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Differential calculus on Hopf Group Coalgebra

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Abstract. In this paper we construct the Differential calculus on the Hopf Group Coalgebra introduced by Turaev [10]. We proved that the concepts introduced by S.L.Woronowicz in constructing Differential calculus on Hopf Compact Matrix Pseudogroups (Quantum Groups) [5] can be adapted to serve again in our construction.

Keywords: Differential calculus, Hopf algebra, Hopf Group Coalgebra

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Introduction

Quantum groups, from a mathematical point of view, may be introduced by making emphasis on their $q-$deformed enveloping algebra aspects [2], [3] which leads to the quantized enveloping algebras, or by making emphasis in the $R-$matrix formalism that describes the deformed group algebra. Also, they are mathematically well defined in the framework of Hopf algebra [1]. Quantum groups provide an interesting example of non-commutative geometry [8]. Non-commutative differential calculus on quantum groups is a fundamental tool needed for many applications [7], [6].

S.L.Woronowicz [5] gave the general framework for bicovariant differential calculus on quantum groups following general ideas of A.Connes. Also, He showed that all important notions and formulae of classical Lie group theory admit a generalization to the quantum group case and he has restricted himself to compact matrix pseudogroups as introduced in [4].In contrast to the classical differential geometry on Lie groups, there is no functorial method to obtain a unique bicovariant differential calculus on a given quantum group [9].

Recently, Quasitriangular Hopf $\pi-$coalgebras are introduced by Turaev [10]. He has showed that they give rise to crossed $\pi-$categories. Virelizier [11] studied the algebraic properties of the Hopf $\pi-$coalgebras, also he has showed that the existence of integrals and trace for such coalgebras and has generalized the main properties of the quasitriangular Hopf algebras to the setting of Hopf $\pi-$coalgebra.

In this paper we will use the concepts introduced by S.L.Woronowicz [5] to construct the Differential calculus on the Hopf group coalgebra(introduced by Turaev [10]). We briefly describe the content of the paper.In section one we give the definition of Hopf group coalgebras [11]. In section two, we give the main definitions and theorems concerning first order differential calculus. Section three contains the construction of the $\pi-$graded Bicovariant bimodules. Finally, in section four we construct the first order differential calculus on the Hopf group coalgebra.

Now let us give some basic definitions about Hopf $\pi-$coalgebra where π is a non-commutative discrete group.

1 Hopf Group Coalgebra

1 Definition. A $\pi-$coalgebra is a family $C = \{C_\alpha\}_{\alpha\in\pi}$ of $k-$linear spaces endowed with a family $\Delta = \{\Delta_{\alpha,\beta} : C_{\alpha\beta} \to C_\alpha \otimes C_\beta\}_{\alpha,\beta\in\pi}$ of $k-$linear maps (the comultiplication) and a $k-$linear map $\varepsilon : C_1 \to k$ such that

- Δ is coassociative in the sense that for any $\alpha, \beta, \gamma \in \pi$,

$$(\Delta_{\alpha,\beta} \otimes \mathrm{id})\Delta_{\alpha\beta,\gamma} = (\mathrm{id} \otimes \Delta_{\beta,\gamma})\Delta_{\alpha,\beta\gamma}\ ,$$

- for all $\alpha \in \pi$,

$$(\mathrm{id} \otimes \varepsilon)\Delta_{\alpha,1} = (\varepsilon \otimes \mathrm{id})\Delta_{1,\alpha}.$$

2 Notation. [Sweedler's notation] In the case of Hopf group coalgebra Sweedler's notations have been extended by Turaev [10] and Virelizier [11] in the following way: for any $\alpha, \beta \in \pi$ and $c \in C_{\alpha\beta}$, they defined

$$\Delta_{\alpha,\beta}(c) = \sum_{(c)} c_{(1,\alpha)} \otimes c_{(2,\beta)} \in C_\alpha \otimes C_\beta.$$

or shortly, if we have the summation implicit

$$\Delta_{\alpha,\beta}(c) = c_{(1,\alpha)} \otimes c_{(2,\beta)}.$$

The coassociativity axiom gives that , for any $\alpha, \beta, \gamma \in \pi$ and $c \in C_{\alpha\beta,\gamma}$

$$c_{(1,\alpha\beta)(1,\alpha)} \otimes c_{(1,\alpha\beta)(2,\beta)} \otimes c_{(2,\gamma)} = c_{(1,\alpha)} \otimes c_{(2,\beta\gamma)(1,\beta)} \otimes c_{(2,\beta\gamma)(2,\gamma)}.$$

Let $C = (\{C_\alpha\}_{\alpha\in\pi}, \Delta, \varepsilon)$ be a $\pi-$coalgebra and A be an algebra with multiplication m and unit element 1_A. The family Δ and the map m induce a map

$$* : \mathrm{conv}(C, A) \otimes \mathrm{conv}(C, A) \to \mathrm{conv}(C, A)$$

defined by the composition

$$\mathrm{Hom}(C_\alpha, A) \otimes \mathrm{Hom}(C_\beta, A) \xrightarrow{\rho} \mathrm{Hom}(C_\alpha \otimes C_\beta, A \otimes A) \xrightarrow{\mathrm{Hom}(\Delta_{\alpha,\beta},m)} \mathrm{Hom}(C_{\alpha\beta}, A)$$

where ρ is the natural injection of $\mathrm{Hom}(C_\alpha, A) \otimes \mathrm{Hom}(C_\beta, A)$ into $\mathrm{Hom}(C_\alpha \otimes C_\beta, A \otimes A)$

The map $*$ is called convolution product of f, g
Also, the maps

$$\varepsilon : C_1 \longrightarrow k \quad \text{and} \quad \eta : k \longrightarrow A$$

induce a map

$$\eta_{\mathrm{Conv}(C,A)} : k \longrightarrow \mathrm{Conv}(C, A)$$

defined by

$$(\eta_{\mathrm{Conv}(C,A)}(\lambda))(c) = \varepsilon(c)\eta(\lambda)$$

for all $c \in C_1$.

