Note di Matematica
Note Mat. 29 (2009), n. 1, 79-88
ISSN 1123-2536, e-ISSN 1590-0932
DOI 10.1285/i15900932v29n1p79

Statistical Convergence in Strong Topology of Probabilistic Normed Spaces

Bernardo Lafuerza-Guillén [i]
Departamento de Estadística y Matemática Aplicada,
Universidad de Almería, E–04120 Almería, Spain
blafuerz@ual.es

Mohd Rafi
School of Applied Mathematics, University of Nottingham Malaysia
Campus Jalan Broga, 43500 Semenyih, Selangor D.E., Malaysia
mohd.rafi@nottingham.edu.my

Received: 03/03/2008; accepted: 22/04/2008.

Abstract. Following the concept of statistical convergence, we define and study statistical analogue concepts of convergence and Cauchy's sequence on a probabilistic normed space that is endowed with a strong topology. Some important properties of statistical convergence were also extended in this new setting.

Keywords: S-statistically–convergence, S–convergence, statistically dense

MSC 2000 classification: Primary 46S50, Secondary 54E70.

1 Introduction

Probabilistic normed (PN) spaces are real linear spaces in which the norm of each vector is an appropriate distribution function rather than a number. Such spaces were first introduced by A. N. Šerstnev in 1963 [17].

In recently, C. Alsina et al in [1] introduced a new definition of PN spaces that includes Šerstnev's and leads naturally to the identification of the principle class of PN spaces, the Menger spaces. In this paper we investigate questions of statistical continuity in PN spaces under the new definition. We recall some notation and terminology used in [20].

Δ^+ denote the set of all one-dimensional probability distributions whose support is the positive half-line, i.e., Δ^+ is the set of all functions such that Dom $F = [0, +\infty]$, Ran $F \subseteq [0, 1]$, $F(0) = 0$, $F(+\infty) = 1$, and F is non-decreasing and left-continuous on $(0, +\infty)$. The subset $\mathcal{D}^+ \subseteq \Delta^+$ is the set

$$\mathcal{D}^+ = \{F \in \Delta^+ \colon l^- F(+\infty) = 1\}.$$

[i]We wish to thank Prof. C. Sempi for several helpful comments. The first author was supported by grants from Ministerio de Ciencia y Tecnologia (MTM2006-12218) of Spain.

Here $l^- f(x)$ denotes the left limit of the function f at the point x, $l^- f(x) = \lim_{t\to x-} f(t)$. The set Δ^+ is ordered by the usual pointwise ordering of functions; and ε_a is a special function in Δ^+ given by

$$\varepsilon_a(x) = \begin{cases} 0, & x \leq a, \\ 1, & x > a. \end{cases}$$

In [18], D. L. Sibley gave a useful modified Lévy metric d_L in Δ^+. For $F, G \in \Delta^+$,

$$d_L(F,G) = \inf\{h \in (0,1] \colon \text{both } [F,G;h] \text{ and } [G,F;h] \text{ hold}\},$$

where $[F,G;h]$ denote the condition $G(x) \leq F(x+h)+h$, for $x \in (0, \frac{1}{h})$. There is a natural topology on Δ^+ that is induced by the modified Lévy metric d_L (see [20], Sect. 4.2). Convergence with respect to this metric is equivalent to weak convergence of distribution functions. i.e., for any sequence $\{F_k\}$ in Δ^+ and $F \in \Delta^+$, the sequence $\{d_L(F_k, F)\}$ converges to 0 if and only if $\{F_k(x)\}$ converges to $F(x)$ at every point of continuity of the limit function F.

1 Lemma. *The following statements hold:*

(i) For any $F \in \Delta^+$, $d_L(F, \varepsilon_0) = \inf\{h \colon F(h+) > 1-h\}$,

(ii) For any $t > 0$, $F(t) > 1-t$ if and only if $d_L(F, \varepsilon_0) < t$,

(iii) If $F, G \in \Delta^+$ and $F \leq G$, then $d_L(G, \varepsilon_0) \leq d_L(F, \varepsilon_0)$.

A *triangle function* is a binary operation on Δ^+ that is commutative, associative, non-decreasing in each place, and has ε_0 as an identity element. Continuity of a triangle function means uniform continuity with respect to the natural product topology on $\Delta^+ \times \Delta^+$.

