{j})} \binom{(t-\widehat{j})}{s},$$

and the length of a $j = 0$-set (the number of nonzero entries of k_W-vectors in the set) varies from $(t - \widehat{j})$ to $n_W(t - \widehat{j})$. For a set of length N, we choose $N - 1$, k_W-spreads arising from translation planes so for a $j = 0$-set $(z_1, z_2, \ldots, z_N)$, where the z_i are k_W-vectors, we write this set in the form $(z_1, z_1 M_1, z_2 M_2, \ldots$ $\ldots, z_N M_{N-1})$, where the M_i are k_W-spreads $\mathcal{M}_i$.

We are partitioning $(q^k-1)^{(t-\widehat{j})}$ non-zero vectors in pieces of size $(q^{k_W}-1)$ so we consider

$$((q^{k_W}-1)+1)^{n_W}-1)^{(t-\widehat{j})} = \left(\sum_{i=1}^{n_W}\binom{n_W}{i}(q^{k_W}-1)^i\right)^{(t-\widehat{j})},$$

from which the total number of k_W-spreads required may be easily determined.

3 Theorem. *I. Let $\mathcal{P}$ denote the partition of the non-zero vectors into the $j=0$-sets. Let $\mathcal{S}_0$ denote the set of $t-1=0$-sets and choose an arbitrary partition $\mathcal{Z}$ of the set of all $j=0$-sets of $\mathcal{P}-\mathcal{S}_0$.*

II. For each element W of the partition $\mathcal{Z}$, choose a divisor k_W of k and if the length of a given $j=0$-set of W is N, choose $N-1$ k_W-spreads arising from translation planes (the spreads need not be distinct).

III. We have then constructed a partition of the tk-dimensional vector spaces into t, $(t-1)=0$-sets $(x_1,0,0,0,\ldots,0)$, $(0,x_2,0,\ldots,0)$, $\ldots$, $(0,0,\ldots,0,x_t)$, which are k-spaces and then partitioned the remaining vectors in $j=0$-sets of k_W-spaces for any collection $\{k_W; W\in\mathcal{Z}\}$.

4 Remark. Note that our construction does not partition individual k-spaces of a k-spread into k_W-subspaces so may be regarded as a completely non-trivial method of obtaining partitions of vector spaces into k_W-subspaces.

4 Fundamental Groups

In this section, we show that all k-spreads defining translation planes admit collineation groups fixing $x=0, y=0$ and $y=x$, and hence whose elements have the form $(x,y)\to(xA,yA)$ generate an isomorphic group in our constructed (t,k)-spread. The idea is simply to identify the lines $y=0, y=0$ in the spreads and show that the group elements $(x_1,x_2,\ldots,x_t)\to(x_1A,x_2A,\ldots,x_tA)$ generate a group in the (t,k)-spread.

5 Theorem. *Referring to Theorem 1, assume that each original k-spread admits a collineation of the form $(x,y)\to(xA,yA)$, where A is a $k\times k$, matrix. Identify $x=0, y=0$ in each spread. Then $(x_1,x_2,\ldots,x_t)\to(x_1A,x_2A,\ldots,x_tA)$ is a collineation of the constructed (t,k)-spread.*

Proof. The defined mapping is a bijective mapping on the vectors that maps $\mathcal{M}_i$ to $A^{-1}\mathcal{M}_iA$, so is a collineation of the new k-spread. $\boxed{QED}$

6 Remark. We now generalize the above theorem to our more general construction as follows. Whenever we take a divisor k_W of k, we assume that A is a diagonal $n_W\times n_W$ matrix with a $k_W\times k_W$ matrix A_{k_W} on the diagonal. If

s is the smallest divisor of k that we use in the general construction then this merely says that $(z_1, z_2, \ldots, z_{ts}) \to (z_1 A_s, z_2 A_s, \ldots, z_{ts} A_s)$ can be used to define a collineation of the constructed generalized spread (partition of the associated vector space).

5 Lifting and Twisting

The idea of 'algebraically lifting' a translation plane of order q^2 with spread in $PG(3, q)$ is generalized to lifting a 2-spread in a $z2$-dimensional subspace. By this technique, a translation plane of order q^4 is constructed (lifted) with spread in $PG(3, q^2)$. The reader is directed to 35.6 of the Handbook [12] for the precise definition. The main points are that there is an elation group of order q^2 and a Baer group of order $q + 1$ in the lifted translation plane. It turns out that it is possible to use the Baer group in combination with the kernel homology group of order $q^2 - 1$ to construct a group of order $q^2 - 1$, containing a scalar group of order $q - 1$, which defines a field K_R isomorphic to $GF(q^2)$ when adjoining the zero mapping and where the associated vector space becomes a K_R-space. Such a group is called a 'retraction' group and when forming the projective 3-space over K_R defined by the lattice of K_R-vector subspaces, there is a constructed subgeometry partition. Specifically, there is a partition of $PG(3, q^2)$ by subgeometries isomorphic to $PG(3, q)$ and $PG(1, q^2)$. The subgeometries isomorphic to $PG(3, q)$ (respectively $PG(1, q^2)$) correspond to orbits of length $q + 1$ (respectively 1).

