Note di Matematica
Note Mat. 29 (2009), n. 1, 61-72
ISSN 1123-2536, e-ISSN 1590-0932
DOI 10.1285/i15900932v29n1p61

Other classes of self-referred equations

Michele Miranda Jr.
Dipartimento di Matematica "Ennio De Giorgi"
Università del Salento
C.P. 193, 73100 Lecce, Italy
michele.miranda@unile.it

Eduardo Pascali
Dipartimento di Matematica "Ennio De Giorgi"
Università del Salento
C.P. 193, 73100 Lecce, Italy
eduardo.pascali@unile.it

Received: 10/09/2007; accepted: 18/04/2008.

Abstract. In a previous note [4] we have established some existence and uniqueness theorem for a new type of differential equation that we have defined hereditary and self-referred type. In this note we continue the investigation by proving local existence and uniqueness theorems for other classes of self-referred differential equations.

The classes of equations that we are going to consider are:

$$\frac{\partial}{\partial t}u(x,t) = u\left(\int_0^t u(x,s)ds + \psi(u(x,t)), t\right),$$

$$\frac{\partial^2}{\partial t^2}u(x,t) = u\left(\int_{x-\delta(x,t)}^{x+\delta(x,t)} \frac{\partial}{\partial t}u(\xi,t)d\xi, t\right).$$

Keywords: Non-linear evolution equations; Self-referred differential equations; Iteration

MSC 2000 classification: 47J35; 45G10

1 Introduction

In a previous note [4] we have started the study of a new class of functional differential equations that can be adapted phenomena which evolution presents self-reference with respect to his history.

Phenomena of hereditary type have been treated since the beginning of XX-th century ([10], [3] and [2]), but the attention for phenomena of self-reference type with respect to the state itself of the phenomenon is recent [1], [9] and [8].

In this note, by continuing the inspection started in [4], and by essentially using the same line of proof, we introduce and study some new type of functional differential equations establishing for them theorems of local existence and uniqueness. Also for these equations there are several open problems, as already signalized in [4].

2 A class of hereditary and self-referred equations

Let X be the space of continuous functions $u : R^2 \to R$ and $\psi : R \to R$.

1 Theorem. *If ψ satisfies the following condition: $\exists L_\psi \geq 0$ such that:*

$$|\psi(\xi_1) - \psi(\xi_2)| \leq L_\psi |\xi_1 - \xi_2| \quad \forall \xi_1, \xi_2 \in R \tag{1}$$

and if u_0 satisfies the following conditions: $\exists L_0 \geq 0$ such that

$$|u_0(x) - u_0(y)| \leq L_0 |x - y|, \quad \forall x, y \in R \tag{2}$$

and

$$\|u_0\|_\infty < \infty, \tag{3}$$

then there exist $T_0 > 0$ and a unique u continuous on $R \times [0, T_0]$ such that:

(1) u is bounded in $R \times [0, T_0]$

(2) u is Lipschitz with respect to x uniformly with respect to $t \in [0, T_0]$ (and Lipschitz in t uniformly with respect to x) and moreover, for $x \in R$ and $t \in [0, T_0]$:

$$u(x,t) = u_0(x) + \int_0^t u\left(\int_0^\tau u(x,s)ds + \psi(u(x,\tau)), \tau\right) d\tau; \tag{4}$$

whence

$$\begin{cases} \dfrac{\partial}{\partial t} u(x,t) = u\left(\displaystyle\int_0^t u(x,s)ds + \psi(x,t), t\right), x \in R, t \in [0, T_0] \\ u(x,0) = u_0(x). \end{cases} \tag{5}$$

Moreover the solution u can be prolonged for all $t \geq T_0$.

Proof. Let us consider the sequence (u_n), defined by recurrence:

$$\begin{aligned} u_1(x,t) &= u_0(x) + \int_0^t u_0\left(\int_0^\tau u_0(x)ds + \psi(u_0(x))\right) d\tau \\ &= u_0(x) + \int_0^t u_0\left(u_0(x)\tau + \psi(u_0(x))\right) d\tau, \end{aligned} \tag{6}$$

and for $n \geq 1$

$$u_{n+1}(x,t) = u_0(x) + \int_0^t u_n\left(\int_0^\tau u_n(x,s)ds + \psi(u_n(x,\tau)), \tau\right) d\tau. \tag{7}$$

