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Normability of probabilistic normed spaces

Bernardo Lafuerza–Guillén [i]
Department of Statistics and Applied Mathematics,
University of Almería
blafuerz@ual.es

José Antonio Rodríguez–Lallena
Department of Statistics and Applied Mathematics,
University of Almería
jarodrig@ual.es

Carlo Sempi
Department of Mathematics "Ennio De Giorgi",
University of Salento
carlo.sempi@unisalento.it

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Abstract. Relying on Kolmogorov's classical characterization of normable topological vector spaces, we study the normability of those probabilistic normed spaces that are also topological vector spaces and provide a characterization of normable Šerstnev spaces. We also study the normability of other two classes of probabilistic normed spaces.

Keywords: probabilistic normed space, normability, Šerstnev space, t-norm, probabilistic radius, $\mathcal{D}$-bounded set, topologically bounded set.

MSC 2000 classification: primary 54E70, secondary 46S50

1 Introduction

Probabilistic normed spaces were introduced by Šerstnev in [15]; their definition was generalized in [1], a paper that revived the study of these spaces. We recall the definition, the properties and the examples of probabilistic normed spaces that will be used in the following.

Let Δ be the space of distribution functions and $\Delta^+ := \{F \in \Delta \mid F(0) = 0\}$ the subset of *distance distribution functions* [12]. The space Δ can be metrized in several equivalent ways [11,13,16,17] in such a manner that the metric topology coincides with the topology of weak convergence for distribution functions. Here, we assume that Δ is metrized by the *Sibley metric* d_S, which is the metric

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denoted by d_L in [12]. We shall also consider the subset $\mathcal{D}^+ \subset \Delta^+$ of the *proper* distance distribution functions, i.e. those $F \in \Delta^+$ for which $\lim_{x\to+\infty} F(x) = 1$.

A *triangle function* [10,12] is a mapping $\tau : \Delta^+ \times \Delta^+ \to \Delta^+$ that is commutative, associative, nondecreasing in each variable and has ε_0 as identity, where ε_a $(a \leq +\infty)$ is the distribution function defined by

$$\varepsilon_a(t) := \begin{cases} 0, & t \leq a, \\ 1, & t > a. \end{cases}$$

Given a nonempty set S, a mapping $\mathcal{F}$ from $S \times S$ into Δ^+ and a triangle function τ, a *probabilistic metric space* (briefly a PM space) is the triple $(S, \mathcal{F}, \tau)$ with the following properties, where we set $F_{p,q} := \mathcal{F}(p, q)$,

(M1) $F_{p,q} = \varepsilon_0$ if, and only if, $p = q$;

(M2) $F_{p,q} = F_{q,p}$ for all p and q in S;

(M3) $F_{p,r} \geq \tau(F_{p,q}, F_{q,r})$ for all p, q, $r \in S$.

A *probabilistic normed space* (briefly a PN space) is a quadruple (V, ν, τ, τ^*), where V is a vector space, τ and τ^* are continuous triangle functions such that $\tau \leq \tau^*$ and ν is a mapping from V into Δ^+, called the *probabilistic norm*, such that for every choice of p and q in V the following conditions hold:

(N1) $\nu_p = \varepsilon_0$ if, and only if, $p = \theta$ (θ is the null vector in V);

(N2) $\nu_{-p} = \nu_p$;

(N3) $\nu_{p+q} \geq \tau(\nu_p, \nu_q)$;

(N4) $\nu_p \leq \tau^*\left(\nu_{\lambda p}, \nu_{(1-\lambda)p}\right)$ for every $\lambda \in [0, 1]$.

If ν satisfies (N2), (N3), (N4) and $\nu_\theta = \varepsilon_0$ (but not necessarily (N1)), then (V, ν, τ, τ^*) is said to be a *probabilistic pseudonormed space* (briefly, a PPN space). The pair (V, ν) is called a *probabilistic seminormed space* (PSN space for short) if ν satisfies (N1) and (N2).

Now we list several special classes of PN spaces (see [7] for details).

