Note di Matematica
Note Mat. 29 (2009), n. 1, 73-78
ISSN 1123-2536, e-ISSN 1590-0932
DOI 10.1285/i15900932v29n1p73

Irrationality for a class of rational series

Eduardo Pascali
Department of Mathematics "'Ennio De Giorgi"',
University of Salento, P.O.B.123, 73100, Lecce, Italy
eduardo.pascali@unile.it

Received: 10/09/2007; accepted: 22/04/2008.

Abstract. The aim of this paper is to prove the irrationality of the sum of some classes of series of positive rational numbers.

Keywords: Irrationality, infinite series, infinite sequences

MSC 2000 classification: primary 11J72

1 Irrationality of rational series

Various criteria for the irrationality of the sum of infinite series of positive rationals are known, see the references. We prove another criterion, which is strictly related to the proof in [6] and to the results in [3], [4], [8].

The first result is an easy step for the final criterion and clarify the role of the hypotheses.

1 Proposition. *Given* $a_1 > 1$, $a_2 > 1$ *integers numbers, let* (a_n) *be the sequence defined by*

$$a_{n+1} = (a_1 a_2 \cdots a_n)^{\gamma_1}, \quad n > 2,$$

with $1 < \gamma_1 \in N$.

Let (b_n) *be a sequence of positive integers such that:*

$$\exists k_1, \gamma_2 \in]0, +\infty[\quad b_n \leq k_1 a_{n-1}^{\gamma_2} \quad \forall n > 2.$$

If $\gamma_2 \in [0, 1]$ *and* $\gamma_1(1 - \gamma_2) \geq 1$, *then the series* $\sum_j \frac{b_j}{a_j}$ *is convergent to an irrational number. Moreover for every non decreasing sequence* (c_j) *of positive integers the series* $\sum_j \frac{b_j}{a_j c_j}$ *is convergent to an irrational number.*

Proof. We remark that (a_n) is an increasing sequence and that by induction it can be checked that $a_{n+2} \geq 2^{(2\gamma_1)^n}$ for all n. Moreover, the following inequalities hold:

$$\begin{aligned} 0 < & \frac{b_j}{a_j} \leq k_1 \frac{(a_{j-1})^{\gamma_2}}{a_j} = k_1 \frac{1}{(a_1 \dots a_{j-2})^{\gamma_1(1-\gamma_2)} (a_{j-1})^{\gamma_1}} \\ \leq & k_1 \frac{1}{a_{j-1}^{\gamma_1}} \leq k_1 \frac{1}{2^{\gamma_1 (2\gamma_1)^{j-3}}} \end{aligned}$$

As a consequence, the series $\sum_j \frac{b_j}{a_j}$ is convergent. Arguing as in [6], assume that the sum is a rational number $\alpha = \frac{p}{q}$, where $p, q \in N$ and $1 \leq q$, and observe that:

$$A_n = \Big(p - q\sum_{j=1}^{n} \frac{b_j}{a_j}\Big) \prod_{j=1}^{n} a_j = q\Big(\prod_{j=1}^{n} a_j\Big) \sum_{j=n+1}^{+\infty} \frac{b_j}{a_j}$$

is a positive integer and $A_n \geq 1$. Now:

$$\begin{aligned} A_n =& q\Big(\prod_{j=1}^{n} a_j\Big) \sum_{j=n+1}^{+\infty} \frac{b_j}{a_j} = q a_{n+1}^{\frac{1}{\gamma_1}} \sum_{j=n+1}^{+\infty} \frac{b_j}{a_j} = q \sum_{j=n+1}^{+\infty} \frac{b_j a_{n+1}^{\frac{1}{\gamma_1}}}{a_j} \\ \leq& q \sum_{j=n+1}^{+\infty} \frac{b_j a_j^{\frac{1}{\gamma_1}}}{a_j} \leq k_1 q \sum_{j=n+1}^{+\infty} \frac{a_{j-1}^{\frac{1}{\gamma_2}}}{a_j^{\frac{\gamma_1 - 1}{\gamma_1}}} \leq k_1 q \sum_{j=n+1}^{+\infty} \frac{1}{(a_1 \ldots a_{j-2})^{\gamma_1 - \gamma_1\gamma_2 - 1} a_{j-1}^{\gamma_1 - 1}} \\ \leq& k_1 q \sum_{j=n+1}^{+\infty} \frac{1}{[2^{\gamma_1 - 1}]^{(2\gamma_1)^{j-3}}}. \end{aligned}$$