Typical continuous triangle functions are the operations τ_T and τ_S, which are, respectively, given by

$$\tau_T(F,G)(x) = \sup_{u+v=x} T(F(u), G(v)),$$

and

$$\tau_S(F,G)(x) = \inf_{u+v=x} S(F(u), G(v)),$$

for all $F, G \in \Delta^+$ and all $x \in \mathbb{R}$ [20]. Here T is a continuous *t-norm* and S is a continuous *t-conorm*, i.e., both are continuous binary operations on $[0,1]$ that are commutative, associative and nondecreasing in each place; T has 1 as identity and S has 0 as identity. If T is a t-norm and T^* is defined on $[0,1]\times[0,1]$ via

$$T^*(x,y) = 1 - T(1-x, 1-y),$$

then T^* is a t-conorm, specifically the t-conorm of T.

The most important t-norm are the functions W, Prod, and M which are defined, respectively, by

$$\begin{aligned} W(a,b) &= \max(a+b-1,0), \\ \mathrm{Prod}(a,b) &= ab, \\ M(a,b) &= \min(a,b). \end{aligned}$$

Their corresponding t-conorms are given, respectively, by

$$\begin{aligned} W^*(a,b) &= \min(a+b,1), \\ \mathrm{Prod}^*(a,b) &= a+b-ab, \\ M^*(a,b) &= \max(a,b). \end{aligned}$$

2 Definition. [1] A probabilistic normed space (briefly, a PN space) is a quadruple (V,ν,τ,τ^*), where V is a real linear space, τ and τ^*, with $\tau \leq \tau^*$ are continuous triangle functions, and the probabilistic norm ν is a mapping from V into Δ^+ (writing ν_p for $\nu(p)$), the following conditions hold:

(N1) $\nu_p = \varepsilon_0$ if and only if $p = \theta$(the zero vector of V);

(N2) $\nu_{-p} = \nu_p$ for all $p \in V$;

(N3) $\nu_{p+q} \geq \tau(\nu_p,\nu_q)$ for all $p,q \in V$;

(N4) $\nu_p \leq \tau^*(\nu_{\alpha p},\nu_{(1-\alpha)q})$ for every $p \in V$ and for every $\alpha \in [0,1]$.

If $\tau = \tau_T$ and $\tau^* = \tau_{T^*}$ for some continuous t-norm T and its associated t-conorm T^*, then $(V,\nu,\tau_T,\tau_{T^*})$ is a *Menger PN space.*

Let (V,ν,τ,τ^*) be a PN space. Since τ is continuous, the system of strong *neighborhoods of zero*

$$\{\mathcal{N}_\theta(\lambda)\colon \lambda > 0\}, \tag{1}$$

where

$$\mathcal{N}_\theta(\lambda) = \{p \in V\colon d_L(\nu_p,\varepsilon_0) < \lambda\}. \tag{2}$$

determines a first countable Hausdorff topology on V, called the *strong topology* (briefly, $\mathcal{S}$-topology). Thus, the $\mathcal{S}$-topology can be completely specified by means of $\mathcal{S}$-convergence of sequences.

3 Theorem. *[20] In the simple space (S,d,G), the strong topology is equivalent to the d-metric topology when $G \in \mathcal{D}^+$. If $G \notin \mathcal{D}^+$, then the strong topology coincides with the discrete topology.*

The following lemma is an immediate consequence of the definition of neighborhood of zero and Lemma 1.1(ii).

4 Lemma. *In a PN space (V, ν, τ, τ^*), for each $p \in V$, we have*

$$\nu_p(t) > 1 - t \quad \Longleftrightarrow \quad p \in \mathcal{N}_\theta(t). \tag{3}$$

5 Definition. [20] (i) A sequence $\{p_k\}$ of elements of V is said to be $\mathcal{S}$-convergent to θ, the null vector of V, in the $\mathcal{S}$-topology if for any $\lambda > 0$ there is an integer $K(\lambda) \in \mathbb{N}$ such that $p_k \in \mathcal{N}_\theta(\lambda)$ whenever $k \geq K(\lambda)$. In this case we write $p_k \xrightarrow{\mathcal{S}} \theta$ or $\mathcal{S} - \lim_k p_k = \theta$.
(ii) A sequence $\{p_k\}$ is said to be a $\mathcal{S}$-Cauchy sequence if for any $\lambda > 0$, there is an integer $M(\lambda) \in \mathbb{N}$ such that $p_k - p_l \in \mathcal{N}_\theta(\lambda)$ whenever $k, l \geq M(\lambda)$