When there is a continuous t–norm T (see [3,12]) such that $\tau = \tau_T$ and $\tau^* = \tau_{T^*}$, where $T^*(x, y) := 1 - T(1 - x, 1 - y)$,

$$\tau_T(F, G)(x) := \sup_{s+t=x} T(F(s), G(t)) \quad \text{and} \quad \tau_{T^*}(F, G)(x) := \inf_{s+t=x} T^*(F(s), G(t))$$

the PN space $(V, \nu, \tau_T, \tau_{T^*})$ is called a *Menger* PN space, and is denoted by (V, ν, T). Recall that the maximum and minimum continuous t–norm are respectively given by $M(x, y) := \min\{x, y\}$ and $W(x, y) := \max\{x+y-1, 0\}$; and another important continuous t–norm is $\Pi(x, y) := xy$.

If a PN space (V, ν, τ, τ^*) satisfies the following condition

$$(\check{\mathrm{S}}) \qquad \forall\, p \in V \quad \forall\, \lambda \in \mathbf{R} \setminus \{0\} \quad \forall\, x > 0 \qquad \nu_{\lambda p}(x) = \nu_p \left(\frac{x}{|\lambda|} \right),$$

then it is called a *Šerstnev* PN space; the condition (Š) implies that the best-possible selection for τ^* is $\tau^* = \tau_M$, which satisfies a stricter version of (N4), namely

$$\forall\, \lambda \in [0, 1] \qquad \nu_p = \tau_M \left(\nu_{\lambda p}, \nu_{(1-\lambda)p} \right).$$

One speaks of an *equilateral* PN space when there is $F \in \Delta^+$ different from both ε_0 and ε_∞ such that, for every $p \neq \theta$, $\nu_p = F$, and when $\tau = \tau^* = \Pi_M$, which is the triangle function defined for G and H in Δ^+ by $\Pi_M(G, H)(x) := M(G(x), H(x))$. This equilateral PN space will be denoted by (V, F, Π_M).

Let $G \in \Delta^+$ be different from ε_0 and from ε_∞ and let $(V, \|\cdot\|)$ be a normed space; then, define, for $p \neq \theta$,

$$\nu_p(x) := G \left(\frac{x}{\|p\|} \right).$$

Then (V, ν, M) is a Menger PN space denoted by $(V, \|\cdot\|, G, M)$. This type of Menger PN spaces are known as *simple* PN spaces. Observe that simple PN spaces belongs to the class of Šerstnev spaces. In the same conditions, if ν is defined by

$$\nu_p(x) := G \left(\frac{x}{\|p\|^\alpha} \right),$$

with $\alpha \geq 0$, then the pair (V, ν) is a PSN space called α*–simple* and it is denoted by $(V, \|\cdot\|, G; \alpha)$. The α–simple spaces can be endowed with a structure of PN space in a very general setting (G should be a continuous and strictly increasing function in $\mathcal{D}^+$, see [7]).

Let $(\Omega, \mathcal{A}, P)$ be a probability space, $(V, \|\cdot\|)$ a normed space and S a vector space of V–valued random variables (possibly, the entire space). For every $p \in S$ and for every $x \in \overline{\mathbf{R}}_+$, let $\nu \colon S \to \Delta^+$ be defined by

$$\nu_p(x) := P\{\omega \in \Omega \mid \|p(\omega)\| < x\};$$

then (S, ν) is called an E*–normed space* (briefly, EN space) with *base* $(\Omega, \mathcal{A}, P)$ and *target* $(V, \|\cdot\|)$. Every EN space (S, ν) is a PPN space under τ_W and τ_M. It is said to be *canonical* if it is a PN space under the same two triangle functions. In this latter case, it is a Šerstnev space.

See [6, 8, 14] for properties of PN spaces.