Directly from the definition of u_1 we can deduce that:

$$|u_1(x,t)| \leq \|u_0\|_\infty + t\|u_0\|_\infty = \|u_0\|_\infty(1+t), \quad \forall x \in R, t \geq 0. \tag{8}$$

and then, in general, for every $n \in N$:

$$|u_n(x,t)| \leq \|u_0\|_\infty \left(1 + t + \cdots + \frac{t^n}{n!}\right) \leq \|u_0\|_\infty e^t, \quad \forall x \in R, t \geq 0. \tag{9}$$

Moreover

$$|u_1(x,t) - u_0(x)| \leq t\|u_0\|_\infty := A_1(t),$$

and also:

$$\begin{aligned}
|u_1(x,t) - u_1(y,t)| \leq & L_0|x-y| + \int_0^t L_0 \Big| \int_0^\tau u_0(x)ds + \\
& + \psi(u_0(x)) - \int_0^\tau u_0(y)ds - \psi(u_0(y)) \Big| d\tau \leq \\
\leq & L_0|x-y| + \int_0^t \Big(L_0 \Big[\int_0^\tau L_0|x-y|ds + L_\psi L_0|x-y| \Big] \Big) d\tau = \\
= & \Big(L_0 + \int_0^t L_0 \Big[\int_0^\tau L_0 ds + L_\psi L_0 \Big] d\tau \Big) |x-y| = \\
= & (L_0 + L_0^2 \frac{t^2}{2} + L_\psi L_0^2 t)|x-y|.
\end{aligned} \tag{10}$$

Now, if we define

$$L_1(t) := L_0 + \frac{1}{2}\left(\int_0^t L_0 d\tau\right)^2 + L_\psi \int_0^t L_0^2 d\tau, \tag{11}$$

or, more generally:

$$L_{n+1}(t) := L_0 + \frac{1}{2}\left(\int_0^t L_n(\tau) d\tau\right)^2 + L_\psi \int_0^t L_n(\tau)^2 d\tau, \tag{12}$$

for every $n \geq 1$, we can prove, by induction:

$$|u_n(x,t) - u_n(y,t)| \leq L_n(t)|x-y| \quad \forall x \in R \quad \forall t \geq 0. \tag{13}$$

Moreover, since:

$$\begin{aligned} u_{n+1}(x,t) - u_n(x,t) = \int_0^t \Big(u_n \left(\int_0^\tau u_n(x,s)ds + \psi(u_n(x,\tau)), \tau \right) + \\ - u_{n-1} \left(\int_0^\tau u_n(x,s)ds + \psi(u_n(x,\tau)), \tau \right) + \\ + u_{n-1} \left(\int_0^\tau u_n(x,s)ds + \psi(u_n(x,\tau)), \tau \right) + \\ - u_{n-1} \left(\int_0^\tau u_{n-1}(x,s)ds + \psi(u_{n-1}(x,\tau)), \tau \right) \Big) d\tau \end{aligned}$$

we can deduce that:

$$\begin{aligned} &|u_{n+1}(x,t) - u_n(x,t)| \leq \\ &\leq \int_0^t \Big(A_n(\tau) + L_{n-1}(\tau) \left(\int_0^\tau A_n(s)ds + L_\psi A_n(\tau) \right) \Big) d\tau, \end{aligned} \tag{14}$$

if we define, by recurrence, for every $n \in N$:

$$A_{n+1}(t) := \int_0^t \Big(A_n(\tau) + L_{n-1}(\tau) \left(\int_0^\tau A_n(s)ds + L_\psi A_n(\tau) \right) \Big) d\tau. \tag{15}$$

We remark that:

$$0 \leq L_{n+1}(t) - L_0 = \frac{1}{2} \left(\int_0^t L_n(\tau) d\tau \right)^2 + L_\psi \int_0^t L_n(\tau)^2 d\tau.$$

We fix $k_0 > 0$ and $h < 1$; we can then choose $T_0 > 0$ in such a way that

$$t \in [0, T_0] \Longrightarrow \begin{cases} \frac{1}{2} L_0^2 t^2 + L_\psi L_0^2 t \leq k_0 \\ \frac{1}{2} k_0^2 t^2 + L_\psi (k_0 + L_0)^2 t \leq k_0 \\ 0 \leq t + (k_0 + L_0) \left(\frac{t^2}{2} + L_\psi t \right) \leq h < 1. \end{cases} \tag{16}$$