6 Remark. Of course, there is nothing special about θ as a limit; if one wishes to consider the convergence of the sequences $\{p_k\}$ to the vector p in the $\mathcal{S}$-topology, then it suffices to consider the sequence $\{p_k - p\}$ and its convergence to θ. In other word, $\mathcal{S} - \lim_k p_k = p$ is equivalent to $\mathcal{S} - \lim_k (p_k - p) = \theta$.

7 Lemma. *[2] For any $\alpha \in \mathbb{R}$, any $p \in V$, and any $\epsilon > 0$, there exists a $\lambda > 0$ such that*

$$\alpha p \in \mathcal{N}_\theta(\epsilon) \quad \textit{whenever} \quad p \in \mathcal{N}_\theta(\lambda). \tag{4}$$

8 Lemma. *[2] If $0 \leq \alpha < 1$, then*

$$\nu_{\alpha p} \geq \nu_p \tag{5}$$

for any $p \in V$.

We observe that, in view of Lemma 1.2 and (N3), we have the following lemma.

9 Lemma. *Let (V, ν, τ, τ^*) be a PN space. For every $p, q, r \in V$,*

$$d_L(\nu_{p-r}, \varepsilon_0) \leq d_L(\tau(\nu_{p-q}, \nu_{q-r}), \varepsilon_0). \tag{6}$$

10 Definition. Let (V, ν, τ, τ^*) and (V', ν', τ, τ^*) be two PN spaces with the S-topology. A map $f \colon V \to V$ is said to be S-continuous at $u \in V$ if for every neighborhood of $f(u) \in V'$, $\mathcal{N}'_{f(u)}(s)$, there exists a neighborhood of $u \in V$, $\mathcal{N}_(\theta)(t)$ such that

$$f(x) \in \mathcal{N}'_{f(u)}(s) \quad \text{whenever} \quad x \in \mathcal{N}_u(t). \tag{7}$$

The map f is said to be a $\mathcal{S}$-continuous map if it is $\mathcal{S}$-continuous at every element of V.

2 Statistical Convergence of real/complex numbers

The idea of the statistical convergence of real numbers ware independently introduced by Fast [6] and Steinhaus [19]. But the rapid developments started after the papers of Šalát [15], Fridy [8] and Connor [5]. This concept was further extended to Banach spaces by Kolk [12], to locally convex spaces by Maddox [13] and to fuzzy numbers by Savaş [16]. Recently, many papers published on the study of statistical convergence in many aspects of real numbers and fuzzy numbers by numerous authors (see ([3,4,9,14]).

In this section, we list some of the basic concepts of statistical convergence of real numbers and we refer to [7–9] for more details.

11 Definition. [7] If K is a subset of the positive integers $\mathbb{N}$, then K_n denotes the set $\{k \in K : k \leq n\}$ and $|K_n|$ denotes the number of elements in K_n.

The natural density of K is given by

$$\delta(K) = \lim_{n\to\infty} \frac{|K_n|}{n}. \tag{8}$$

12 Remark. Clearly, finite subsets have zero natural density and $\delta(K^c) = 1 - \delta(K)$ where $K^c = \mathbb{N} \backslash K$.

13 Definition. K is said to be statistically dense if

$$\delta(K) = 1. \tag{9}$$

The set $\{k \in K : k \neq m^2, m = 1, 2, \dots\}$ is statistically dense, while the set $\{3k : k = 1, 2, \cdots\}$ is not. A subsequence of a sequence is called statistically dense if the set of all indices of its elements is statistically dense.

We will be particularly concerned with integer sets having natural density zero. So,

14 Definition. If $\{x_k\}$ is a sequence such that x_k satisfies property P for all k except a set of natural density zero, then we say that $\{x_k\}$ satisfies P for "almost all k", and we abbreviate this by "$a.a.k$".