If (V, ν, τ, τ^*) is a PN space, a mapping $\mathcal{F}\colon V \times V \to \Delta^+$ can be defined through

$$\mathcal{F}(p, q) := \nu_{p-q}. \tag{1}$$

This function $\mathcal{F}$ makes $(V, \mathcal{F}, \tau)$ a PM space. Every PM space can be endowed with the strong topology, i.e., the topology generated by the *strong neighbourhoods*. For $(V, \mathcal{F}, \tau)$, the strong neighbourhoods are defined as follows: for every $t > 0$, the neighbourhood $N_p(t)$ at a point p of V is defined by

$$N_p(t) := \{q \in V \mid d_S(F_{p,q}, \varepsilon_0) < t\} = \{q \in V \mid \nu_{p-q}(t) > 1 - t\}\,.$$

It is known (see [12]) that $(V, \mathcal{F}, \tau)$, where $\mathcal{F}$ is defined by (1), and therefore (V, ν, τ, τ^*), is a Hausdorff space, and hence, a T_1 space; moreover, it is metrizable. But we do not know whether (V, ν, τ, τ^*) is normable.

2 PN spaces and topological vector spaces

A result from [2] can be rephrased for the purpose of the present paper in the following form:

1 Theorem. *[Alsina, Schweizer, Sklar] Every PN space (V, ν, τ, τ^*), when it is endowed with the strong topology induced by the probabilistic norm ν, is a topological vector space if, and only if, for every $p \in V$ the map from $\mathbf{R}$ into V defined by*

$$\lambda \mapsto \lambda\, p \tag{2}$$

is continuous.

It was proved in [2, Theorem 4] that, if the triangle function τ^* is Archimedean, i.e. if τ^* admits no idempotents other than ε_0 and ε_∞ [12], then the mapping (2) is continuous and, as a consequence, the PN space (V, ν, τ, τ^*) is a topological vector space.

The following theorem studies whether certain classes of spaces are topological vector spaces.

2 Theorem. (a) *No equilateral space (V, F, Π_M) is a topological vector space.*

(b) *A Šerstnev space (V, ν, τ) is a topological vector space if, and only if, the probabilistic norm ν maps V into $\mathcal{D}^+$ rather than into Δ^+, viz. $\nu(V) \subseteq \mathcal{D}^+$.*

(c) *A simple space $(V, \|\cdot\|, G, M)$ is a topological vector space if, and only if, G belongs to $\mathcal{D}^+$.*

(d) *If G is a distribution function different form ε_0 and ε_∞, then the α–simple space $(V, \|\cdot\|, G; \alpha)$ is a topological vector space, if, and only if, G belongs to $\mathcal{D}^+$.*

(e) *An EN space (S, ν) is a topological vector space if, and only if, ν_p belongs to $\mathcal{D}^+$ for every $p \in S$.*

Proof. Let θ denote the null vector of the vector space V. Since any PM space and, hence, any PN space, can be metrized, one can limit oneself to investigating the behaviour of sequences. Moreover, because of the linear structure of V, one can take $p \neq \theta$ and an arbitrary sequence (λ_n) with $\lambda_n \neq 0$ $(n \in \mathbf{N})$ such that $\lambda_n \to 0$ as n tends to $+\infty$.

(a) For every $n \in \mathbf{N}$, one has $\nu_{\lambda_n p} = F$ while $\nu_\theta = \varepsilon_0$. Therefore the map (2) is not continuous.

(b) If ν maps V into $\mathcal{D}^+$, then, for every $t > 0$, one has

$$\nu_{\lambda_n p}(t) = \nu_p\left(\frac{t}{|\lambda_n|}\right) \xrightarrow[n\to+\infty]{} 1,$$

whence the assertion. Conversely, if there exists at least one $p \in V$ such that $\nu_p \in \Delta^+ \setminus \mathcal{D}^+$, namely such that $\nu_p(x) \xrightarrow[x\to+\infty]{} \gamma < 1$, then, for $x > 0$,

$$\nu_{\lambda_n p}(x) = \nu_p\left(\frac{x}{|\lambda_n|}\right) \xrightarrow[n\to+\infty]{} \gamma < 1,$$

so that the mapping $\lambda \mapsto \lambda p$ is not continuous.

(c) It is a trivial consequence of part (b), since every simple space is a Šerstnev space.