Hence:

$$0 \leq L_1(t) - L_0 = \frac{1}{2} L_0^2 t^2 + L_\psi L_0^2 t \leq k_0,$$

$$\begin{aligned} 0 \leq L_2(t) - L_0 &= \frac{1}{2} (\int_0^t L_1(\tau) d\tau)^2 + L_\psi \int_0^t L_1(\tau)^2 d\tau \\ &\leq \frac{1}{2} k_0^2 t^2 + L_\psi (k_0 + L_0)^2 t \leq k_0 \end{aligned}$$

and then, by induction:

$$0 \le L_n(t) - L_0 \le k_0 \quad \forall n \in N, \forall t \in [0, T_0].$$

Then

$$0 \le L_n(t) \le L_0 + k_0 := k_1 \quad \forall n \in N. \tag{17}$$

From (15), (16) and (17) there follows:

$$\begin{aligned} 0 \le A_{n+1}(t) &\le \int_0^t (\|A_n\| + k_1(\|A_n\|\tau + L_\psi \|A_n\|))\, d\tau \\ &= \|A_n\| \left(t + k_1 \left(\frac{t^2}{2} + L_\psi t \right) \right) \le h\|A_n\|. \end{aligned}$$

As a consequence,

$$\frac{\|A_{n+1}\|}{\|A_n\|} \le h, \quad \forall n \in N;$$

then there follows that the series with general term given by $A_n(t)$ is totally convergent and so there exists $u_\infty : R \times [0, T_0] \to R$ such that:

$$\lim_{n \to +\infty} u_n(x,t) = u_\infty(x,t)$$

uniformly in $R \times [0, T_0]$; moreover u_∞ is bounded and Lipschitz continuous with respect to the variable x with Lipschitz constant $L_\infty(t) \le k_1$ (and u_∞ is, of course, Lipschitz in t uniformly with respect to x). Minding that:

$$\begin{aligned} &\left| u_n \left(\int_0^t u_n(x,s)ds + \psi(u_n(x,t)), t \right) - u_\infty \left(\int_0^t u_\infty(x,s)ds + \psi(u_\infty(x,t)), t \right) \right| \\ &\le \|u_n - u_\infty\|_{L^\infty(R\times[0,T_0])} + \\ &\quad + L_\infty(t) k \int_0^t |u_n(x,s) - u_\infty(x,s)| ds + |\psi(u_n(x,t)) - \psi(u_\infty(x,t))| \\ &\le \|u_n - u_\infty\|_{L^\infty(R\times[0,T_0])} + k_1 \int_0^t \|u_n - u_\infty\|_{L^\infty(R\times[0,T_0])} ds + \\ &\quad + L_\psi \|u_n - u_\infty\|_{L^\infty(R\times[0,T_0])}, \end{aligned}$$

from (16) and (17), we can deduce that:

$$u_\infty(x,t) = u_0(x) + \int_0^t u_\infty \left(\int_0^\tau u_\infty(x,s)ds + \psi(u_\infty(x,t)), \tau \right) d\tau \tag{18}$$

and then u_∞ is a solution of the problem:

$$\begin{cases} \dfrac{\partial}{\partial t} u_\infty(x,t) = u_\infty \left(\displaystyle\int_0^t u_\infty(x,s)ds + \psi(u_\infty(x,t)), t \right) \\ u_\infty(x,0) = u_0(x), \quad x \in R, t \in [0, T_0]. \end{cases}$$

In order to prove the uniqueness, we assume that there exists a real continuous function on $R \times [0, T_0]$, say v, such that, for every $x \in R$ and $t \in [0, T_0]$:

$$v(x,t) = u_0(x) + \int_0^t v\left(\int_0^\tau v(x,s)ds + \psi(v(x,s)), \tau\right) d\tau.$$