Let π be a translation plane of order q^2 with spread in $PG(3, q)$ (a $(2, 2)$-spread). The spread for the lifted translation plane π^L of order q^4 has the general form

$$x = 0, y = x \begin{bmatrix} u & F(t) \\ t & u^q \end{bmatrix}, \text{ for all } u, t \in GF(q^2),$$

where F is a function defined by the functions defining the original spread for π. In Jha and Johnson [6], it has been shown that the previously mentioned 'retraction group' has the following form:

$$\left\langle \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a^q & 0 & 0 \\ 0 & 0 & a^q & 0 \\ 0 & 0 & 0 & a \end{bmatrix} ; a \in GF(q^2)^* \right\rangle .$$

Now we shall 'twist' the representation by the mapping $(x, y) \to (x, y \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix})$

to change the lifted spread to the form

$$x = 0, y = x \begin{bmatrix} F(t) & u^q \\ u & t \end{bmatrix}, \text{ for all } u, t \in GF(q^2),$$

and with the retraction group now represented in the form

$$\left\langle \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a^q & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a^q \end{bmatrix}; a \in GF(q^2 \right\rangle.$$

Furthermore, there is a Baer group of order $q+1$ which originally had the form

$$\left\langle \begin{bmatrix} e & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e \end{bmatrix}; e \text{ has order dividing } q+1 \right\rangle.$$

Hence, now there is a diagonal group with diagonal elements $h = \begin{bmatrix} e & 0 \\ 0 & 1 \end{bmatrix}$. If $g = \begin{bmatrix} a & 0 \\ 0 & a^q \end{bmatrix}$ then we have the representation of the group elements as $(x, y) \to (xg, yg)$ and $(x, y) \to (xh, yh)$.

We therefore may take any vector space V_{z2} of dimension $z2$ over $GF(q)$ and construct a 2-spread by the use of $\sum_{j=0}^{z-2} \binom{z}{j}(z - j - 1) = N_z$ spreads in $PG(3, q)$. Now we may algebraically lift each of these N_z spreads to obtain N_z spreads in $PG(3, q)$. It is important to note that the algebraic lifting process is really defined on the non-zero matrices so the $x = 0, y = 0$ components of the original translation plane with spread in $PG(3, q)$ can be essentially ignored in the construction.

When we do this, we change the $(z - 1) = 0$-sets so that $(x_1, 0, \ldots, 0)$ now assumes that $x_1 \in GF(q^2)$. The vector space becomes $z2$-dimensional over $GF(q^2)$ and $z4$-dimensional over $GF(q)$.

Therefore, if we twist the lifted spreads as above then our constructed 2-spread in the $z2$-dimensional vector space will admit the group whose elements are: $(x_1, x_2, \ldots, x_z) \to (x_1g, x_2g, \ldots, x_zg)$, where the x_i are 2-vectors over $GF(q^2)$. We note that the group fixes a Baer subplane componentwise of each of the N_z spreads used in the construction and hence fixes $q^2 - 1$ components other than $x = 0, y = 0$, in each spread. How this works in the $z2$-dimensional constructed 2-spreads is that there are $q^2 - 1$ non-zero vectors fixed in each component. For example, if we would have a resulting fixed component of the form

$$y = (x_1 \begin{bmatrix} F_1(t) & u^q \\ u & t \end{bmatrix}, x_1 \begin{bmatrix} F_2(s) & s^q \\ s & s \end{bmatrix}, \ldots, x_1 \begin{bmatrix} F_{z-j-1}(h) & r^q \\ r & h \end{bmatrix})$$

then the group G would fix each of the spread components where $u = w = \ldots = r = 0$. Then letting $x_1 = (0, x_{12})$, for x_{12} in $GF(q^2)$, we see that that we have the points $(0, x_{12}, 0, x_{12}t, 0, x_{12}s, \ldots, 0, x_{12}h)$ (again suppressing the zero 2-vectors). What this means is that the group fixes a set $(q^{2z} - 1)/(q^2 - 1)$ of the total $(q^{4z} - 1)/(q^4 - 1)$ components and there is a corresponding group B of order $q + 1$ that fixes all vectors $(x_1, x_2, \ldots, x_z)$, where $x_i = (x_{i1}, x_{i2})$ and $x_{i1} = 0$ for all $i = 1, 2, \ldots, t$ and hence fixes a vector subspace pointwise of dimension $2z$. In a vector space of dimension $4z$, we call any subspace of dimension $2z$ a 'Baer subspace'. Let $FixB$ denote the Baer subspace pointwise fixed by the group B. So, taking any component $y = (x_1 \begin{bmatrix} F_1(t) & 0 \\ 0 & t \end{bmatrix}, x_1 \begin{bmatrix} F_2(s) & 0 \\ 0 & s \end{bmatrix}, \ldots, x_1 \begin{bmatrix} F_{z-j-1}(h) & 0 \\ 0 & h \end{bmatrix})$ in a typical $j = 0$-set, then this component contains $(q^2-1)+1$ (the zero vector) points fixed by B. More generally, the components of intersection intersect $FixB$ in 1-dimensional $GF(q^2)$-subspaces.