(d) Let (λ_n) be a sequence of real numbers that tends to 0, when n goes to $+\infty$. Then, for all $p \in V$ and $x > 0$, one has, for every $n \in \mathbf{N}$,

$$\nu_{\lambda_n p}(x) = G\left(\frac{x}{\|\lambda_n p\|^\alpha}\right) = G\left(\frac{x}{|\lambda_n|^\alpha \|p\|^\alpha}\right).$$

Therefore $\lim_{n\to+\infty} \nu_{\lambda_n p}(x) = 1$ if, and only if, G belongs to $\mathcal{D}^+$.

(e) The proof is analogous to that of part (b).

$\boxed{QED}$

For every PN space (V, ν, τ, τ^*), if $p \in V$ and $x \geq 0$, then $\nu_p(x)$ may be thought of as the probability $P(\|p\| < x)$, where $\|\cdot\|$ is a norm for V. So the fact that ν_p does not belong to $\mathcal{D}^+$ means that $P(\|p\| < +\infty) < 1$; this is to be regarded as being "odd". Therefore we shall call *strict* any PN space (V, ν, τ, τ^*) such that $\nu(V) \subseteq \mathcal{D}^+$, i.e., such that ν_p belongs to $\mathcal{D}^+$ for every $p \in V$. This definition can be extended to PPN and PSN spaces. Thus, Theorem 2 (b), (c), (d) and (e) can be rephrased as follows.

3 Theorem. *Šerstnev spaces, simple spaces, α–simple spaces and EN spaces are topological vector spaces if, and only if, they are strict.*

However, in general PN spaces, the condition $\nu(V) \subseteq \mathcal{D}^+$ is not necessary to obtain a topological vector space: see Theorem 11 below.

3 Normability of PN spaces

If (V, ν, τ, τ^*) is a topological vector space, the question naturally arises of whether it is also normable; in other words, whether there is a norm on V that generates the strong topology. This question had been broached by Prochaska [9] in the case of Šerstnev PN spaces. For this case, we shall provide a complete characterization of those strict Šerstnev PN spaces that are indeed normable (see Theorem 6 further on). In the process, we shall need Kolmogorov's classical characterization of normability for T_1 spaces [4].

4 Theorem. *[Kolmogorov] A T_1 topological vector space is normable if, and only if, there is a neighbourhood of the origin θ that is convex and topologically bounded.*

Here, we have called *topologically bounded* a set A in a topological vector space E when, for every sequence (λ_n) of real numbers that converges to 0 as n tends to $+\infty$ and for every sequence (p_n) of elements of A, one has $\lambda_n p_n \to \theta$ in the topology of E.

We recall that the *probabilistic radius* of a set A in a PN space (V, ν, τ, τ^*) is the distance distribution function R_A given by

$$R_A(x) := \ell^- \Phi_A(x) \, (:= \lim_{u \to x^-} \Phi_A(u)) \quad \text{for all } x \in]0, \infty[\, ,$$

where $\Phi_A(u) := \inf\{\nu_p(u) \mid p \in A\}$ for all $u \in]0, \infty[$ (see [8]). A subset A of a PN space (V, ν, τ, τ^*) is said to be *$\mathcal{D}$–bounded* if, and only if, there exists a distribution function $G \in \mathcal{D}^+$ such that $\nu_p \geq G$ for every $p \in A$. One can take $G = R_A$, when R_A belongs to $\mathcal{D}^+$.

3.1 The case of Šerstnev spaces

In characterizing normable Šerstnev spaces we shall need the following result.

5 Theorem. *In a strict Šerstnev space (V, ν, τ) the following statements are equivalent for a subset A of V:*

(a) *A is $\mathcal{D}$–bounded;*

(b) *A is topologically bounded.*

Proof. (a) $\Longrightarrow$ (b) Let A any $\mathcal{D}$–bounded subset of V and let (p_n) be any sequence of elements of A and (λ_n) any sequence of real numbers that converges to 0; there is no loss of generality in assuming $\lambda_n \neq 0$ for every $n \in \mathbf{N}$. Then, for every $x > 0$, and for every $n \in \mathbf{N}$,

$$\nu_{\lambda_n p_n}(x) = \nu_{p_n}\left(\frac{x}{|\lambda_n|}\right) \geq R_A\left(\frac{x}{|\lambda_n|}\right) \xrightarrow[n \to +\infty]{} 1.$$

Thus $\lambda_n p_n \to \theta$ in the strong topology and A is topologically bounded.