Then for every compact $K \subset R$, if $x \in K$:

$$\begin{aligned}
|v(x,t) - u_\infty(x,t)| \leq & \int_0^t \Big| v(\int_0^\tau v(x,s)ds + \psi(v(x,\tau)), \tau) + \\
& - u_\infty(\int_0^\tau v(x,s)ds + \psi(v(x,\tau)), \tau) + \\
& + u_\infty(\int_0^\tau v(x,s)ds + \psi(v(x,\tau)), \tau) + \\
& - u_\infty(\int_0^\tau u_\infty(x,s)ds + \psi(u_\infty(x,\tau)), \tau) \Big| d\tau \\
\leq & \int_0^t \Big(\|v - u_\infty\|_{L^\infty(K\times[0,T_0])} + \\
& + k_1 \Big| \int_0^\tau (\|v - u_\infty\|_{L^\infty(K\times[0,T_0])} + \\
& + L_\psi \|v - u_\infty\|_{L^\infty(K\times[0,T_0])}) \Big| \Big) d\tau \leq \\
\leq & \|v - u_\infty\|_{L^\infty(K\times[0,T_0])} \int_0^t (1 + k_1\tau + L_\psi L_\infty) d\tau \\
\leq & h\|v - u_\infty\|_{L^\infty(K\times[0,T_0])} < \|v - u_\infty\|_{L^\infty(K\times[0,T_0])}.
\end{aligned}$$

Therefore $u_\infty = v$ in $K \times [0, T_0]$, and then the uniqueness follows.

We remark that: $v_0(x) = u(x, T_0)$ verifies the following conditions:

$$|v_0(x) - v_0(y)| \geq k_1|x - y|; \qquad ||v_o||_\infty \leq ||u_0||\infty e^{T_0};$$

hence the local solution $u = u(x,t)$ can be prolonged. QED

3 A class of self-referred equation

In this section we establish a theorem, analogue to the one proved in Section 2, concerning the following class of equations:

$$\frac{\partial^2}{\partial t^2} u(x,t) = u\left(\int_{x-\delta(x,t)}^{x+\delta(x,t)} \frac{\partial}{\partial t} u(\xi,t) d\xi, t\right). \tag{19}$$

The Theorem that we are going to prove is the following:

2 Theorem. *Let* $\alpha \in L^\infty(R,R) \cap Lip(R,R)$ *and* $\beta \in L^\infty(R,R)$ *be given functions and let* $\delta : R^2 \to]0,+\infty[$ *be a real and bounded function, Lipschitz continuous in the* x *variable uniformly with respect to* t*, that is there exists* $L_\delta > 0$ *such that*

$$|\delta(x,t) - \delta(y,t)| \le L_\delta |x-y|, \quad \forall x, y \in R, \forall t;$$

then there exists $T_0 > 0$ *and there exists a unique* $u : R \times [0,T_0] \to R$ *continuous, bounded and Lipschitz in* x *uniformly with respect to* t *(and Lipschitz in* t *uniformly with respect to* x*) such that:*

$$u(x,t) = \alpha(x) + t\beta(x) + \int_0^t \int_0^\tau u\left(\int_{x-\delta(x,s)}^{x+\delta(x,s)} \frac{\partial}{\partial s} u(\xi,s) d\xi, s\right) ds d\tau,$$

that is

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} u(x,t) = u\left(\displaystyle\int_{x-\delta(x,t)}^{x+\delta(x,t)} \frac{\partial}{\partial t} u(x,t) d\xi, t\right) \\ u(x,0) = \alpha(x) \\ \dfrac{\partial}{\partial t} u(x,0) = \beta(x), \end{cases} \qquad x \in R, t \in [0,T_0]$$

Moreover the solution u *can be prolonged for all* $t \ge T_0$*.*

Proof. Let us consider the following sequence, defined by recurrence:

$$u_{n+1}(x,t) = \alpha(x) + t\beta(x) + \\ + \int_0^t \int_0^\tau u_n\left(\int_{x-\delta(x,s)}^{x+\delta(x,s)} \frac{\partial}{\partial s} u_n(\xi,s) d\xi, s\right) ds d\tau \tag{20}$$

with:

$$u_1(x,t) = \alpha(x) + t\beta(x) + \int_0^t \int_0^\tau \alpha\left(\int_{x-\delta(x,s)}^{x+\delta(x,s)} \beta(\xi) d\xi\right) ds d\tau.$$