Hence, B, a $2z$-dimensional vector space, is partitioned by 1-dimensional $GF(q^2)$-subspaces and may be considered partitioned by 2-spaces over $GF(q)$.

So, in general a component fixed the group in the z-$(2, 2)$-spread has the form $y = (x_1 \begin{bmatrix} F_1(t) & 0 \\ 0 & t \end{bmatrix}, x_1 \begin{bmatrix} F_2(t) & 0 \\ 0 & t \end{bmatrix}, \ldots, x_1 \begin{bmatrix} F_{z-j-1}(t) & 0 \\ 0 & t \end{bmatrix})$ in a typical $j = 0$-set and there are $(q^2 - 1)^{z-j-1}$ such elements. That is, In a typical $j = 0$-set, we have $(q^4 - 1)^{z-j-1}$ components, which means that we have

$$(q^4 - 1)^{z-j-1} - (q^2 - 1)^{z-j-1} = (q^2 - 1)^{z-j-1}((q^2 + 1)^{z-j-1} - 1)$$

remaining components in orbits of lengths $q+1$. The retraction group G produces a $GF(q^2)$-field F and we have a $4z$-dimensional F-vector space fixing $(q^{2z} - 1)/(q^2 - 1)$ components and and having orbits of length $q + 1$ on the remaining $(q^{4z} - 1)/(q^4 - 1) - (q^{2z} - 1)/(q^2 - 1)$ components and having.

$$\begin{aligned}
&(q^{4z} - 1)/(q^4 - 1) - (q^{2z} - 1)/(q^2 - 1) \\
&= (1 + q^4 + q^8 + \cdots + q^{4(2z-1)}) - (1 + q^2 + \cdots + q^{2(2z-1)}) \\
&= (q^4 - q^2) + (q^8 - q^4) + \cdots + (q^{4(2z-1)} - q^{2(2z-1)}) \\
&= q^2(q^2 - 1) + q^4(q^4 - 1) + \cdots + q^{2(2z-1)}(q^{2(2t-1)} - 1),
\end{aligned}$$

which is divisible by $q+1$.

Hence, we have the following theorem.

7 Theorem. *Choose any set of* $\sum_{j=0}^{z-2}\binom{z}{j}(z-j-1)=N_z$ *spreads in* $PG(3,q)$*. Algebraically lift and twist each of these spreads. Apply the* 2*-spread construction to construct a* 2*-spread over* $GF(q^2)$ *of an* $8z$ *dimensional* $GF(q)$*-space, with*

$$(q^{4z}-1)/(q^4-1)$$

2*-spaces over* $GF(q^2)$*.*

(1) This 2*-spread over* $GF(q^2)$ *admits a retraction group* G *of order* q^2-1*, such that the union of the zero vector gives a field* F *over which the ambient space is an* F*-space.*

(2) The group G *fixes exactly* $(q^{2z}-1)/(q^2-1)$ *components. If* K^* *denotes the kernel homology group of order* q^2-1*, then in* GK^**, there is a group of order* $q+1$ *that fixes pointwise a subspace of dimension* $4z$ *over* $GF(q)$*. We call this a 'Baer subspace'.*

(3) Since the group G *contains the scalar homology group of order* $q-1$*, we may 'retract' the spread to produce a subgeometry partition of* $PG(4z-1,q^2)$ *by subgeometries isomorphic to* $PG(3,q)$ *and* $PG(1,q^2)$*. In particular, there are exactly*

$$(q^{2z}-1)/(q^2-1)$$

$PG(1,q^2)$*'s and*

$$\left((q^{4z}-1)/(q^4-1)-(q^{4t}-1)/(q^2-1)\right)/(q+1)$$

$PG(3,q)$*'s.*

(4) We may derive any of the 2*-spreads by the subspace given by*

$$x=0, y=0, y=x\begin{bmatrix}0 & u^q\\ u & 0\end{bmatrix}; u\in GF(q^2)^*.$$

We note that the group G *leaves this spread invariant, fixes two components and has* $(q-1)$ *orbits of length* $q+1$*. The group fixes two Baer subplanes of the net and has* $q-1$ *orbits of length* $q+1$*. To re-represent this spread, we obtain the derived spread in the form:*

$$x=0, y=0, y=x^q\begin{bmatrix}0 & u\\ u & 0\end{bmatrix}; u\in GF(q^2)^*.$$

The spread

$$x = 0, y = x \begin{bmatrix} u & F(t) \\ t & u^q \end{bmatrix}, \textit{ for all } u, t \in GF(q^2),$$