Then from the convergence of the series $\sum_{j=n+1}^{+\infty} \frac{1}{[2^{\gamma_1-1}]^{(2\gamma_1)^{j-3}}}$, we have $A_n < 1$ for n large and a contradiction follows. The last statement easily follows from the inequalities:

$$\frac{b_j}{c_j a_j} \leq \frac{b_j}{a_j}$$

and, since (c_j) is non a decreasing sequence:

$$\sum_{j=n+1}^{+\infty} \frac{b_j}{c_j a_j} \prod_{j=1}^{n} (c_j a_j) \leq \sum_{j=n+1}^{+\infty} \frac{c_{n+1} b_j}{c_j a_j} \prod_{j=1}^{n} a_j \leq \sum_{j=n+1}^{+\infty} \frac{b_j}{a_j} \prod_{j=1}^{n} a_j.$$

QED

Now we can state our main result.

2 Theorem. *Given $a_1 > 1$, $a_2 > 1$, and let (a_n) be a sequence of positive integers such that:*

$$\exists k_1, k_2, \alpha_1, \alpha_2 \in]0, +\infty[\quad k_1 (a_1 a_2 \cdots a_n)^{\alpha_1} \leq a_{n+1} \leq k_2 (a_1 a_2 \cdots a_n)^{\alpha_2} \tag{1}$$

for all $n > 3$. Let (b_n) be a sequence of positive integers such that

$$\exists k_3, \gamma \in [0, +\infty[\qquad b_n \leq k_3 a_{n-1}^{\gamma} \quad \forall n > 3. \tag{2}$$

If the following conditions hold:

$$\alpha_1 - \gamma\alpha_2 \geq 0 \tag{3}$$

$$(\alpha_1+\alpha_2)\frac{\alpha_1}{\alpha_2}>1;\quad \frac{k_2^{\frac{1}{\alpha_2}}}{k_1^{\alpha_1}}\frac{1}{2^{\alpha_1^2((\alpha_1+\alpha_2)\frac{\alpha_1}{\alpha_2}-1)}}<1 \tag{4}$$

$$\alpha_1>1 \tag{5}$$

$$\frac{k_1}{k_2^{\frac{\alpha_1}{\alpha_2}}}\geq 1, \tag{6}$$

then the series $\sum_j \frac{b_j}{a_j}$ *converges to an irrational number; moreover if* (c_j) *is an increasing sequences of positive integers then also* $\sum_j \frac{b_j}{c_j a_j}$ *converges to an irrational number.*

Proof. By (1) and (6) we have:

$$a_{n+1}\geq \frac{k_1}{k_2^{\frac{\alpha_1}{\alpha_2}}}a_n^{(\alpha_1+\alpha_2)\frac{\alpha_1}{\alpha_2}}\geq a_n.$$

Hence the sequence (a_n) is non decreasing. Moreover from (1) we deduce:

$$a_{n+3}\geq \frac{k_1^{\sum_{j=1}^{n}(\frac{\alpha_1}{\alpha_2}(\alpha_1+\alpha_2))^j}}{k_2^{\frac{\alpha_1}{\alpha_2}\sum_{j=1}^{n-1}(\frac{\alpha_1}{\alpha_2}(\alpha_1+\alpha_2))^j}}2^{2\alpha_1((\alpha_1+\alpha_2)\frac{\alpha_1}{\alpha_2})^n}\equiv D_n$$

and the following inequalities hold:

$$\frac{b_{n+4}}{a_{n+4}}\leq\frac{k_3a_{n+3}^{\gamma}}{a_{n+4}}\leq\frac{k_3k_2^{\gamma}}{k_1}\frac{1}{(a_1a_2\dots a_{n+2})^{\alpha_1-\gamma\alpha_2}a_{n+3}^{\alpha_1}}\leq\frac{k_3k_2^{\gamma}}{k_1}\frac{1}{a_{n+3}^{\alpha_1}}\leq\frac{k_3}{k}{}^{\gamma}_{2}k_1D_n^{-1}.$$