Since:

$$|u_1(x,t)| \le \|\alpha\|_\infty + \|\beta\|_\infty t + \|\alpha\|_\infty \frac{t^2}{2},$$

we get that:

$$\begin{aligned}|u_2(x,t)| \leq & \|\alpha\|_\infty + \|\beta\|_\infty t \\ & + \int_0^t \int_0^\tau \left(\|\alpha\|_\infty + \|\beta\|_\infty s + \|\alpha\|_\infty \frac{s^2}{2} \right) ds \\ \leq & \|\alpha\|_\infty \left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} \right) + \|\beta\|_\infty \left(t + \frac{t^3}{3!} \right) \\ \leq & (\|\alpha\|_\infty + \|\beta\|_\infty) \left(1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + \frac{t^4}{4!} \right).\end{aligned} \tag{21}$$

and then, in general:

$$|u_{n+1}(x,t)| \leq (\|\alpha\|_\infty + \|\beta\|_\infty)e^t, \quad \forall n \in N, \forall t > 0. \tag{22}$$

In the same way it is possible to prove that

$$\left| \frac{\partial}{\partial t} u_n(x,t) \right| \leq (\|\alpha\|_\infty + \|\beta\|_\infty)e^t, \quad \forall n \in N, \forall t > 0. \tag{23}$$

We remark that:

$$|u_1(x,t) - \alpha(x)| \leq t\|\beta\|_\infty + \|\alpha\|_\infty \frac{t^2}{2} := A_1(t),$$

$$\left| \frac{\partial}{\partial t} u_1(x,t) - \beta(x) \right| \leq \|\alpha\|_\infty t := B_1(t).$$

Moreover:

$$\begin{aligned}&|u_2(x,t) - u_1(x,t)| \leq \\ &\leq \int_0^t \int_0^\tau \left(A_1(s) + L_\alpha \int_{x-\delta(x,s)}^{x+\delta(x,s)} \left| \frac{\partial}{\partial s} u_1(\xi, s) - \beta(\xi) \right| d\xi \right) ds d\tau \\ &\leq \int_0^t \int_0^\tau \left(A_1(s) + L_\alpha \int_{x\delta(x,s)}^{x+\delta(x,s)} B_1(s) d\xi \right) ds d\tau \\ &= \int_0^t \int_0^\tau (A_1(s) + L_\alpha 2\delta(x,s) B_1(s)) ds d\tau \\ &\leq \int_0^t \int_0^\tau (A_1(s) + L_\alpha M B_1(s)) ds d\tau := A_2(t),\end{aligned} \tag{24}$$

where L_α denotes the Lipschitz constant Lipschitz of the function α e $M = 2\|\delta\|_\infty$. In general we define, for $n \geq 1$:

$$A_{n+1}(t) = \int_0^t \int_0^\tau (A_n(s) + L_{n-1}(s) M B_n(s)) ds d\tau \tag{25}$$

and

$$B_{n+1}(t) = \int_0^t (A_n(s) + ML_{n-1}(s)B_n(s))ds, \tag{26}$$

and we note that:

$$\begin{aligned}
&\left|\frac{\partial}{\partial t}u_2(x,t) - \frac{\partial}{\partial t}u_1(x,t)\right| = \\
&= \Big| \int_0^t u_1\left(\int_{x-\delta(x,s)}^{x+\delta(x,s)} \frac{\partial}{\partial s}u_1(\xi,s)d\xi, s\right) ds - \int_0^t \alpha\left(\int_{x-\delta(x,s)}^{x+\delta(x,s)} \beta(\xi)d\xi\right) ds\Big| \\
&\qquad\qquad \le \int_0^t (A_1(s) + L_\alpha MB_1(s))ds. \qquad (27)
\end{aligned}$$

Minding that, if we denote by L_β the Lipschitz constant of the function β:

$$\begin{aligned}
|u_1(x,t) - u_1(y,t)| \le& (L_\alpha + tL_\beta)|x-y| \\
&+ \int_0^t \int_0^\tau L_\alpha \Big| \int_{x-\delta(x,s)}^{x+\delta(x,s)} \beta(\xi)d\xi + \\
&- \int_{y-\delta(y,s)}^{y+\delta(y,s)} \beta(\xi)d\xi \Big| dsd\tau \\
\le& (L_\alpha + tL_\beta)|x-y| + \\
&+ \int_0^t \int_0^\tau L_\alpha 2(1+L_\delta)\|\beta\|_\infty |x-y| \\
=& \left(L_\alpha + tL_\beta + 2L_\alpha(1+L_\delta)\|\beta\|_\infty \frac{t^2}{2}\right)|x-y|,
\end{aligned}$$

we can define the Lipschitz constant of u_1 by:

$$L_1(t) = L_\alpha + tL_\beta + 2L_\alpha(1+L_\delta)\|\beta\|_\infty \frac{t^2}{2}.$$