15 Definition. A sequence $\{x_k\}$ of (real or complex) numbers is said to be statistically convergent to some number L, if for every $\epsilon > 0$, the set $K_\epsilon = \{k \in \mathbb{N} : |x_k - L| \geq \epsilon\}$ has natural density zero, viz.

$$\delta(K_\epsilon) = 0. \tag{10}$$

In this case we write $stat - \lim_k x_k = L$ or $x_k \xrightarrow{stat} L$.

16 Definition. A sequence $\{x_k\}$ is said to be a *statistically Cauchy's sequence* if for each $\epsilon > 0$, there exists a number $N = N(\epsilon)$ such that

$$\delta(\{k \in \mathbb{N} \colon |x_k - x_N| \geq \epsilon\}) = 0, \tag{11}$$

i.e.,$|x_k - x_N| < \epsilon$ for $a.a.k$.

Friday [8] proved that a sequence $\{x_k\}$ is statistically convergent if and only if it is statistically Cauchy's sequence.

3 Statistical Convergence on PN spaces

Recently, Karakus [11] has introduced statistical convergence in Šerstnev PN spaces. In this section, we extend the idea of statistical convergence to the setting of sequences in a PN space endowed with $\mathcal{S}$-topology.

17 Definition. Let (V, ν, τ, τ^*) be a PN space, let $\{x_k\}$ be an V-valued sequence, and $L \in V$. The sequence $\{p_k\}$ is $\mathcal{S}$-statistically convergent to θ provided that for every $t > 0$

$$\delta(\{k \colon p_k \notin \mathcal{N}_\theta(t)\}) = 0, \tag{12}$$

or equivalently by (7)

$$\lim_n \frac{1}{n} |\{k \leq n \colon p_k \notin \mathcal{N}_\theta(t)\}| = 0,$$

i.e., $p_k \in \mathcal{N}_\theta(t)$, for $a.a.k$. In this case, we write $p_k \overset{\mathcal{S}-stat}{\longrightarrow} \theta$ or $\mathcal{S}$-stat-$\lim p_k = \theta$, where θ is called the $\mathcal{S}$-statistical limit (briefly, a $\mathcal{S}$-stat-limit) of $\{p_k\}$.

The following lemma is an immediate consequence of above definition and the well-known density properties.

18 Lemma. *Let (V, ν, τ, τ^*) be a PN space. Then, for every $t > 0$ the following statements are equivalent:*
(i) $sst - \lim_k(p_k) = \theta$,
(ii) $\delta(\{k \in \mathbb{N} \colon p_k \notin \mathcal{N}_\theta(t)\}) = 0$,
(iii) $\delta(\{k \in \mathbb{N} \colon d_L(\nu_{p_k}, \varepsilon_0) \geq t)) = 0$,
(iv) $\delta(\{k \in \mathbb{N} \colon d_L(\nu_{p_k}, \varepsilon_0) < t)) = 1$.

19 Theorem. *Let (V, ν, τ, τ^*) be a PN space. If a sequence $\{p_k\}$ is $\mathcal{S}$-statistically convergent in the $\mathcal{S}$- topology, then $\mathcal{S}$-stat-limit is unique.*

Proof. Assume that $\mathcal{S} - stat - limp_k = p$ and $\mathcal{S} - stat - limp_k = q$ with $p \neq q$. For any $t > 0$, define the following sets:

$$K_1(t) = \{k \in \mathbb{N} \colon p_k - p \notin \mathcal{N}_\theta(t)\},$$

$$K_2(t) = \{k \in \mathbb{N} \colon p_k - q \notin \mathcal{N}_\theta(t)\}.$$

Let $K(t) = K_1(t) \cap K_2(t)$. We observe that, since $\delta(K_1(t)) = 0$ and $\delta(K_2(t)) = 0$ for all $t > 0$, we have $\delta(K(t)) = 0$ for all $t > 0$ which implies that $\delta(\mathbb{N} \setminus K(t)) = 1$ for all $t > 0$. Let $k_i \in \mathbb{N} \setminus K(t)$, then $p_{k_i} - p \in \mathcal{N}_\theta(t)$. Let $d_L(v_{p_{k_i}-p}, \varepsilon_0) = \alpha$. Then $t - \alpha > 0$ and the uniform continuity of τ implies that there exists a $t' > 0$ such that