(b) $\Longrightarrow$ (a) Let A be a subset of V which is not $\mathcal{D}$–bounded. Then

$$R_A(x) \xrightarrow[x \to +\infty]{} \gamma < 1.$$

By definition of R_A, for every $n \in \mathbf{N}$ there is $p_n \in A$ such that

$$\nu_{p_n}(n^2) < \frac{1+\gamma}{2} < 1.$$

If $\lambda_n = 1/n$, then, for every $n \in \mathbf{N}$,

$$\nu_{\lambda_n p_n}(1/2) \leq \nu_{\lambda_n p_n}(n) = \nu_{p_n}(n^2) < \frac{1+\gamma}{2} < 1,$$

which shows that $(\nu_{\lambda_n p_n})$ does not tend to ε_0, even if it has a weak limit, viz. $(\lambda_n p_n)$ does not tend to θ in the strong topology; in other words, A is not topologically bounded. $\boxed{QED}$

As a consequence of the previous results, it is now possible to characterize normability for strict Šerstnev spaces according to the following criterion.

6 Theorem. *A strict Šerstnev space (V, ν, τ) is normable if, and only if, the null vector θ has a convex $\mathcal{D}$–bounded neighbourhood.*

The following (restrictive) sufficient condition is in [9]; we prove it here not only for the sake of completeness, but also because Prochaska's thesis is not easily accessible and, moreover, because the notation there adopted is different from the one that has become usual after the publication of [12].

7 Theorem. *[Prochaska] A strict Šerstnev space (V, ν, τ) with $\tau = \tau_M$ is locally convex.*

Proof. It suffices to consider the family of neighbourhoods of the origin θ, $N_\theta(t)$, with $t > 0$. Let $t > 0$, $p, q \in N_\theta(t)$ and $\lambda \in [0, 1]$. Then

$$\begin{aligned} \nu_{\lambda p+(1-\lambda)q}(t) &\geq \tau_M \left(\nu_{\lambda p}, \nu_{(1-\lambda)q}\right)(t) \\ &= \sup_{\mu\in[0,1]} M\left(\nu_{\lambda p}(\mu t), \nu_{(1-\lambda)q}\left((1-\mu)t\right)\right) \\ &\geq M\left(\nu_{\lambda p}(\lambda t), \nu_{(1-\lambda)q}\left((1-\lambda)t\right)\right) = M\left(\nu_p(t), \nu_q(t)\right) > 1 - t. \end{aligned}$$

Thus $\lambda p + (1 - \lambda)q$ belongs to $N_\theta(t)$ for every $\lambda \in [0, 1]$. QED

As a consequence of Theorems 4 and 7, every simple PN space $(V, \|\cdot\|, G, M)$ with $G \in \mathcal{D}^+$ is trivially normable, since their strong topology coincides with the topology of their classical norm. In general, it is to be expected that most of the PN spaces considered in Theorem 7 will be normable, as shown by the following corollary.

8 Corollary. *Let (V, ν, τ_M) be a strict Šerstnev space. If $N_\theta(t)$ is $\mathcal{D}$–bounded for some $t \in]0, 1[$, then (V, ν, τ_M) is normable.*

3.2 Other cases

Apart from the Šerstnev spaces, we can also determine whether an α–simple space is normable, as the following result shows.