3 Lemma. *The k-space*

$$\mathrm{Conv}(C, A) = \bigoplus_{\alpha\in\pi} \mathrm{Hom}(C_\alpha, A)$$

endowed with the convolution product $*$ and the unit element $\varepsilon 1_A$ is a $\pi-$graded algebra called the convolution algebra.

4 Remark. If we put $A = k$ in the above lemma the $\pi-$graded algebra $\mathrm{Conv}(C, k) = \bigoplus_{\alpha\in\pi} C^*_\alpha$ is called dual to C and denoted by C^*.

5 Definition. A Hopf $\pi-$ coalgebra is a $\pi-$coalgebra $H = (\{H_\alpha\}_{\alpha\in\pi}, \Delta, \varepsilon)$ endowed with a family

$$S = \{S_\alpha : H_\alpha \to H_{\alpha^{-1}}\}_{\alpha\in\pi}$$

of $k-$linear maps called the antipode such that

(1) Each H_α is an algebra with multiplication m_α and unit element $1_\alpha \in H_\alpha$,

(2) The linear maps

$$\Delta_{\alpha,\beta} : H_{\alpha\beta} \to H_\alpha \otimes H_\beta,$$
$$\varepsilon : H_1 \to k.$$

are algebra maps for all $\alpha, \beta \in A$,

(3) For any $\alpha \in \pi$

$$m_\alpha (S_{\alpha^{-1}} \otimes \mathrm{id}) \Delta_{\alpha^{-1},\alpha} = m_\alpha (\mathrm{id} \otimes S_{\alpha^{-1}}) \Delta_{\alpha,\alpha^{-1}}.$$

6 Remark. If $H = (\{H_\alpha\}_{\alpha\in\pi}, \Delta, \varepsilon, S)$ is a Hopf $\pi-$coalgebra then axiom (3) says that S_α is the inverse of $I_{H_{\alpha^{-1}}}$ in the convolution algebra $\mathrm{Conv}(H, H_{\alpha^{-1}})$.

7 Remark. $(H_1, \Delta_{1,1}, \varepsilon, S_1)$ is a classical Hopf algebra

8 Lemma. *Let $H = (\{H_\alpha\}_{\alpha\in\pi}, \Delta, \varepsilon, S)$ be a Hopf $\pi-$coalgebra. Then*

(1) $\Delta_{\beta^{-1},\alpha^{-1}} S_{\alpha\beta} = \sigma_{H^{\alpha^{-1}}, H^{\beta^{-1}}} (S_\alpha \otimes S_\beta) \Delta_{\alpha,\beta}$ *for any* $\alpha, \beta \in \pi$,

(2) $\varepsilon(S_1) = \varepsilon$,

(3) $S_\alpha(ab) = S_\alpha(b) S_\alpha(a)$ *for any* $\alpha \in \pi$ *and* $a, b \in A$,

(4) $S_{1_\alpha} = 1_{\alpha^{-1}}$ *for any* $\alpha \in \pi$.

9 Definition. Let $C = (\{C_\alpha\}_{\alpha\in\pi}, \Delta, \varepsilon)$ be a $\pi-$coalgebra. A right $\pi-$comodule over C is a family $M = \{M_\alpha\}_{\alpha\in\pi}$ of $k-$linear spaces endowed with a family $\rho = \{\rho_{\alpha,\beta} : M_{\alpha\beta} \longrightarrow M_\alpha \otimes C_\beta\}$ of $k-$linear maps (the structure maps) such that

- For any $\alpha, \beta, \gamma \in \pi$

$$(\rho_{\alpha,\beta} \otimes \mathrm{id}) \rho_{\alpha\beta,\gamma} = (\mathrm{id} \otimes \Delta_{\beta,\gamma}) \rho_{\alpha,\beta\gamma} \qquad *$$

- For any $\alpha \in \pi$

$$(\mathrm{id} \otimes \varepsilon) \rho_{\alpha,1} = \mathrm{id} \qquad **$$

10 Definition. A $\pi-$subcomodule of M is a family $N = \{N_\alpha\}_{\alpha\in\pi}$ where N_α is a $k-$linear subspace of M_α such that for all $\alpha, \beta \in \pi$

$$\rho_{\alpha,\beta}(N_{\alpha\beta}) \subset N_\alpha \otimes C_\beta$$

11 Definition. A $\pi-$comodule morphism between to right $\pi-$comodules M and M' over a $\pi-$coalgebra C (with structure maps ρ and ρ', respectively) is a family $f = \{f_\alpha : M_\alpha \longrightarrow M'_\alpha\}$ of $k-$linear maps such that for all $\alpha, \beta \in \pi$

$$\rho'_{\alpha,\beta}(f_{\alpha\beta}) = (f_\alpha \otimes \mathrm{id})\rho_{\alpha,\beta}$$

12 Notation. [Sweedler's notation] For any $\alpha, \beta \in \pi$ and $m \in M_{\alpha,\beta}$ we write

$$\rho_{\alpha,\beta}(m) = m_{(0,\alpha)} \otimes m_{(1,\beta)} \in M_\alpha \otimes C_\beta$$

also the axiom

$$(\rho_{\alpha,\beta} \otimes \mathrm{id})\rho_{\alpha\beta,\gamma} = (\mathrm{id} \otimes \Delta_{\beta,\gamma})\rho_{\alpha,\beta\gamma}$$

can be written as

$$m_{(0,\alpha\beta)(0,\alpha)} \otimes m_{(0,\alpha\beta)(1,\beta)} \otimes m_{(1,\gamma)} = m_{(0,\alpha)} \otimes m_{(1,\beta\gamma)(1,beta)} \otimes m_{(1,\beta\gamma)(2,\gamma)}$$