Consider now the series $\sum_n D_n^{-1}$, and remark that:

$$\frac{D_{n+1}^{-1}}{D_n^{-1}}=\left[\frac{k_2^{\frac{1}{\alpha_2}}}{k_1^{\alpha_1}}\frac{1}{2^{2\alpha_1((\alpha_1+\alpha_2)\frac{\alpha_1}{\alpha_2})}}\right]^{[\frac{\alpha_1}{\alpha_2}(\alpha_1+\alpha_2]^n}$$

Then by (4), the series $\sum_n D_n^{-1}$ is convergent; hence also $\sum_j \frac{b_j}{a_j}$ is convergent.

Assume that the sum is a rational number $\frac{p}{q}$, where $p,q\in N$ and $1\leq q$, as in the previous proposition. Observe that

$$A_n=\Big(p-q\sum_{j=1}^{n}\frac{b_j}{a_j}\Big)\prod_{j=1}^{n}a_j=q\Big(\prod_{j=1}^{n}a_j\Big)\sum_{j=n+1}^{+\infty}\frac{b_j}{a_j}$$

is a positive integer and $A_n \geq 1$. But we remark that:

$$
\begin{aligned}
1 \leq A_n &\leq \frac{q}{k_1^{\frac{1}{\alpha_1}}} \sum_{j=n+1}^{+\infty} \frac{b_j a_j^{\frac{1}{\alpha_1}}}{a_j} \leq \frac{k_3 q}{k_1^{\frac{1}{\alpha_1}}} \sum_{j=n+1}^{+\infty} \frac{a_{j-1}^{\gamma}}{a_j^{\frac{\alpha_1-1}{\alpha_1}}} \\
&\leq \frac{k_3 q}{k_1^{\frac{1}{\alpha_1}}} \sum_{j=n+1}^{+\infty} \frac{k_2^{\gamma}(a_1 \dots a_{j-2})^{\gamma\alpha_2}}{k_1^{\frac{\alpha_1-1}{\alpha_1}}(a_1 \dots a_{j-1})^{\alpha_1-1}} \\
&\leq \frac{q k_3 k_2^{\gamma}}{k_1} \sum_{j=n+1}^{+\infty} \frac{1}{(a_1 \dots a_{j-2})^{\alpha_1-\alpha_2\gamma} a_{j-1}^{\alpha_1-1}} \leq \frac{q k_3 k_2^{\gamma}}{k_1} \sum_{j=n+1}^{+\infty} \frac{1}{a_{j-1}^{\alpha_1-1}} \\
&\leq \frac{q k_3 k_2^{\gamma}}{k_1} \sum_{j=n+1}^{+\infty} \Big[\frac{k_2^{\frac{\alpha_1}{\alpha_2}\sum_{i=0}^{j-5}(\frac{\alpha_1}{\alpha_2}(\alpha_1+\alpha_2))^i}}{k_1^{\sum_{i=0}^{j-4}(\frac{\alpha_1}{\alpha_2}(\alpha_1+\alpha_2))^i}} \frac{1}{2^{2\alpha_1(\frac{\alpha_1}{\alpha_2}(\alpha_1+\alpha_2))^{j-4}}} \Big]^{\alpha_1-1} \\
&= \frac{q k_3 k_2^{\gamma}}{k_1} \sum_{j=n+1}^{+\infty} \Big[\frac{k_2^{\frac{1}{\alpha_2}\sum_{i=0}^{j-5}(\frac{\alpha_1}{\alpha_2}(\alpha_1+\alpha_2))^i}}{k_1^{\alpha_1\sum_{i=0}^{j-4}(\frac{\alpha_1}{\alpha_2}(\alpha_1+\alpha_2))^i}} \frac{1}{2^{2\alpha_1{}^2(\frac{\alpha_1}{\alpha_2}(\alpha_1+\alpha_2))^{j-4}}} \Big]^{\frac{\alpha_1-1}{\alpha_1}} \\
&= \frac{q k_3 k_2^{\gamma}}{k_1} \sum_{j=n+1}^{+\infty} (D_{j-4}^{-1})^{\frac{\alpha_1-1}{\alpha_1}}.
\end{aligned}
$$

Since the series $\sum_j (D_j^{-1})^{\frac{\alpha_1-1}{\alpha_1}}$ is convergent, we have $\lim_n A_n = 0$, a contradiction follows and the irrationality of the sum is proved.