when derived and twisted is

$$x = 0, y = 0, y = x \begin{bmatrix} F(t) - t^{-1}u^{q+1} & -ut^{-1} \\ t^{-1}u^q & t^{-1} \end{bmatrix}, \textit{ for all } u, t \neq 0 \in GF(q^2),$$

$$y = x^q \begin{bmatrix} 0 & u \\ u & 0 \end{bmatrix}; u \in GF(q^2)^*, \textit{ where } x^q \textit{ means apply the automorphism}$$

to the elements of the 2*-vector.*

The group G *acts on the derived spread and has the same form. The group maps*

$$y = x \begin{bmatrix} F(t) - t^{-1}u^{q+1} & -ut^{-1} \\ t^{-1}u^q & t^{-1} \end{bmatrix} \textit{ to } y = x \begin{bmatrix} F(t) - t^{-1}u^{q+1} & -(a^{q-1}u)t^{-1} \\ t^{-1}(a^{q-1}u)^q & t^{-1} \end{bmatrix}.$$

Hence, any of these general spreads also admit the retraction group G *and produce additional examples of subgeometry partitions of* $PG(4t-1, q^2)$ *by* $PG(3, q)$*'s and* $PG(1, q^2)$*'s.* G *acts as a 'Baer group' of the 'multiply-derived' spreads.*

Note that if we derive any of the spreads, we obtain a set of 4*-dimensional subspaces over* $GF(q)$*. Hence, the derivation of any spread produces a* 4*-spread in the* $8z$*-dimensional space over* $GF(q)$*.*

(5) *If we choose the* N_z *spreads in* $PG(3, q)$ *to be mutually non-isomorphic then the lifted and twisted spreads are mutually non-isomorphic. Therefore, the collineation group of the constructed* 2*-spreads or* 4*-spreads must leave invariant each of the* 2*-spreads (*4*-spreads).*

(6) *To construct the spreads, we use the notion of a* $j = 0$*-set. We may choose any ordering and the constructed* 2*-spreads will not be isomorphic, provided the spreads are mutually non-isomorphic. Hence, we obtain* $N_z!$ 2*-spreads of dimension* $8z$ *over* $GF(q)$*. Then for each of these spreads, we may choose any set to derive. Hence, there are*

$$2^{N_z!}$$

possible ways to do this, for each of the $N_z!$ *spreads. Therefore, we obtain* $N_z!2^{N_z!}$ *mutually non-isomorphic spreads, all of which give rise to subgeometry partitions of* $PG(4z-1, q^2)$ *by subgeometries isomorphic to* $PG(3, q)$*'s and* $PG(1, q^2)$*'s.*

6 More Subgeometry Partitions

In the previous section, we have used our fundamental construction technique to construct subgeometry partitions of $PG(4z-1,q^2)$ by $PG(3,q)$'s and $PG(1,q^2)$'s. Our procedure for the construction of subgeometry partitions is very general and we shall give more applications here and in the next section. The more general theorem relating to our theory is the following and constructs subgeometry partitions of $PG(zk-1,q^2)$ by $PG(k-1,q)$'s and $PG(k/2-1,q^2)$'s.

8 Theorem. *Assume that we have* $\sum_{j=0}^{z-2}\binom{z}{j}(z-j-1)=N_z$ *k-spreads of an $2k$-dimensional vector space over $GF(q)$, for k even, each of which admits a retraction group G of order q^2-1 leaving invariant at least two components, which we identify as $x=0,y=0$. Let M_i for $i=1,\dots,N_z$ denote a matrix spread set of non-singular $k\times k$ matrices, where the difference of any distinct two matrices is non-singular. Finally, assume that the group G has elements $(x,y)\to(xg,yg)$, whether or not G fixes three components of the individual k-spreads. Then G acts on the constructed k-spread of the associated zk-vector space over $GF(q)$. Furthermore, there are associated subgeometry partitions of $PG(zk/2-1,q^2)$ by subgeometries isomorphic to $PG(k-1,q)$ and $PG(k/2-1,q^2)$.*

Proof. This result follows fairly directly by rereading the proofs of the previous section. $\boxed{QED}$

7 Using the Extended André Spreads

In the previous two sections, we have constructed subgeometry partitions using a retraction group of order q^2-1 and our subgeometries are restricted to two possible types. The general theory of partitions of vector spaces is more generally valid for larger order groups, groups of order q^s-1, but the partition objects, while isomorphic to projective spaces are not always strictly subgeometries. The more general objects are called 'quasi-subgeometry partitions' and this theory is developed in Johnson [9], [7], [8]. Furthermore, it is possible to find quasi-subgeometry partitions that are actually subgeometry partitions by stringing together certain André spreads (arising from translation planes). Also, Johnson [10] used similar ideas as presented here but only using André and Desarguesian spreads. The following theorem illustrates what was done in this previous work. The reader is directly to the article for background.