This elements of $M_\alpha \otimes C_\beta \otimes C_\gamma$ is written as $m_{(0,\alpha)} \otimes m_{(1,\beta)} \otimes m_{(2,\gamma)}$

2 Basic Definitions of differential calculus

13 Definition. Let $A = \{A_\alpha\}_{\alpha\in\pi}$ be a Hopf group coalgebra ,$\Gamma = \{\Gamma_\alpha\}_{\alpha\in\pi}$ be a $\pi-$ graded bimodule over A , and

$$d = \{d_\alpha : A_\alpha \longrightarrow \Gamma_\alpha\} \tag{2.1}$$

be a family of linear maps. We say that (Γ, d) is a $\pi-$graded first order differential calculus over A if for any $\alpha \in \pi$

(1) For any $a, b \in A_\alpha$

$$d_\alpha(ab) = d_\alpha(a)b + \mathrm{ad}_\alpha(b) \tag{2.2}$$

(2) Any element $\rho \in \Gamma_\alpha$ is of the form

$$\rho = \sum_{k=1}^{n} a_k d_\alpha b_k \ \ , \ \ a_k, b_k \in A_\alpha$$

14 Definition. Two $\pi-$graded first order differential calculi are said to be isomorphic if there exists a family of bimodule isomorphisms
$i = \{i_\alpha : \Gamma_\alpha \longrightarrow \Gamma'_\alpha\}$ such that

$$i_\alpha(d_\alpha a) = d'_\alpha a, \text{ for all } a \in A_\alpha, \alpha \in \pi.$$

Let $A = \{A_\alpha\}_{\alpha\in\pi}$ be a Hopf group coalgebra , $m_\alpha : A_\alpha \otimes A_\alpha \longrightarrow A_\alpha$ be the multiplication defined on A_α for each α. Define $A^2 = \left\{A^2_\alpha\right\}_{\alpha\in\pi}$ such that

$$A^2_\alpha = \{q \in A_\alpha \otimes A_\alpha, m_\alpha(q) = 0\} \tag{2.3}$$

By definition A^2_α is a linear subspace of $A_\alpha \otimes A_\alpha$ for each $\alpha \in \pi$. On A^2 define an $A-$bimodule structure as

For any $\alpha \in \pi, c \in A_\alpha, \sum_k a_k \otimes b_k \in A^2_\alpha$

$$c\left(\sum_k a_k \otimes b_k\right) = \sum_k ca_k \otimes b_k \tag{2.4}$$

$$\left(\sum_k a_k \otimes b_k\right) c = \sum_k a_k \otimes b_k c \tag{2.5}$$

Define $D = \{D_\alpha\}_{\alpha\in\pi}$ by

$$D_\alpha(b) = 1_\alpha \otimes b - b \otimes 1_\alpha,$$

for all $b \in A_\alpha, \alpha \in \pi$

It is clear that $m_\alpha(D_\alpha(b)) = 0$,i.e.$D_\alpha(b) \in A^2_\alpha$. Moreover

$$D_\alpha(ab) = D_\alpha(a)b + aD_\alpha(b)$$

15 Proposition. *Let $N = \{N_\alpha\}_{\alpha\in\pi}$ be a $\pi-$graded sub-bimodule of $A^2, \Gamma = A^2/N$, $\pi = \{\pi_\alpha : A^2_\alpha \longrightarrow \Gamma_\alpha\}$ be the family of canonical epimorphisms , and $d = \{d_\alpha = \pi_\alpha \circ D_\alpha\}_{\alpha\in\pi}$. Then $\Gamma = (\{\Gamma_\alpha\}_{\alpha\in\pi}, d)$ is a first order differential calculus over A. Any other $\pi-$graded first order differential calculus over A can be obtained in this way.*

Proof. By definition of $\Gamma = \{\Gamma_\alpha\}_{\alpha\in\pi}, \Gamma$ is a $\pi-$graded bimodule over A. Moreover, by definition of $d = \{d_\alpha = \pi_\alpha \circ D_\alpha\}_{\alpha\in\pi}$ we find that $\Gamma = (\{\Gamma_\alpha\}_{\alpha\in\pi}, d)$ is a $\pi-$graded first order differential calculus over A. It remains to show that any $\pi-$graded first order differential calculus over A can be obtained in this way.

Let $\Gamma = (\{\Gamma_\alpha\}_{\alpha\in\pi}, d)$ be any other $\pi-$graded first order differential calculus over A. We have for each $\alpha \in \pi$, $\sum_k a_k \otimes b_k \in A^2_\alpha, c \in A_\alpha$

$$\sum_k ca_k d_\alpha b_k = c\left(\sum_k a_k d_\alpha b_k\right)$$

and

$$\sum_k a_k d_\alpha(b_k c) = \left(\sum_k a_k d_\alpha b_k\right) c$$

i.e. the family $\pi = \{\pi_\alpha : A_\alpha^2 \longrightarrow \Gamma_\alpha\}$defined by the formula

$$\pi_\alpha \left(\sum_k a_k \otimes b_k \right) = \sum_k a_k d_\alpha b_k \tag{2.6}$$

is a bimodule morphism. We will show that π_α is surjective for each $\alpha \in \pi$.