$$d_L(\tau(\nu_{p_{k_i}-p}, G), \nu_{p_{k_i}-p}) < t - \alpha$$

whenever $d_L(G, \varepsilon_0) < t'$. Now let $p_{k_i} - q \in \mathcal{N}_\theta(t')$, then $d_L(v_{p_{k_i}-q}, \varepsilon_0) < t'$. Thus, by equation (6), we have

$$\begin{aligned} d_L(\nu_{p-q}, \varepsilon_0) &\leq d_L(\tau(\nu_{p_{k_i}-p}, \nu_{p_{k_i}-q}), \varepsilon_0) \\ &\leq d_L(\tau(\nu_{p_{k_i}-p}, \nu_{p_{k_i}-q}), \nu_{p_{k_i}-p}) + d_L(\nu_{p_{k_i}-p}, \varepsilon_0) \\ &< t - \alpha + \alpha \\ &= t. \end{aligned}$$

Hence $p - q \in \mathcal{N}_\theta(t)$. Since $t > 0$ is arbitrary, by (3), we get $\nu_{p-q} = \varepsilon_0$ which yields $p - q = 0$, i.e., $p = q$. This completes the proof. $\boxed{QED}$

20 Lemma. *Let (V, ν, τ, τ^*) be a PN space. If $\mathcal{S}$-stat-$\lim p_k = p$ and $f : V \to V'$, defined for all point in V, is a $\mathcal{S}$-continuous function on V, then $\mathcal{S}$-stat-$\lim f(p_k) - f(p)) = \theta$.*

Proof. Since f is $\mathcal{S}$-continuous function, for every $\epsilon > 0$ there exists a $\lambda > 0$ such that

$$p_k - p \in \mathcal{N}_\theta(\lambda) \quad \text{implies} \quad f(p_k) - f(p) \in \mathcal{N}_{\theta'}(\epsilon).$$

But then

$$f(p_k) - f(p) \notin \mathcal{N}_{\theta'}(\epsilon) \quad \text{implies} \quad p_k - p \notin \mathcal{N}_\theta(\lambda).$$

Thus

$$\{k \in \mathbb{N} \colon f(p_k) - f(p) \notin \mathcal{N}_{\theta'}(\epsilon)\} \subseteq \{k \in \mathbb{N} \colon p_k - p \notin \mathcal{N}_\theta(\lambda)\}$$

and therefore

$$\delta(\{k \in \mathbb{N} \colon f(p_k) - f(p) \notin \mathcal{N}_\theta(\epsilon))\} \leq \delta(\{k \in \mathbb{N} \colon p_k - p \notin \mathcal{N}_\theta(\lambda)\}) = 0$$

because $\mathcal{S}$-stat-$\lim_k p_k - p = 0$. This proves $\mathcal{S}$-stat-$\lim_k f(p_k) - f(p) = 0)$. $\boxed{QED}$

21 Theorem. *Let (V, ν, τ, τ^*) be a PN space. If $S - \lim(p_k) = \theta$, then $S - stat - \lim(p_k) = \theta$.*

Proof. By hypothesis, for every $t > 0$, there is an integer $K \in \mathbb{N}$ such that $p_k \in \mathcal{N}_\theta(t)$ whenever $k \geq N(t)$. This guaranties that for every $t > 0$ the set $\{k \in \mathbb{N}\colon p_k \notin \mathcal{N}_\theta(t)\}$ has at most finitely many elements. Thus, by the property of density, we get $\delta(\{k \in \mathbb{N}\colon p_k \notin \mathcal{N}_\theta(t)\}) = 0$. Therefore $\mathcal{S}$-stat-$\lim(p_k) = \theta$, the conclusion. $\boxed{QED}$

22 Theorem. *Let (V, ν, τ, τ^*) be a PN space. A sequence $\{p_k\}$ in V is $\mathcal{S}$-statistically convergent to θ if and only if there exists a statistically dense subset $K = \{k_1 < k_2 < \cdots\} \subseteq \mathbb{N}$ such that $\mathcal{S} - \lim_n(p_{k_n}) = \theta$.*