9 Theorem. *[Johnson [10], [11]] Let the lattice of vector subspaces of Σ over $K_s^+\simeq GF(q^s)$ be denoted by $PG(rn-1,q^s)$. Considering elements $(x_1,x_2,\dots$*

$\ldots, x_z)$, for x_i in $GF(q^n)$, take any

$$A^{(\lambda_1,\ldots,\lambda_{r-j-1})}_{(n_1,\ldots,n_{r-j-1})} = \\ = \left\{ y = (x_1^{*q^{\lambda_1}} n_1 d^{1-q^{\lambda_1}}, \ldots, x_1^{*q^{\lambda_{r-j-1}}} n_{r-j-1} d^{1-q^{\lambda_{r-j-1}}}); d \in GF(q^{sn})^* \right\}.$$

There are

$$\Big((q^{sn}-1)/(q^{(\lambda_1,\ldots,\lambda_{r-j-1})}-1)\Big) \,/\, \Big((q^s-1)/(q^{(\lambda_1,\lambda_2,\ldots,\lambda_{r-j-1},s)}-1)\Big)$$

component orbits of length

$$(q^s-1)/(q^{(\lambda_1,\lambda_2,\ldots,\lambda_{r-j-1},s)}-1)$$

under K_s.

Let $s^* = (\lambda_1, \lambda_2, \ldots, \lambda_{r-j-s})$. Assume that within $(\lambda_1, \lambda_2, \ldots, \lambda_{r-j-1})$, there are elements λ_l and λ_k such that $\lambda_l = s^*$ and $\lambda_k = s^*(s-1)$. If $s = 2$, λ_l and λ_k are equal and it is then possible that $r-j-1 = 1$. Each such orbit becomes a projective subgeometry of $PG(rn-1, q^s)$, isomorphic to $PG(sn/(\lambda_1, \lambda_2, \ldots, \lambda_{r-j-1}, s) - 1, q^{(\lambda_1,\lambda_2,\ldots,\lambda_{r-j-1},s)})$.

What is going on in the previous theorem is that the group arises from a Desarguesian kernel homology group and acts on any André plane. Then André and Desarguesian spreads are used in the manner of our more general construction procedure to construct a variety of k-spreads in zk-dimensional vector spaces. However, what could be done more generally is to have translation planes of order q^k admitting a group of order q^s-1 that acts like a retraction group and would then construct a quasi-subgeometry partition of a corresponding suitable projective space. Then choose $\sum_{j=0}^{z-2} \binom{z}{j}(z-j-1) = N_z\ (2,k)$-spreads admitting the same retraction group, where it is assumed the group elements fix two components, which we identify in each spread by $x = 0, y = 0$. Now form the k-spread in the zk-dimensional vector space and hope to prove that in this more general context the group action on a more general type of component might show that the quasi-subgeometry obtained in the zk-space is actually a subgeometry. One way of guaranteeing that this will work is to include in any component at least two André spreads of the correct type according to Theorem 9. Of course, now the question is besides André planes of order q^n that always admit a putative retraction group of order any order q^z-1, for z dividing n, what other types of translation planes admit such groups? It turns out that there are non-André net replacements of André nets due to Johnson [9] producing translation planes of order q^n that can be configured to admit a particular group of order q^z-1 that satisfies the retraction conditions.

10 Theorem. *Using Theorem 3 for the construction of vector space partitions of dimension tn from translation planes of order q^n and admitting a collineation group of order $q^z - 1$ for z dividing n, such that by adjoining the zero vector, a field F is obtained such the ambient vector space is a F-space then by taking orbits of different lengths of the various n-spaces, it is possible to obtain a very large number of true subgeometry partitions of $PG(tn/z - 1, q^z)$ by subgeometries isomorphic to $PG(w, q^b)$ for any set $\{b;\ b$ a divisor of $z\}$.*

Proof. As long as two André subspreads of the correct configuration is used per subspace in each $j = 0$-set with at least two spreads from translation planes, we will obtain a subgeometry partition, provided for the $j = 0$-sets with exactly one spread, we merely required that the group leaves the spread invariant componentwise (that is, we take the spreads included in $(t - 2) = 0$-sets to have this property and we may do this by extending the field F to a field isomorphic K to $GF(q^n)$ and taking the spread involved to be Desarguesian and coordinatized by K). $\boxed{QED}$

8 Focal-Spreads

Our main construction of k-spreads in sk-dimensional subspaces involved covering the so-called '$j = 0$-sets by k-spreads. We also gave a generalization of this process by which we constructed partitions of vector spaces by subspaces of various dimensions k_i. There is an alternative way to get to our k-spaces and along the way, we are able to construct 'focal-spreads' of various dimensions as well as interesting related vector-space partitions.