Let $\rho \in \Gamma_\alpha$ such that

$$\rho = \sum_k a_k d_\alpha b_k, \qquad a_k, b_k \in A_\alpha$$

Define an element $q \in A_\alpha \otimes A_\alpha$ by

$$q = \sum_k a_k \otimes b_k - a_k b_k \otimes 1_\alpha$$

It is clear that $m_\alpha q = 0$, i.e. $q \in A_\alpha^2$. Moreover,

$$\pi_\alpha(q) = \rho$$

therefore π_α is surjective for each $\alpha \in \pi$.

$$\begin{aligned} \ker \pi &= \{\ker \pi_\alpha\}_{\alpha \in \pi} \\ &= \left\{ \sum_k a_k \otimes b_k \in A_\alpha^2, \sum_k a_k d_\alpha b_k = 0 \right\}_{\alpha \in \pi} \end{aligned}$$

Taking

$$N = \{N_\alpha = \ker \pi_\alpha = \{\sum_k a_k \otimes b_k \in A_\alpha^2 \sum_k a_k d_\alpha b_k = 0\}\}_{\alpha \in \pi} \tag{2.7}$$

then Γ can be identified by A^2/N and for any $b \in A_\alpha$

$$\pi_\alpha D_\alpha(b) = d_\alpha b.$$

QED

16 Definition. Let $\Gamma = (\{\Gamma_\alpha\}_{\alpha \in \pi}, d)$ be a $\pi-$graded first order differential calculus over A. We say that $\Gamma = (\{\Gamma_\alpha\}_{\alpha \in \pi}, d)$ is left covariant if for any $\alpha, \beta \in \pi$

$$\sum_k a_k d_{\alpha\beta} b_k = 0 \Longrightarrow \sum_k \Delta_{\alpha,\beta}(a_k)(\mathrm{id} \otimes d_\beta)\Delta_{\alpha,\beta}(b_k) = 0 \tag{2.8}$$

for any $a_k, b_k \in A_{\alpha\beta}, k = 1, 2, \ldots, n$.

17 Proposition. *Let $\Gamma = (\{\Gamma_\alpha\}_{\alpha\in\pi}, d)$ be a left covariant $\pi-$graded first order differential calculus over A. Then there exists a family of linear mappings*

$$\Delta^l = \left\{\Delta^l_{\alpha,\beta} : \Gamma_{\alpha\beta} \longrightarrow A_\alpha \otimes \Gamma_\beta\right\} \tag{2.9}$$

such that

(1) For any $a \in A_{\alpha\beta}, \rho \in \Gamma_{\alpha\beta}$

$$\Delta^l_{\alpha,\beta}(a\rho) = \Delta_{\alpha,\beta}(a)\Delta^l_{\alpha,\beta}(\rho) \tag{2.10}$$

$$\Delta^l_{\alpha,\beta}(\rho a) = \Delta^l_{\alpha,\beta}(\rho)\Delta_{\alpha,\beta}(a) \tag{2.11}$$

(2) For any $\alpha, \beta, \gamma \in \pi$

$$(\Delta_{\alpha,\beta} \otimes \mathrm{id})\Delta^l_{\alpha\beta,\gamma} = (\mathrm{id} \otimes \Delta^l_{\beta,\gamma})\Delta^l_{\alpha,\beta\gamma} \tag{2.12}$$

(3) For any $\rho \in \Gamma_\alpha$

$$(\varepsilon \otimes \mathrm{id})\Delta^l_{1,\alpha}(\rho) = \rho \tag{2.13}$$

(4) For any $\alpha, \beta \in \pi$

$$\Delta^l_{\alpha,\beta} d_{\alpha\beta} = (\mathrm{id} \otimes d_\beta)\Delta_{\alpha,\beta}(a)$$

Proof. Let $\Delta^l = \left\{\Delta^l_{\alpha,\beta}\right\}_{\alpha,\beta\in\pi}$ where $\Delta^l_{\alpha,\beta} : \Gamma_{\alpha\beta} \longrightarrow A_\alpha \otimes \Gamma_\beta$ is defined by

$$\Delta^l_{\alpha,\beta}\left(\sum_{k=1}^n a_k d_{\alpha\beta} b_k\right) = \sum_{k=1}^n \Delta_{\alpha,\beta}(a_k)(\mathrm{id} \otimes d_\beta)\Delta_{\alpha,\beta}(b_k)$$

where $a_k, b_k \in A_{\alpha\beta}, \alpha, \beta \in \pi$. Then by definition for each $\alpha, \beta \in \pi$ $\Delta^l_{\alpha,\beta}$ is a well defined linear map.

(1) Let $a \in A_{\alpha\beta}$, $\rho \in \Gamma_{\alpha\beta}$, $\rho = \sum_{k=1}^n a_k d_{\alpha\beta} b_k, a_k, b_k \in A_{\alpha\beta}$

$$\begin{aligned}
\Delta^l_{\alpha,\beta}(\rho a) &= \Delta^l_{\alpha,\beta}\left(\sum_k a_k d_{\alpha\beta}(b_k a) - \sum_k a_k b_k d_{\alpha\beta} a\right) \\
&= \sum_k \Delta_{\alpha,\beta}(a_k)(b_{k(1,\alpha)} a_{(1,\alpha)} \otimes d_\beta b_{k(2,\beta)} a_{(2,\beta)}) \\
&= \left(\sum_k \Delta_{\alpha,\beta}(a_k)(\mathrm{id} \otimes d_\beta)\Delta_{\alpha,\beta}(b_k)\right)\Delta_{\alpha,\beta}(a) \\
&= \Delta^l_{\alpha,\beta}(\rho)\Delta_{\alpha,\beta}(a)
\end{aligned}$$