Proof. The proof of sufficiency is easy and can be omitted. Suppose that $\mathcal{S}$-stat-$\lim(p_k) = \theta$. Put $K_m = \{n \in \mathbb{N}\colon p_n \in \mathcal{N}_\theta(\frac{1}{m}), m \in \mathbb{N}\}$. Since $\mathcal{N}_\theta(t_1) \supseteq \mathcal{N}_\theta(t_2)$ whenever $t_1 \geq t_2$, for each $m \in \mathbb{N}$ we have

$$K_1 \supseteq K_2 \supseteq \cdots \supseteq K_m \supseteq K_{m+1} \supseteq \cdots, \tag{13}$$

and $\mathcal{S}$-stat-$\lim(p_k) = \theta$ implies

$$\delta(K_m) = 1 \quad \text{for each} \quad m \in \mathbb{N}. \tag{14}$$

Now, choose $k_1 \in K_1$. According to (14), there exists a $k_2 > k_1$, $k_2 \in K_2$, such that, for every $n \geq k_2$

$$\frac{1}{n}|\{k \leq n\colon p_k \in \mathcal{N}_\theta(\frac{1}{2})\}| > \frac{1}{2}.$$

Again by (14) there exists a $k_3 > k_2$, $k_3 \in K_3$, such that, for every $n \geq k_3$

$$\frac{1}{n}|\{k \leq n\colon p_k \in \mathcal{N}_\theta(\frac{1}{3})\}| > \frac{2}{3}$$

and so on. So by induction we get an increasing index sequence

$$k_1 < k_2 < \cdots < k_j \in K_j \quad (j = 1, 2, \ldots),$$

such that for every $n \geq k_j$

$$\frac{|K_j|}{n} = \frac{1}{n}|\{k \leq n\colon p_k \in \mathcal{N}_\theta(\frac{1}{j})\}| > \frac{j-1}{j} \quad (j = 2, 3, \ldots). \tag{15}$$

Now, we construct the subset $K \subseteq \mathbb{N}$ as follows:

$$K = \{n \in \mathbb{N}\colon 1 \leq n < k_1\} \cup [\bigcup_{j \in \mathbb{N}} \{n \in K_j\colon k_j \leq n < k_{j+1}\}] \tag{16}$$

We show that $K \subseteq \mathbb{N}$ is statistically dense,i.e., $\delta(K) = 1$. Let $n \geq k_1$. Then n belongs to $\{n \in K_j\colon k_j \leq n < k_{j+1}\}$ for some $j \in \mathbb{N}$. Thus by (13), (14) we conclude that,

$$\frac{|K|}{n} \geq \frac{|K_j|}{n} > \frac{j-1}{j} \tag{17}$$

and it follows that $\delta(K) = 1$. Let $\lambda > 0$ and choose an integer $j \in \mathbb{N}$ such that $\frac{1}{j} < \lambda$. Let $n \geq k_j$ and $n \in K$. Then, by definition of K, there exists an integer $m \geq j$ such that $k_m \leq n < k_{m+1}$ and $n \in K_j$. Hence, for every $\lambda > 0$, we get

$$p_n \in \mathcal{N}_\theta(\frac{1}{j}) \subseteq \mathcal{N}_\theta(\lambda), \tag{18}$$

for every $n \geq k_j$ and $n \in K$. This proves that $\mathcal{S} - \lim_n p_n = \theta$ $\boxed{QED}$

23 Lemma. *Let $(V, \nu, \tau_T, \tau_{T^*})$ be a Menger PN space with $T(x, x) \geq x$ for every $x \in [0, 1]$. If $\mathcal{S}$-stat-$\lim_k p_k = p$, $\mathcal{S}$-stat-$\lim_k q_k = q$ and α is a real number, then*

(i) $\mathcal{S}$-stat-$\lim_k(p_k + q_k) = p + q$,

(ii) $\mathcal{S}$-stat-$\lim_k(p_k - q_k) = p - q$,

Proof. (i) For every $\epsilon > 0$, define the following sets:

$$P = \{n \in \mathbb{N} \colon p_k - p \notin \mathcal{N}_\theta(\epsilon)\},$$

$$Q = \{n \in \mathbb{N} \colon q_k - q \notin \mathcal{N}_\theta(\epsilon)\}.$$

Then, $\delta(P) = 0$ and $\delta(Q) = 0$. Now, let $K \subseteq (P \cap Q)$. Then, clearly $\delta(K) = 0$ which implies $\delta(\mathbb{N}/K) = 1$. If $k \in \mathbb{N}/K$, then we get

$$\nu_{(p_k+q_k)-(p+q)}(\epsilon) \geq \sup_{u+v=\epsilon} T(\nu_{p_k-p}(u), \nu_{q_k-q}(v)).$$