9 Theorem. *Let G be a continuous and strictly increasing distribution function in $\mathcal{D}^+$. Then, the α–simple space $(V, \|\cdot\|, G; \alpha)$ is normable.*

Proof. It follows from the assumptions that the α–simple space $(V, \|\cdot\|, G; \alpha)$ is a Menger space under a suitable t–norm T (see [7]). Let $N_\theta(t)$ be a neighbourhood of the origin θ with $t \in]0, 1[$; then

$$N_\theta(t) = \left\{p \in V \mid G\left(\frac{t}{\|p\|^\alpha}\right) > 1 - t\right\} = \left\{p \in V \mid \|p\| < \left(\frac{t}{G^{-1}(1-t)}\right)^{1/\alpha}\right\}.$$

Since $h(t) = \left(t/G^{-1}(1-t)\right)^{1/\alpha}$ is a continuous function such that $\lim_{t\to 0+} h(t) = 0$ and $\lim_{t\to 1-} h(t) = \infty$, then it is clear that the strong topology for V coincides with the topology of the norm $\|\cdot\|$ in V. Therefore, $(V, \|\cdot\|, G; \alpha)$ is normable. QED

It is natural to ask whether results similar to that of Theorem 6 hold for general PN spaces. The conditions of Theorem 5 need not be equivalent; for, there are PN spaces in which a set A may be topologically bounded without being $\mathcal{D}$–bounded. On the other hand, even in those cases, sometimes it is possible to establish directly whether a PN space that is also a topological vector space is normable. To illustrate both facts, we next introduce a new class of PN spaces whose interest goes deeper than just to provide an example to this point. Recall that only a few types of PN spaces are known: finding a new type is useful in order to deepen our knowledge of these spaces.

Before introducing the new class of PN spaces we need the following technical lemma.

10 Lemma. *Let $f\colon [0,+\infty[\to [0,1]$ be a right–continuous nonincreasing function. Define $f^{[-1]}(1) := 0$ and $f^{[-1]}(y) := \sup\{x \mid f(x) > y\}$ for all $y \in [0,1[$ ($f^{[-1]}(y)$ might be infinite). For $x_0 \in [0,+\infty[$ and $y_0 \in [0,1]$, the following facts are equivalent:* (a) $f(x_0) > y_0$; (b) $x_0 < f^{[-1]}(y_0)$.

Proof. If $f(x_0) > y_0$ then $f^{[-1]}(y_0) = \sup\{x \mid f(x) > y_0\} \geq x_0$. If we suppose that $\sup\{x \mid f(x) > y_0\} = x_0$, then $f(x) \leq y_0$ for every $x > x_0$. Thus $f(x_0) = \ell^+ f(x_0) \leq y_0$ ($\ell^+ f(x_0) = \lim_{x \to x_0^+} f(x)$), against the assumption; whence (a) $\Longrightarrow$ (b). The converse result is an immediate consequence of the monotonicity of f. $\boxed{QED}$

The following theorem introduces a new class of PN spaces that generalizes an example in [5] and which also provides some properties of the spaces in that class. As has been said above, such properties are interesting for the purposes of this paper. It may be useful to recall that $\tau_{T^*} \geq \tau_{M^*} = \tau_M$ for every t–norm T (see [12]).

11 Theorem. *Let $(V, \|\cdot\|)$ a normed space and let T be a continuous t–norm. Let f be a function as in Lemma 10, and satisfying the following two properties:*

(a) $f(x) = 1$ *if, and only if,* $x = 0$;

(b) $f(\|p+q\|) \geq T(f(\|p\|), f(\|q\|))$ *for all* $p, q \in V$.

If $\nu\colon V \to \Delta^+$ is given by

$$\nu_p(x) = \begin{cases} 0, & x \leq 0, \\ f(\|p\|), & x \in \,]0,+\infty[\,, \\ 1, & x = +\infty, \end{cases} \tag{3}$$

for every $p \in V$, then (V, ν, τ_T, τ_M) is a Menger PN space satisfying the following properties:

(F1) (V, ν, τ_T, τ_M) *is a topological vector space;*

(F2) (V, ν, τ_T, τ_M) *is normable;*

(F3) *If $p \in V$ and $t > 0$, then the strong neighbourhood $N_p(t)$ in (V, ν, τ_T, τ_M) is not $\mathcal{D}$–bounded, but $N_p(t)$ is topologically bounded whenever $N_p(t) \neq V$;*

(F4) *(V, ν, τ_T, τ_M) is not a Šerstnev space;*

(F5) *(V, ν, τ_T, τ_M) is not a strict PN space.*

PROOF. First, we prove that (V, ν, τ_T, τ_M) is a Menger PN space:

(N1) $\nu_p = \varepsilon_0 \Longleftrightarrow f(\|p\|) = 1 \Longleftrightarrow \|p\| = 0 \Longleftrightarrow p = \theta$.