11 Definition. A 'focal-spread' is a partition of a vector space of dimension $t+k$ over $GF(q)$, $t \neq k$, where there is a unique subspace of dimension t, called the 'focus' of the partition and the remaining vectors are covered by a partial k-spread of degree q^t.

8.1 Focal-Spread Sets

Let B be a focal-spread of dimension $t + k$ over $GF(q)$ with focus L of dimension t. Fix any k-component M. We may choose a basis so that the vectors have the form

$$(x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_t).$$

Let $x = (x_1, x_2, \ldots, x_k)$ and $y = (y_1, y_2, \ldots, y_t)$, where the focus L has equation $x = 0 = (0, 0, 0, \ldots, 0)$ (k-zeros) and the fixed k-component M has equation $y = 0 = (0, 0, 0, \ldots, 0)$ (t-zeros). We note that $q^{t+k} - q^t = q^t(q^k - 1)$, which implies that there are exactly q^t k-subspaces in the focal-spread. Consider any

k-component N distinct from $y = 0$. There are k-basis vectors over $GF(q)$, which we represent as follows: $y = xZ_{k,t}$, where $Z_{k,t}$ is a $k \times t$ matrix over $GF(q)$, whose k rows are a basis for the k-component. Hence, we obtain a set of q^t k-components, which we also represent as follows: Row 1 shall be given by $[u_1, u_2, \ldots, u_t]$, as u_i vary independently over $GF(q)$. Then the rows $2, \ldots, k$ have entries that are functions of the u_i.

Let $f_{i,j}(u_1, \ldots, u_t)$, denote the entry in the i, j position.

12 Theorem. *[Jha and Johnson [2]] Let V_{t+k} be a $t+k$-dimensional vector space over $GF(q)$ and let S be a set of $q^t - 1$ $k \times t$ matrices of rank k such that the difference of any two distinct matrices also has rank k. Then there is an associated focal-spread constructed as*

$$x = 0, y = 0, y = xM;\ M \in S,$$

where x is a k-vector and y is a t-vector over $GF(q)$, where the focus is $x = 0$.

Therefore, we have:

13 Theorem. *[Jha and Johnson [2]]The set of focal-spreads of dimension $t+k$, for $t \neq k$ and focus of dimension t is equivalent to the set of sets of $q^t - 1$ $k \times t$ matrices of rank k, whose distinct differences are also of rank k.*

We may always choose one k-space to have 1's in the (i,i), position and 0's elsewhere in the $k \times t$ matrices.

8.2 Beutelspacher's Construction

Let V_{t+k} be a vector space of dimension $t + k$ over $GF(q)$ for $t > k$ and let L be a subspace of dimension t. Let V_{2t} be a vector space of dimension $2t$ containing V_{t+k} ($t > k$ required here) and let F_t be a t-spread containing L. There are always at least Desarguesian t-spreads with this property. Let M_t be a component of the spread F_t not equal L. Then $M_t \cap V_{t+k}$ is a subspace of V_{t+k} of dimension at least k. But, since M_t is disjoint from L, the dimension is precisely k and we then obtain a focal-spread with focus L. This construction is due to Beutelspacher [1]

From the matrix spread set point of view, this means that the $k \times t$ matrices for a focal-spread have been extended to a set of $t \times t$ matrices of rank t whose differences are also of rank t. We therefore ask the following questions:

Can any focal-spread be extended to a spread set for a translation plane?

In this section, we show how to construct chains of focal-spreads of dimension zk.

However, before we give our construction, we wish to generalize the notation of a focal-spread. Our definition of focal-spreads requires three properties: (1) there is a partition with a unique subspace of dimension t, (2) the remaining vectors are partitioned by a partial k-spread and (3) the dimension of the vector space is $t+k$. In previous work, all focal-spreads were strongly related to translation plane spreads, as noted by the Buetelspacher construction, but in our general definition, we shall not require condition (3).

14 Definition. Let V be a vector space of dimension $t+k$ that admits a partition of exactly one subspace of dimension t' and the remaining vectors are partitioned by a partial k-spread. Then this partition shall be called a '(t,t',k)-focal-spread' and the unique subspace of dimension t' shall be called the 'focus. When $t=t'$, we shall use the term 'focal-spread' as before. Note that $q^{t+k}-q^{t'}=Z(q^k-1)$, so for $t\geq t'$, we must have $q^{t'}(q^{t-t'}-1)=Z(q^k-1)$, implying that k divides $t-t'$.