Similarly,

$$\Delta^l_{\alpha,\beta}(a\rho) = \Delta_{\alpha,\beta}(a)\Delta^l_{\alpha,\beta}(\rho)$$

(2) Let $\mathrm{ad}_{\alpha\beta\gamma}\, b \in \Gamma_{\alpha\beta\gamma}$,with $a, b \in A_{\alpha\beta\gamma}$, then we have

$$(\Delta_{\alpha,\beta} \otimes \mathrm{id})\Delta^l_{\alpha\beta,\gamma}(\mathrm{ad}_{\alpha\beta\gamma}\, b) = \\ (\Delta_{\alpha,\beta} \otimes \mathrm{id})\Delta_{\alpha\beta,\gamma}(a)(\mathrm{id} \otimes \mathrm{id} \otimes d_\gamma)(\Delta_{\alpha,\beta} \otimes \mathrm{id})\Delta_{\alpha\beta,\gamma}(b)$$

On the other hand

$$(\mathrm{id} \otimes \Delta^l_{\beta,\gamma})\Delta^l_{\alpha,\beta\gamma}(\mathrm{ad}_{\alpha\beta\gamma}\, b) = \\ (\mathrm{id} \otimes \Delta_{\beta,\gamma})\Delta_{\alpha,\beta\gamma}(a)(\mathrm{id} \otimes \mathrm{id} \otimes d_\gamma)(\mathrm{id} \otimes \Delta_{\beta,\gamma})\Delta_{\alpha,\beta\gamma}(b)$$

(3) For $\alpha \in \pi$ let $\mathrm{ad}_\alpha\, b \in \Gamma_\alpha$, $a, b \in A_\alpha$

$$\begin{aligned}(\varepsilon \otimes \mathrm{id})\Delta^l_{1,\alpha}(\mathrm{ad}_\alpha\, b) &= \varepsilon(a_{(1,1)})a_{(2,\alpha)}\varepsilon(b_{(1,1)})d_\alpha b_{(2,\alpha)} \\ &= \mathrm{ad}_\alpha\, b.\end{aligned}$$

(4) Let $a \in A_{\alpha\beta}$

$$\begin{aligned}\Delta^l_{\alpha,\beta} d_{\alpha\beta}(a) &= \Delta_{\alpha,\beta}(1_{\alpha\beta})(\mathrm{id} \otimes d_\beta)\Delta_{\alpha,\beta}(a) \\ &= (\mathrm{id} \otimes d_\beta)\Delta_{\alpha,\beta}(a)\end{aligned}$$

QED

18 Definition. Let $\Gamma = (\{\Gamma_\alpha\}_{\alpha\in\pi}, d)$ be a $\pi-$graded first order differential calculus over A. We say that $\Gamma = (\{\Gamma_\alpha\}_{\alpha\in\pi}, d)$ is right covariant if for any $\alpha, \beta \in \pi$

$$\sum_{k=1}^{n} a_k d_{\alpha\beta} b_k = 0 \Longrightarrow \sum_{k=1}^{n} \Delta_{\alpha,\beta}(a_k)(d_\alpha \otimes \mathrm{id})\Delta_{\alpha,\beta}(b_k) = 0 \tag{2.14}$$

We say that $\Gamma = (\{\Gamma_\alpha\}_{\alpha\in\pi}, d)$ is bicovariant if it is left and right covariant.

19 Proposition. *Let $\Gamma = (\{\Gamma_\alpha\}_{\alpha\in\pi}, d)$ be a right covariant $\pi-$graded first order differential calculus over A. Then there exists a family of linear mappings*

$$\Delta^r = \left\{\Delta^r_{\alpha,\beta} : \Gamma_{\alpha\beta} \longrightarrow \Gamma_\alpha \otimes A_\beta\right\} \tag{2.15}$$

such that

(1) For any $a \in A_{\alpha\beta}, \rho \in \Gamma_{\alpha\beta}$

$$\begin{aligned}\Delta^r_{\alpha,\beta}(a\rho) &= \Delta_{\alpha,\beta}(a)\Delta^r_{\alpha,\beta}(\rho)\\ \Delta^r_{\alpha,\beta}(\rho a) &= \Delta^r_{\alpha,\beta}(\rho)\Delta_{\alpha,\beta}(a)\end{aligned} \tag{2.16}$$

(2) for any $\alpha, \beta, \gamma \in \pi$

$$(\mathrm{id} \otimes \Delta_{\beta,\gamma})\Delta^r_{\alpha,\beta\gamma} = (\Delta^r_{\alpha,\beta} \otimes \mathrm{id})\Delta^r_{\alpha\beta,\gamma} \tag{2.17}$$

(3) For any $\rho \in \Gamma_\alpha$

$$(\mathrm{id} \otimes \varepsilon)\Delta^r_{\alpha,1}(\rho) = \rho \tag{2.18}$$

(4) for any $\alpha, \beta, \gamma \in \pi$

$$\Delta^r_{\alpha,\beta} d_{\alpha,\beta} = (d_\alpha \otimes \mathrm{id})\Delta_{\alpha,\beta}$$

Proof. Similar to that of proposition 17 , where for any $\alpha, \beta, \gamma \in \pi$ $a_k, b_k \in A_{\alpha\beta}$

$$\Delta^r_{\alpha,\beta}\left(\sum_k a_k d_{\alpha\beta} b_k\right) = \sum_k \Delta_{\alpha,\beta}(a_k)(d_\alpha \otimes \mathrm{id})\Delta_{\alpha,\beta}(b_k) \tag{2.19}$$