(N2) Trivial.

(N3) For all $p, q \in V$, the inequality $\nu_{p+q} \geq \tau_T(\nu_p, \nu_q)$ means that one has

$$\nu_{p+q}(x) \geq \tau_T(\nu_p, \nu_q)(x) = \sup_{s+t=x} T(\nu_p(s), \nu_q(t))$$

for all $x \in]0, +\infty[$, or, equivalently,

$$f(\|p + q\|) \geq T(f(\|p\|), f(\|q\|)),$$

as assumed.

(N4) Let $p \in V$ and let $\lambda \in [0, 1]$. Then, the inequality $\nu_p \leq \tau_M\left(\nu_{\lambda p}, \nu_{(1-\lambda)p}\right)$ is equivalent, for all $x \in]0, +\infty[$, to

$$\begin{aligned} f(\|p\|) = \nu_p(x) &\leq \tau_M\left(\nu_{\lambda p}, \nu_{(1-\lambda)p}\right)(x) = \sup_{s+t=x} M\left(\nu_{\lambda p}(s), \nu_{(1-\lambda)p}(t)\right) \\ &= \sup_{s+t=x} M\left(f(\lambda\|p\|), f\left((1-\lambda)\|p\|\right)\right) \\ &= M\left(f(\lambda\|p\|), f\left((1-\lambda)\|p\|\right)\right) = \min\{f(\lambda\|p\|), f((1-\lambda)\|p\|)\}. \end{aligned}$$

Therefore, one has, for all $p \in V$ and for all $\lambda \in [0, 1]$, $\nu_p \leq \tau_M\left(\nu_{\lambda p}, \nu_{(1-\lambda)p}\right)$ if, and only if, $f(\|p\|) \leq f(\alpha\|p\|)$ for all $\alpha \in [0, 1]$, namely if, and only if, f is nonincreasing.

Now we prove properties (F1) through (F5):

(F1) Let $p \in V$. We only have to prove that the map from $\mathbf{R}$ into V defined by $\lambda \mapsto \lambda p$ is continuous at every $\lambda \in \mathbf{R}$. Let $\eta > 0$ (we shall suppose, without loss of generality, that $\eta \leq 1$). We must prove that there exists a number $\delta > 0$ such that $d_S(\nu_{\lambda' p - \lambda p}, \varepsilon_0) < \eta$ whenever $|\lambda' - \lambda| < \delta$; or, equivalently, such that $d_S(\nu_{\beta p}, \varepsilon_0) < \gamma$ whenever $|\beta| < \delta$. Since $d_S(\nu_q, \varepsilon_0) = \inf\{h \mid \ell^+\nu_q(h) > 1 - h\} = 1 - f(\|q\|)$, then one has $d_S(\nu_{\beta p}, \varepsilon_0) < \gamma$ if, and only if, $1 - f(|\beta|\|p\|) < \gamma$, viz. $f(|\beta|\|p\|) > 1 - \gamma$, or, again, by Lemma 10, if, and only if, $|\beta| < \delta := f^{[-1]}(1 - \gamma)/\|p\|$.