We fix k_0 and we choose $T_0 > 0$ in such a way that

$$tL_\beta + 4(1+L_\delta)e^{T_0}(\|\alpha\|_\infty + \|\beta\|_\infty)(L_\alpha + k_0)\frac{t^2}{2} \le k_0. \tag{28}$$

Now in general, we can deduce that for $n \ge 1$:

$$|u_{n+1}(x,t) - u_{n+1}(y,t)| \le L_{n+1}(t)|x-y|, \quad \forall x,y \in R, \forall t > 0,$$

where:

$$\begin{aligned}
L_{n+1}(t) =& L_\alpha + tL_\beta + \\
&+ 2(1+L_\delta)\int_0^t \int_0^\tau L_n(s) \left\|\frac{\partial}{\partial t}u_n\right\|_{L^\infty(R\times[0,T_0])} dsd\tau. \qquad (29)
\end{aligned}$$

Then:

$$L_{n+1}(t) \leq L_\alpha + tL_\beta + \\ + 4(1+L_\delta)(\|\alpha\|_\infty + \|\beta\|_\infty)\int_0^t\int_0^\tau e^{T_0}L_n(s)dsd\tau \leq \\ \leq L_\alpha + tL_\beta + 4(1+L_\delta)e^{T_0}(\|\alpha\|_\infty + \|\beta\|_\infty)\int_0^t\int_0^\tau L_n(s)dsd\tau. \quad (30)$$

From (28), we obtain that:

$$0 \leq L_1(t) - L_\alpha \leq tL_\beta + 4(1+L_\delta)e^{T_0}(\|\alpha\|_\infty + \|\beta\|_\infty)L_\alpha\frac{t^2}{2} \leq k_0, \quad (31)$$

for $t \in [0, T_0]$. In the same way:

$$\begin{aligned} 0 \leq L_2(t) - L_\alpha \leq & tL_\beta + 4(1+L_\delta)(\|\alpha\|_\infty + \|\beta\|_\infty)e^{T_0}\int_0^t\int_0^\tau L_1(s) \\ \leq & tL_\beta + 4e^{T_0}(1+L_\delta)(\|\alpha\|_\infty + \|\beta\|_\infty)(k_0 + L_\alpha)\frac{t^2}{2} \\ \leq & k_0, \end{aligned}$$

$t \in [0, T_0]$. Hence, by induction, we can prove that:

$$0 \leq L_n(t) \leq k_0 + L_\alpha := k_1, \quad \forall n \in N, \forall t \in [0, T_0]. \quad (32)$$

Using the estimate (32), from (25) we can deduce that, $\forall n \in N$ and $\forall t \in [0, T_0]$:

$$A_{n+1}(t) \leq \int_0^t\int_0^\tau (A_n(s) + M(k_0 + L_\alpha)B_n(s))ds; \quad (33)$$

and from (26), $\forall n \in N, \forall t \in [0, T_0]$:

$$B_{n+1}(t) \leq \int_0^t (A_n(s) + M(k_0 + L_\alpha)B_n(s))ds, \quad (34)$$

for every $n \in N$. We have obtained then that:

$$A_{n+1}(t) \leq (\|A_n\| + Mk_1\|B_n\|)\frac{t^2}{2}, \quad \forall n \in N, \forall t \in [0, T_0]. \quad (35)$$

$$B_{n+1}(t) \leq (\|A_n\| + Mk_1\|B_n\|)t, \quad \forall n \in N, \forall t \in [0, T_0]. \quad (36)$$

If at this point we assume that T_0 has been choosen in such a way that for $t \in [0, T_0]$ we have that

$$0 \leq (1 + Mk_1)\left(\frac{t^2}{2} + t\right) \leq 2h < 1, \quad (37)$$

we obtain that

$$A_{n+1}(t) + B_{n+1}(t) \le \left(\frac{t^2}{2} + t\right)(\|A_n\| + Mk_1\|B_n\|) \le$$
$$\le \left(\frac{t^2}{2} + t\right)(Mk_1 + 1)(\|A_n\| + \|B_n\|) \le h(\|A_n\| + \|B_n\|). \quad (38)$$