Let V be a vector space of dimension sk, where vectors are written in the form $(x_1,x_2,\ldots,x_s)$, where the x_i are k-vectors. Note that $x_1=0$ is an $(s-1)k$-dimensional $GF(q)$ space. Let $y_1=(x_1,\ldots,x_s)$, so that we have also the representation (x_1,y_1). Choose $s-1$ k-spreads $\mathcal{M}_i^1$, $i=2,\ldots,s$ (sets of q^k $k\times k$ matrices including the zero matrix, where the difference of any two is identically zero or nonsingular). Then we claim that the

$$(x_1,y_1=(x_1M_{12},x_1M_{13},\ldots,x_1M_{1s}));$$

x_1 is a k-vector over $GF(q)$, which we call

$$y_1=(x_1M_{12},x_1M_1,\ldots,x_1M_{1s}),$$

for $M_{1i}\in\mathcal{M}_i^1$, $i=2,\ldots,s$, is a partial k-spread. To see this, simply note that for fixed M_i, $y_1=(x_1M_{11},x_1M_{12},\ldots,x_1M_{1s})$ is clearly a k-dimensional $GF(q)$-subspace. Since we have an intersection of two such subspaces if and only if there exists a k-vector x_1 such that say

$$y_1=(x_1M_{12},x_1M_{12},\ldots,x_1M_{1s})=(x_1M_{12}',x_1M_{12}',\ldots,x_1M_{1s}'),$$

then we have

$$x_1(M_{1i}-M_{1i}')=0$$

for all $i=2,\ldots,s$. Since $\mathcal{M}_i^1$ is a spread for all i, it follows that $M_{1i}-M_{1i}'$ is non-singular for $M_{1i}\neq M_{1i}'$, which implies that $x_1=0$ and that the only intersections is $(0,0,\ldots,0)$. Hence, this proves that a partial k-spread of degree $q^{(s-1)k}$.

This proves the following theorem.

15 Theorem. *Let V be an sk-dimensional vector space over $GF(q)$, where $s-1>k$ and let $\left\{\mathcal{M}_i^1;i=2,\ldots,s\right\}$ be a set of $s-1$ k-spreads (including the* *zero matrix in each k-spread). Represent the vector space by* (x_1, y)*, where* x_1 *is a k-vector and* y *is a* $(s-1)k$*-vector. Then*

$$x_1 = 0, y_1 = (x_1M_{12}, x_1M_{13}, \ldots, x_1M_{1s});\ M_{1i} \in \mathcal{M}_i^1,\ i = 2, \ldots, s$$

is a focal-spread of dimension $(s-1)k + k$*, with focus of dimension* $(s-1)k$*.*

Now consider the vectors of $x_1 = 0$, $(x_2, x_3, \ldots, x_s)$. Note that $x_2 = 0$ is a $(s-2)k$-dimensional subspace of $x_1 = 0$ and let $y_2 = (x_3, \ldots, x_s)$, with the same notation as before. Choose $s-2$ k-spreads $\mathcal{M}_i^2$, for $i = 3, \ldots, s$ and construct the following focal-spread of $x_1 = 0$:

$$x_2 = 0,\ y_2 = (x_2M_{23}, x_2M_{24}, \ldots, x_2M_{2s})$$

of dimension $(s-2)k + k$ with focus of dimension $(s-2)k$ with partial k-spread of dimension $q^{k(s-2)}$. Assume that $s-2 > k$. What this means is that we have constructed a partition of a vector space with exactly one subspace of dimension $(s-2)k$ and a partial k-spread of degree $q^{(s-1)k} + q^{(s-2)k}$. In terms of counting the partition numbers, we have

$$q^{sk} - 1 = q^{(s-2)k} - 1 + q^{(s-1)k}(q^k - 1) + q^{(s-2)k}(q^k - 1).$$

Therefore, we have constructed a $((s-1)k, (s-2)k, k)$-focal-spread.

$$\begin{aligned} x_1 &= x_2 = 0, \\ y_1 &= (x_1M_{12}, x_1M_{13}, \ldots, x_1M_{1s});\ M_{1i} \in \mathcal{M}_i^1,\ i = 2, \ldots, s, \\ &\quad \text{referring to } (x_1, y_1) \\ y_2 &= (x_2M_{23}, x_2M_{24}, \ldots, x_2M_{2s}); M_{2i} \in \mathcal{M}_i^2, i = 3, \ldots, s, \\ &\quad \text{referring to } (0, x_2, y_2). \end{aligned}$$

16 Remark. Clearly, the process may be repeated by the additional choice of $(s-3)$ k-spreads $\mathcal{M}_i^3$, $i = 4, \ldots, s$. If we wish to construct a $((s-1)k, (s-w)k, k)$-focal-spread, for $s - w > k$, we would choose

$$\begin{aligned} &(s-1) + (s-2) + (s-3) + \cdots + (s-w) \\ =\ &(s-1)(s-2)/2 - (s-w-1)(s-w-2)/2 \end{aligned}$$

spreads and continue the process described above. If we continue this process until we end with a $((s-1)k, 2k, k)$-focal-spread then the next step produces a k-spread of the sk-dimensional vector space. For the $2k$-dimensional subspace, we would use one k-spread in the next step. Hence, the number of spreads used in this method is $1 + 2 + \cdots + s - 1 = (s-1)s/2$.