QED

20 Proposition. *Let $\Gamma = (\{\Gamma_\alpha\}_{\alpha\in\pi}, d)$ be a bicovariant $\pi-$graded first order differential calculus overA, Δ^l, Δ^r be the families of linear mappings introduced in proposition 17 and 19. Then we have*

$$(\mathrm{id} \otimes \Delta^r_{\beta,\gamma})\Delta^l_{\alpha,\beta\gamma}(\mathrm{ad}_{\alpha\beta\gamma}\, b) = (\Delta^l_{\alpha,\beta} \otimes \mathrm{id})\Delta^r_{\alpha\beta,\gamma}(\mathrm{ad}_{\alpha\beta\gamma}\, b) \tag{2.20}$$

Proof. Let $a, b \in A_{\alpha\beta\gamma}$

$$(\mathrm{id} \otimes \Delta^r_{\beta,\gamma})\Delta^l_{\alpha,\beta\gamma}(\mathrm{ad}_{\alpha\beta\gamma}\, b) = \\ (\mathrm{id} \otimes \Delta_{\beta,\gamma})\Delta_{\alpha,\beta\gamma}(a)(\mathrm{id} \otimes d_\beta \otimes \mathrm{id})(\mathrm{id} \otimes \Delta_{\beta,\gamma})\Delta_{\alpha,\beta\gamma}(b)$$

On the other hand

$$(\Delta^l_{\alpha,\beta} \otimes \mathrm{id})\Delta^r_{\alpha\beta,\gamma}(\mathrm{ad}_{\alpha\beta\gamma}\, b) = \\ (\Delta_{\alpha,\beta} \otimes \mathrm{id})\Delta_{\alpha\beta,\gamma}(a)(\mathrm{id} \otimes d_\beta \otimes \mathrm{id})(\Delta_{\alpha,\beta} \otimes \mathrm{id})\Delta_{\alpha,\beta\gamma}(b)$$

Using the coassociativity property we find that equation 2.20 holds. QED

3 $\pi-$graded Bicovariant bimodules

Throughout this section let $A = \{A_\alpha\}_{\alpha\in\pi}$ be a hopf group coalgebra

21 Definition. let $\Gamma = \{\Gamma_\alpha t\}_{\alpha\in\pi}$ be a $\pi-$graded bimodule over A , $\Delta^l = \{\Delta^l_{\alpha,\beta} : \Gamma_{\alpha\beta} \longrightarrow A_\alpha \otimes \Gamma_\beta\}_{\alpha,\beta\in\pi}$ be a family of linear maps. We say that $\Gamma = (\{\Gamma_\alpha\}_{\alpha\in\pi}, \Delta^l)$ is a left covariant π-graded bimodule over A if

(1) For any $a \in A_{\alpha\beta}$, $\rho \in \Gamma_{\alpha\beta}$ $\alpha, \beta \in \pi$

$$\Delta^l_{\alpha,\beta}(a\rho) = \Delta_{\alpha,\beta}(a)\Delta^l_{\alpha,\beta}(\rho) \tag{3.1}$$

$$\Delta^l_{\alpha,\beta}(\rho a) = \Delta^l_{\alpha,\beta}(\rho)\Delta_{\alpha,\beta}(a) \tag{3.2}$$

(2) For all $\alpha, \beta, \gamma \in \pi$.

$$(\Delta_{\alpha,\beta} \otimes \mathrm{id})\Delta^l_{\alpha\beta,\gamma} = (\mathrm{id} \otimes \Delta^l_{\beta,\gamma})\Delta^l_{\alpha,\beta\gamma} \tag{3.3}$$

(3) For any $\rho \in \Gamma_\alpha$,$\alpha \in \pi$

$$(\varepsilon \otimes \mathrm{id})\Delta^l_{1,\alpha}(\rho) = \rho \tag{3.4}$$

22 Definition. Let $\Gamma = \{\Gamma_\alpha\}_{\alpha\in\pi}$be a $\pi-$graded bimodule over A, $\Delta^r = \{\Delta^r_{\alpha,\beta} : \Gamma_{\alpha\beta} \longrightarrow \Gamma_\alpha \otimes A_\beta\}$ be a family of linear maps. We say that $\Gamma = (\{\Gamma_\alpha\}_{\alpha\in\pi}, \Delta^r)$ is a right covariant $\pi-$graded bimodule over A if

(1) For any $a \in A_{\alpha\beta}, \rho \in \Gamma_{\alpha\beta}$

$$\Delta^r_{\alpha,\beta}(a\rho) = \Delta_{\alpha,\beta}(a)\Delta^r_{\alpha,\beta}(\rho) \tag{3.5}$$

$$\Delta^r_{\alpha,\beta}(\rho a) = \Delta^r_{\alpha,\beta}(\rho)\Delta_{\alpha,\beta}(a) \tag{3.6}$$

(2) For $\alpha, \beta, \gamma \in \pi$.

$$(\Delta_{\alpha,\beta} \otimes \mathrm{id})\Delta^r_{\alpha\beta,\gamma} = (\mathrm{id} \otimes \Delta^r_{\beta,\gamma})\Delta^r_{\alpha,\beta\gamma} \tag{3.7}$$

(3) For any $\rho \in \Gamma_\alpha$,$\alpha \in \pi$

$$(\mathrm{id} \otimes \varepsilon)\Delta^r_{\alpha,1}(\rho) = \rho \tag{3.8}$$

23 Definition. let $\Gamma = \{\Gamma_\alpha\}_{\alpha\in\pi}$ be a $\pi-$graded bimodule over A , $\Delta^l = \{\Delta^l_{\alpha,\beta} : \Gamma_{\alpha\beta} \longrightarrow A_\alpha \otimes \Gamma_\beta\}_{\alpha,\beta\in\pi}$, and $\Delta^r = \{\Delta^r_{\alpha,\beta} : \Gamma_{\alpha\beta} \longrightarrow \Gamma_\alpha \otimes A_\beta\}$be two families of linear maps. We say that $\Gamma = (\{\Gamma_\alpha\}_{\alpha\in\pi}, \Delta^l, \Delta^r)$ is a bicovariant π-graded bimodule over A if

(1) $\Gamma = (\{\Gamma_\alpha\}_{\alpha\in\pi}, \Delta^l)$ is a left covariant π-graded bimodule over A.