We have then proved that the series

$$\sum (A_{n+1}(t) + B_{n+1}(t)), \quad t \in [0, T_0]$$

is totally convergent and, then, the same is true for the series

$$\sum A_{n+1}(t), \quad \sum B_{n+1}(t).$$

There follows that:

$$\begin{array}{l} u_n \rightrightarrows u_\infty \\ \frac{\partial}{\partial t} u_n \rightrightarrows \frac{\partial}{\partial t} u_\infty, \end{array} \quad \text{in } R \times [0, T_0].$$

On the other hand, u_∞ is Lipschitz continuous in x uniformly with respect to $t \in [0, T_0]$ with Lipschitz constant $L_\infty(t) \le k_1$ (and Lipschitz in t uniformly with respect to x); moreover

$$\left| u_n \left(\int_{x-\delta(x,t)}^{x+\delta(x,t)} \frac{\partial}{\partial t} u_n(\xi, t) d\xi, t \right) - u_\infty \left(\int_{x-\delta(x,t)}^{x+\delta(x,t)} \frac{\partial}{\partial t} u_\infty(\xi, t) d\xi, t \right) \right| \le$$
$$\le \|u_n - u_\infty\|_{L^\infty(R \times [0,T_0])} + k_1 \int_{x-\delta(x,t)}^{x+\delta(x,t)} \left| \frac{\partial}{\partial t} u_n(\xi, t) - \frac{\partial}{\partial t} u_\infty(\xi, t) \right| d\xi$$
$$\le \|u_n - u_\infty\|_{L^\infty(R \times [0,T_0])} + k_1 \left\| \frac{\partial}{\partial t} u_n - \frac{\partial}{\partial t} u_\infty \right\|_{L^\infty(R \times [0,T_0])} 2M \longrightarrow 0.$$

We can then easily deduce that:

$$u_\infty(x, t) = \alpha(x) + t\beta(x) + \int_0^t \int_0^\tau u_\infty \left(\int_{x-\delta(x,s)}^{x+\delta(x,s)} \frac{\partial}{\partial t} u_\infty(\xi, s) d\xi, s \right) ds d\tau$$

for every $x \in R$, $t \in [0, T_0]$. Uniqueness of the solution follows easily, arguing as in Theorem 1.

We remark that:

$$|u(x, T_0)| \le (||\alpha|| + ||\beta||)e^{T_0}; \quad |\frac{\partial}{\partial t} u(x, T_0)| \le (||\alpha|| + ||\beta||)e^{T_0} \le (||\alpha|| + ||\beta||)e^{T_0}$$

and

$$|u(x, T_0) - u(y, T_0)| \le k_1 |x - y|,$$

hence the solution $u = u(x, t)$ can be prolonged. QED

References

[1] E. Eder: *The functional-differential equation $x'(t) = x(x(t))$*, J. Differential Equations, no. 3 **54** (1984), 390–400.

[2] G. Fichera: *Having a long memory creates serious problems*, Arch. Rational Mech. Anal., no. 2 **70** (1979), 101–112.

[3] G. Fichera: *Analytic problems of materials with memory*, In Theoretical and applied mechanics (Proc. 15th Internat. Congr., Univ. Toronto, Toronto, Ont., 1980) North-Holland, Amsterdam, (1980), 223–230.

[4] M. Jr. Miranda, E. Pascali: *On a type of evolution of self-referred and hereditary phenomena*, Aequationes Math., **71** (2006), 253–268.

[5] M. Jr. Miranda, E. Pascali: *On a class of differential equations with self-reference*, Rend. Matem, **25** (2005), 155–164.

[6] E. Pascali: *Existence of solutions to a self-referred and hereditary system of differential equations*, Electronic J. Differential Equations, **6** (2006), 1–7.

[7] E. Pascali, Le V. Ut: *An existence theorem for self-referred and hereditary differential equations*, Adv. differential Equations and Control Processes, in press.

[8] Jian-Guo Si, Sui Sun Cheng: *Analytic solutions of a functional-differential equation with state dependent argument*, Taiwanese J. Math., no. 1 **4** (1997), 471–480.

[9] Jian-Guo Si, Xin-Ping Wang, Sui Sun Cheng: *Analytic solutions of a functional-differential equation with a state derivative dependent delay*, Aequationes Math., no. 1 **57** (1999), 75–86.

[10] V. Volterra: *Opere matematiche: Memorie e note*, Pubblicate a cura dell'Accademia Nazionale dei Lincei col concorso del Consiglio Nazionale delle Ricerche. Accademia Nazionale dei Lincei, Rome, **V** (1962), 1926–1940.