The last statement can be proved as in Proposition 1. QED

Let us show some examples to which our result can be applied. If we choose $\alpha_1 = 2$ and $\alpha_2 = 4$, we can select $0 < \gamma \leq \frac{1}{2}$ and $0 < k_2 \leq k_1^2$.

Theorem 2 ensures the following abstract scheme for the irrationality of the sum of a rational series.

3 Theorem. *Consider a pair* $((a_n), \psi)$*, where* (a_n) *is a strictly increasing sequence of positive integers and* $\psi : [0, +\infty[\to]0, +\infty[$ *is an increasing function. Assume that*

$$\prod_{j=1}^{n} a_j \leq \psi(a_{n+1}). \tag{7}$$

and take $h : [1, +\infty[\to [0, +\infty[$ *non increasing such that*

$$\int_1^{+\infty} h(\xi) d\xi < +\infty. \tag{8}$$

Then for every sequence (b_j) *of positive integers for which:*

$$\frac{b_j}{a_j}\psi(a_j) \le h(a_j), \tag{9}$$

the series $\sum_j \frac{b_j}{a_j}$ *is convergent and the sum is an irrational number.*

Proof. Since

$$\frac{b_j}{a_j} \le \frac{b_j}{a_j}\psi(a_j)\frac{1}{\psi(a_j)} \le \frac{h(a_j)}{\psi(a_j)} \le \frac{h(a_j)}{\psi(1)},$$

for $n > a_1$ we have

$$\begin{aligned}\sum_{j=1}^{n}\frac{b_j}{a_j} \le &\sum_{j=1}^{n}\frac{h(a_j)}{\psi(1)} \le \frac{1}{\psi(1)}\sum_{j=1}^{n}\frac{h(a_j)(a_{j+1}-a_j)}{a_{j+1}-a_j}\\ \le &\frac{1}{\psi(1)}\sum_{j=1}^{n}h(a_j)(a_{j+1}-a_j) \le \frac{1}{\psi(1)}\int_{a_1}^{n}h(\xi)d\xi.\end{aligned}$$

Then the series $\sum_j \frac{b_j}{a_j}$ is convergent. Arguing as in the proof of Theorem 2, we have:

$$\begin{aligned}1 \le A_n =&q(\prod_{j=1}^{n}a_j)\sum_{j=n+1}^{+\infty}\frac{b_j}{a_j} \le q\psi(a_{n+1})\sum_{j=n+1}^{+\infty}\frac{b_j}{a_j} \le q\sum_{j=n+1}^{+\infty}\frac{b_j\psi(a_j)}{a_j}\\ \le &\sum_{j=n+1}^{+\infty}h(a_j) \le \sum_{j=n+1}^{+\infty}h(a_j)(a_{j+1}-a_j) \le q\int_{a_n}^{+\infty}h(\xi)d\xi.\end{aligned}$$

From (8), $A_n \to 0$ and we get a contradiction. QED

References

[1] Badea C.: *The irrationality of certain infinite series*; Glasgow Math. J. **29** (1987), 221–228.

[2] Badea C.: *A theorem on irrationality of infinite series and applications*; Acta Arith. XIII, **4** (1993), 313–323.

[3] Erdös P.: *Some problems and results on the irrationality of the sum of infinite series*; J. Math. Sci., **10** (1975), 1–7.

[4] Erdös P. and Strauss E.G.: *On the irrationality of certain Ahmes series*; J. Indian Math. Soc., **27** (1963), 129–133.

[5] Hančl J.: *Criterion for irrational sequences*; J. Number Theory, no. 1 **43** (1993), 88–92.

[6] Hančl J.: *Irrational rapidly convergent series*; Rend. Sem. Mat. Univ. Padova **107** (2002), 225–231.

[7] Hančl J.: *Irrational sequences of rational numbers*; Note di Mat. (2004), 1–10.

[8] Sándor J.: *Some classes of irrational numbers*; Studia Univ. Babes-Bolyai, Math. **XXIX** (1984), 3–12.