(2) $\Gamma = (\{\Gamma_\alpha\}_{\alpha\in\pi}, \Delta^r)$ is a right covariant π-graded bimodule over A.

(3) For all $\alpha, \beta, \gamma \in \pi$.

$$(\Delta^l_{\alpha,\beta} \otimes \mathrm{id})\Delta^r_{\alpha\beta,\gamma} = (\mathrm{id} \otimes \Delta^r_{\beta,\gamma})\Delta^l_{\alpha,\beta\gamma} \tag{3.9}$$

24 Definition. Let $\Gamma = (\{\Gamma_\alpha\}_{\alpha\in\pi}, \Delta^l)$ be a left covariant π-graded bimodule over A. For any $\alpha \in \pi$ an element $\rho \in \Gamma_\alpha$ is said to be left invariant if

$$\Delta^l_{1,\alpha}(\rho) = 1_1 \otimes \rho \tag{3.10}$$

Denote by ${}_{\mathrm{inv}}\Gamma = \{{}_{\mathrm{inv}}\Gamma_\alpha\}_{\alpha\in\pi}$ the set of all left invariant elements of Γ. Clearly , ${}_{\mathrm{inv}}\Gamma_\alpha$ is a linear subspace of Γ_α for each $\alpha \in \pi$.

25 Lemma. *Let $\Gamma = (\{\Gamma_\alpha\}_{\alpha\in\pi}, \Delta^l)$ be a left covariant π-graded bimodule over A,${}_{\mathrm{inv}}\Gamma = \{{}_{\mathrm{inv}}\Gamma_\alpha\}_{\alpha\in\pi}$ be the linear subspace of all left invariant elements of Γ. Then there exists a family*

$$P = \{P_\alpha : \Gamma_1 \longrightarrow \Gamma_\alpha\}_{\alpha\in\pi} \tag{3.11}$$

of mappings such that

$$P_\alpha(b\rho) = \varepsilon(b)P_\alpha(\rho) \tag{3.12}$$

for any b$\in A_1, \rho \in \Gamma_1, \alpha \in \pi$.
Moreover, for any $\rho \in \Gamma_\alpha$, $\alpha \in \pi$ we have

$$\rho = \sum_k a_k P_\alpha(\rho_k) \tag{3.13}$$

where a_k , ρ_k are elements of A_α , Γ_1 respectively such that

$$\Delta^l_{\alpha,1}(\rho) = \sum_k a_k \otimes \rho_k \tag{3.14}$$

and equation 3.13 can be uniquely written in this form.

PROOF. For any $\alpha \in \pi$, $\rho \in \Gamma_1$ set

$$P_\alpha(\rho) = \sum_k S_{\alpha^{-1}}(a_k)\rho_k \tag{3.15}$$

where

$$\Delta^l_{\alpha^{-1},\alpha}(\rho) = \sum_{k=1}^{n} a_k \otimes \rho_k$$

Recall that for any $\alpha, \beta \in \pi$, $a \in A_{\beta^{-1}}$ where $\Delta_{\beta^{-1}\alpha^{-1},\alpha}(a) = a_{(1,\beta^{-1}\alpha^{-1})} \otimes a_{(2,\alpha)}$we have

$$\Delta_{\alpha,\beta}(S_{\beta^{-1}\alpha^{-1}}(a_{(1,\beta^{-1}\alpha^{-1})}))(a_{(2,\alpha)} \otimes 1_\beta) = 1_\alpha \otimes S_{\beta^{-1}}(a) \tag{3.16}$$

For any $\rho \in \Gamma_1$, $\alpha \in \pi$ set

$$\begin{aligned}
\Delta^l_{\alpha^{-1},\alpha}(\rho) &= \sum_k a_k \otimes \rho_k \\
\Delta^l_{1,\alpha}(\rho_k) &= \sum_l b_{kl} \otimes \rho_{kl} \\
\Delta_{\alpha^{-1},1}(a_k) &= \sum_m c_{km} \otimes d_{km}
\end{aligned}$$

Using equation 3.3 we have

$$\sum_{k,l} a_k \otimes b_{kl} \otimes \rho_{kl} = \sum_{k,m} c_{km} \otimes d_{km} \otimes \rho_k \tag{3.17}$$

We compute

$$\begin{aligned}
\Delta^l_{1,\alpha}(P_\alpha(\rho)) &= \sum_k \Delta_{1,\alpha}(S_{\alpha^{-1}}(a_k))\Delta^l_{1,\alpha}(\rho_k) \\
&= \sum_{k,l} \Delta_{1,\alpha}(S_{\alpha^{-1}}(c_{km}))(d_{km} \otimes \rho_k) \\
&= \sum_k (1_1 \otimes S_{\alpha^{-1}}(a_k))(1_1 \otimes \rho_k) \\
&= 1_1 \otimes P_\alpha(\rho)
\end{aligned}$$

This shows that $P_\alpha(\rho)$ is left invariant element in Γ_α for each $\alpha \in \pi$.