(F2) Let $p \in V$. Let $t > 0$ (we shall suppose, without loss of generality, that $t < 1 - \lim_{x\to\infty} f(x)$). Then, because of Lemma 10,

$$\begin{aligned} N_p(t) &= \{q \in V \mid d_S(\nu_{p-q}, \varepsilon_0) < t\} = \{q \in V \mid f(\|p-q\|) > 1-t\} \\ &= \{q \in V \mid \|p-q\| < f^{[-1]}(1-t)\} = B(p, f^{[-1]}(1-t)), \end{aligned}$$

i.e., the strong neighbourhood $N_p(t)$ is a ball in $(V, \|\cdot\|)$ with centre at p. Conversely, let $r > 0$. If $t = 1 - f(r)$, then $f^{[-1]}(1-t) < r$, whence

$$N_p(t) = B(p, f^{[-1]}(1-t)) \subset B(p, r).$$

Therefore, the strong topology for $(V, \nu, \tau_T, \tau_{T^*})$ coincides with the topology of the norm in $(V, \|\cdot\|)$.

(F3) If $p \in V$ and $0 < t < 1 - \lim_{x\to\infty} f(x)$, then $N_p(t) = B(p, f^{[-1]}(1-t))$ is a ball in $(V, \|\cdot\|)$, whence $N_p(t)$ is topologically bounded. On the other hand, if $0 < x < \infty$ then

$$\begin{aligned} \Phi_{N_p(t)}(x) &= \inf\{\nu_q(x) \mid q \in N_p(t)\} = \inf\{f(\|q\|) \mid \|p-q\| < f^{[-1]}(1-t)\} \\ &= f\left(\|p\| + f^{[-1]}(1-t)\right). \end{aligned}$$

Thus, $\lim_{x\to\infty} R_{N_p(t)}(x) = f\left(\|p\| + f^{[-1]}(1-t)\right) < 1$, i.e., $N_p(t)$ is not $\mathcal{D}$–bounded.

(F4) It is immediate to check that $(V, \nu, \tau_T, \tau_{T^*})$ is a Šerstnev space if, and only if, the function f is constant on $]0, \infty[$. From assumption (a) this constant should be less than 1, which contradicts the right–continuity of f at $x = 0$. Thus, (V, ν, τ_T, τ_M) is not a Šerstnev space.

(F5) It is immediate that $\nu(V \setminus \{\theta\}) \subset \Delta^+ \setminus \mathcal{D}^+$.

$\boxed{QED}$

Now we consider some special cases and use the preceding theorem in order to give some examples.

12 Example. Suppose that, in Theorem 11, $T = \Pi$. Then, property (b) reads $f(\|p+q\|) \geq f(\|p\|)\, f(\|q\|)$ for all $p, q \in V$. It is not difficult to prove that, under the given assumptions on f, property (b) is equivalent to the following one:

$$f(x+y) \geq f(x) f(y), \qquad \text{for all } x, y \in [0, \infty[\,. \tag{4}$$

The following are examples of functions f satisfying the assumptions of Theorem 11 in this case:

$$f_{\alpha,\beta}(x) := 1 - \frac{\beta}{\alpha} + \frac{\beta}{x+\alpha}, \qquad 0 \le \beta \le \alpha,$$
$$g_{\alpha,\beta}(x) := 1 - \alpha + \alpha \exp\left(-x^{\beta}\right), \qquad 0 < \alpha \le 1,\ \beta > 0.$$

13 Example. Take $T = W$ in Theorem 11. In this case property (b) reads

$$\forall\, p, q \in V \qquad f(\|p+q\|) \ge f(\|p\|) + f(\|q\|) - 1,$$

which is equivalent to the following one

$$\forall\, x, y \in [0, +\infty[\qquad 1 + f(x+y) \ge f(x) + f(y),$$

namely to the fact that the function $x \mapsto f(x) - 1$ is superadditive. For instance, the following functions satisfy these properties but not those considered in Example 12, since they do not satisfy (4):

$$h_{\alpha,\beta}(x) := \begin{cases} 1 - \alpha x, & 0 \le x \le \beta, \\ 1 - \alpha\beta, & x > \beta, \end{cases} \qquad 0 < \beta \le 1/\alpha.$$

4 Conclusion

In what precedes we have been able to characterize those Šerstnev spaces that are normable topological vector spaces. Several questions remain open: to give at least sufficient conditions under which a general PN space is normable; more, to characterize (rather than just having a sufficient condition) the class of PN spaces that are also topological vector spaces, and, once this has been achieved, to study normability in the class thus determined.

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