Now, we choose $\lambda < \epsilon$ such that $\lambda = \min(u, v)$. We note that $\mathcal{N}_\theta(\lambda) \subseteq \mathcal{N}_\theta(\epsilon)$. Then the above inequality becomes

$$\begin{aligned}
\nu_{(p_k+q_k)-(p+q)}(\epsilon) &\geq \sup_{\lambda>0} T(\nu_{p_k-p}(\lambda), \nu_{q_k-q}(\lambda)) \\
&\geq T(\nu_{p_k-p}(\lambda), \nu_{q_k-q}(\lambda)) \\
&> T(1-\lambda, 1-\lambda) > 1-\lambda \\
&> 1-\epsilon.
\end{aligned}$$

This shows that

$$\delta(\{k \in \mathbb{N} \colon (p_k + q_k) - (p + q) \notin \mathcal{N}_\theta(\epsilon)\},$$

i.e., $\mathcal{S}$-stat-$\lim_k(p_k + q_k) = p + q$.

(ii) Similar to (i). $\boxed{QED}$

References

[1] C. Alsina, B. Schweizer, A. Sklar: *On the definition of probabilistic normed space*, Aequationes Math. **46** (1993), 91–98.

[2] C. Alsina, B. Schweizer, A. Sklar: *Continuity properties of probabilistic norms*, J. Math. Anal. Appl. **208** (1997), 446–452.

[3] S. Aytar, S. Pehlivan: *Statistically monotonic and statistically bounded sequences of fuzzy numbers*, Information Sciences, **176** (2006), 734–744.

[4] C. Çakan, B. Altay: *Statistically boundedness and statistical core of double sequences*, J. Math. Anal. Appl. **317** (2006), 690–697.

[5] J. Connor: *The statistical and strong p-Cesaro convergence of sequences*, Analysis **8** (1988), 47–63.

[6] H. Fast: *Sur la convergence statistique*, Colloq. Math. **2** (1951), 241–244.

[7] A. R. Freedman, J. J. Sember: *Densities and Summability*, Pasific J. Math. **95** (1981), 293–305.

[8] J. A. Fridy: *On statistical convergence*, Analysis **5** (1985), 301–313.

[9] J. A. Fridy: *Statistical limit points*, Proc. Amer. Math. Soc. **118** (1993), 1187–1192.

[10] J. A. Fridy, C. Orhan: *Statistical limit superior and limit inferior*, Proc. Amer. Math. Soc. **125** (1997), 3625–3631.

[11] S. Karakus: *Statistical convergence on probabilistic normed spaces*, Math. Comm. **12** (2007), 11–23.

[12] E. Kolk: *The statistical convergence in Banach spaces*, Acta Comment. Univ. Tartu. **928** (1991), 41–52.

[13] I. J. Maddox: *Statistical convergence in a locally convex space*, Math. Proc. Cambridge Phil. Soc. **104** (1988), 141–145.

[14] Mursaleen, O. H. H. Edely: *Statistical convergence of double sequences*, J. Math. Anal. Appl. **288** (2003), 223–231.

[15] T. Šalát: *On statistical convergent sequences of real numbers*, Math. Slovaca, **30** (1980), 139–150.

[16] E. Savaş: *On statistically convergent sequences of fuzzy numbers*, Information Sciences, **137** (2001), 277–282.

[17] A. N. Šerstnev: *On the notion of a random normed spaces*, Dokl. Akad. nauk SSSR, **149** (1963), 280–283.

[18] D. A. Sibley: *A metric for weak convergence of distribution functions*, Rocky Mountain J. Math. **1** (1971), 427–430.

[19] H. Steinhaus: *Sur la convergence ordinaire et la convergence asymptotique*, Collog. Math. **2** (1951), 73–74.

[20] B. Schweizer, A. Sklar: Probabilistic metric spaces, Elsevier/North-Holand, New York, 1983.