17 Remark. In our main construction technique, we would have used additional spreads than given here. Furthermore, we could have modified the present construction process by separating out the $j = 0$-sets and using different sets of k-spreads.

To give an example of partitions possibly obtained illustrating the above remark, we recall again our general construction. Let $\mathcal{S}_t$ be an ordered sequence of $\sum_{j=0}^{t-2} \binom{t}{j}(t-j-1) = N_t$ $(2,k)$-spreads over $GF(q)$. We may represent each of the $(2,k)$-spreads , where we identify two common components $x = 0, y = 0$, and where x and y are k-vectors. Then if $\mathcal{M}_i$ is a k-spread, there is a set of $q^k - 1$ nonsingular matrices $M_{i,z}$, for $z = 1, 2, \ldots, q^k - 1$, $i = 1, 2, \ldots, N_t$, whose distinct differences are also non-singular. Hence, we represent the k-spread by $y = xM_{iz}$, for $z = 1, 2, \ldots, q^k - 1$. Consider the $j = 0$-sets for $j = 0, 1, 2, \ldots, t-2$, and assume an ordering for the N_t sets. For each such $j = 0$-set, for $j > 2$, we eliminate the zero elements and write vectors in the form $(x_1, x_2, \ldots, x_{t-j})$, where x_w for $w = 1, 2, \ldots, t-j$ are all non-zero vectors. We partition this set by

$$(x_1, x_1M_{1,z_1}, x_1M_{2,z_2}, x_1M_{3,z_3}, \ldots, x_1M_{t-j,z_{t-j}})$$

where M_{i,z_i} varies over $\mathcal{M}_i$, and where the indices z_i are independent of each other. By adjoining the zero vector, we then may consider

$$y = (x_1M_{1,z_1}, x_1M_{2,z_2}, x_1M_{3,z_3}, \ldots, x_1M_{t-j,z_{t-j}})$$

as a k-subspace for fixed M_{i,z_i}, for each $i = 1, 2, \ldots, t-j$. Then we obtain a spread of k-spaces of a tk-dimensional vector space over $GF(q)$ as a union of $j = 0$-spreads.

Now consider again the vectors $(x_1, x_2, \ldots, x_t)$, where the x_i are k-vectors. Write the space as (x_1, y_1) so that $x_1 = 0$ is a $(t-1)k$-vector space. Now pull out of the k-spread given just above by all subspaces that have the first component of a general vector as the zero k-vector. This forms a k-space partition of a $(t-1)k$-vector space. Now ignore this partition and consider simply this space as a $(t-1)k$-dimensional subspace in a putative focal-spread. Now take the remaining k-subspaces that are disjoint from $x_1 = 0$ to form the partial k-spread. Note by this method we obtain $\sum_{j=0}^{t-2} \binom{t}{j}(t-j-1) = N_t - \sum_{j=0}^{t-3} \binom{t-1}{j}(t-j-2) = N_{t-1}$ $(2,k)$-spreads required for the construction. For example, we would have used $t-1$ spreads by the method first given in this section, whereas we see that by a modification of the ideas of the first section, we would have used $N_t - N_{t-1}$. For example, for $t = 4$, then we would have used 8 spreads from translation planes by the first method mentioned in this section and $17 - 5 = 12$ spreads from translation planes by the modified method.

18 Theorem. *By a modification of the ideas of Theorem 1, it is possible to construct focal-spreads of dimension* $(t-1)k+k$*, with focus of dimension* $(t-1)k$ *as well as* $((t-1)k,(t-w)k,k)$*-focal spreads with focus of dimension* $(t-w)k$*, for any* $(t-w)>k$*.*

19 Theorem. *By modifying the ideas of Theorem 3, it is possible to construct partitions of* tk*-dimensional vector spaces by subspaces of varying degrees* k_i *dividing* k *and one subspace of dimension* $(t-w)k$*.*

Proof. One way to see this last theorem is to choose k_i-spreads at the various stages say when partitioning $x_1=0$. $\boxed{QED}$

9 Irreducibility

Finally, we offer a few words on what makes the k-spreads constructed in this article 'new' in some sense. Given a rk-spread, it is trivial to obtain a k-spread simply by finding a k-spread of each component. We shall say that a k-spread is 'irreducible' if it cannot be obtained in this way. All of the k-spreads of tk-dimensional vector spaces that are constructed in this article are almost certainly irreducible. If a less restrictive definition of irreducible is taken, perhaps say that the k-spread cannot be obtained from any other vector space partition regardless of dimensions of the various subspaces in the partition then our k-spreads would not then be considered irreducible as for example, they could be obtained from certain $((t-1)k,(t-w)k,k)$-focal-spreads. Since the ideas of focal-spreads and their generalizations are important in various contexts so we will consider the connections with such partitions to our k-spreads to be of independent interest and allow the definition of an irreducible k-spread to be one that cannot be obtained from another z-spread by refining each component. The reader is directly to the authors' work [4], [5], [3], [2] for connections to the present article.

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