To prove Equation 3.12 , let $b \in A_1$, $\rho \in \Gamma_1$, set

$$\begin{aligned}
\Delta_{\alpha^{-1},\alpha}(b) &= \sum_k b_k \otimes d_k \\
\Delta^l_{\alpha^{-1},\alpha}(\rho) &= \sum_l c_l \otimes \rho_l \\
\Delta^l_{\alpha^{-1},\alpha}(b\rho) &= \Delta_{\alpha^{-1},\alpha}(b)\Delta^l_{\alpha^{-1},\alpha}(\rho) \\
&= \sum_{k,l} b_k c_l \otimes d_k \rho_l
\end{aligned}$$

Then

$$\begin{aligned}
P_\alpha(b\rho) &= \sum_{k,l} S_{\alpha^{-1}}(c_l) S_{\alpha^{-1}}(b_k) d_k \rho_l \\
&= \varepsilon(b) P_\alpha(\rho)
\end{aligned}$$

To prove equation 3.13. Let $\alpha \in \pi, \rho \in \Gamma_\alpha$. Set

$$\begin{aligned}
\Delta^l_{1,\alpha}(\rho) &= \sum_m d_m \otimes \varrho_m \\
\Delta^l_{\alpha^{-1},\alpha}(\rho_k) &= \sum_n b_{kn} \otimes \rho_{kn} \\
\Delta_{\alpha,\alpha^{-1}}(d_m) &= \sum_l d_{ml} \otimes c_{ml}
\end{aligned}$$

where

$$\Delta^l_{\alpha,1}(\rho) = \sum_k a_k \otimes \rho_k \tag{3.18}$$

using equation 3.3 we have

$$(\Delta_{\alpha,\alpha^{-1}} \otimes \mathrm{id})\Delta^l_{1,\alpha} = (\mathrm{id} \otimes \Delta^l_{\alpha^{-1},\alpha})\Delta^l_{\alpha,1}$$

i.e.

$$\sum_{m,l} d_{ml} \otimes c_{ml} \otimes \varrho_m = \sum_{k,n} a_k \otimes b_{kn} \otimes \rho_{kn} \tag{3.19}$$

Then using equation 3.4 we have

$$\begin{aligned}
\rho &= \sum_m \varepsilon(d_m)\varrho_m \\
&= \sum_{k,n} a_k S_{\alpha^{-1}}(b_{kn})\rho_{kn} \\
&= \sum_m a_k P_\alpha(\rho_k)
\end{aligned}$$

Finally, to prove the uniqueness of expression 3.13 let $P^{'} = \{P^{'}_{\alpha} : \Gamma_1 \longrightarrow_{\text{inv}} \Gamma_{\alpha}\}$ be another family of mappings satisfying that for $\rho \in \Gamma_{\alpha}$

$$\rho = \sum_k a_k P'_{\alpha}(\rho_k) \tag{3.20}$$

where a_k , ρ_k are elements of A_{α} , Γ_1 respectively such that

$$\Delta^l_{\alpha,1}(\rho) = \sum_k a_k \otimes \rho_k$$

Let $\rho \in \Gamma_{\alpha}$ such that $\Delta^l_{\alpha,1}(\rho) = \sum_k a_k \otimes \rho_k$. Then using 3.13

$$\rho = \sum_k a_k P_{\alpha}(\rho_k)$$

But using equation 3.20 we have

$$\rho = \sum_k a_k P^{'}_{\alpha}(\rho_k)$$

Subtracting the above two equations we obtain

$$0 = \sum_k a_k(P_{\alpha}(\rho_k) - P^{'}_{\alpha}(\rho_k))$$

Assuming that all $a_k^{'}s$ all linearly independent we get

$$P_{\alpha}(\rho_k) = P^{'}_{\alpha}(\rho_k) \qquad k = 1, 2, \ldots, n.$$

which proves the uniqueness of the expression 3.13.

26 Lemma. *Let $\Gamma = (\{\Gamma_{\alpha}\}_{\alpha\in\pi}, \Delta^l)$ be a left covariant π-graded bimodule over A. Then, for any $\alpha, \beta \in \pi, \rho \in_{\text{inv}} \Gamma_{\alpha\beta}$ we have*

$$\Delta^l_{\alpha,\beta}(\rho) = 1_{\alpha} \otimes \varrho \tag{3.21}$$

where $\varrho \in_{\text{inv}} \Gamma_{\beta}$.

Proof. Let $\alpha, \beta \in \pi, \rho \in_{\text{inv}} \Gamma_{\alpha\beta}$, then using lemma 3, and since the mappings P_{α} are onto for each $\alpha \in \pi$ then there exists an element $\xi \in \Gamma_1$ such that

$$\rho = P_{\alpha\beta}(\xi) \tag{3.22}$$

Set

$$\Delta^l_{\beta^{-1}\alpha^{-1},\alpha\beta}(\xi) = \sum_k a_k \otimes \xi_k$$
$$\Delta^l_{\alpha,\beta}(\xi_k) = \sum_l c_{kl} \otimes \xi_{kl}$$
$$\Delta^l_{\beta^{-1},\beta}(\xi) = \sum_m b_m \otimes \rho_m$$

and

$$\Delta^l_{\beta^{-1}\alpha^{-1},\alpha}(b_m) = \sum_n b_{mn} \otimes d_{mn} \tag{3.23}$$

Using equation 3.3

$$\sum_{k,l} a_k \otimes c_{kl} \otimes \xi_{kl} = \sum_{m,n} b_{mn} \otimes d_{mn} \otimes \rho_m \tag{3.24}$$

Applying $\Delta^l_{\alpha,\beta}$ to both sides of equation 3.22, using equations 3.24 and 3.16, we get

$$\begin{aligned}\Delta^l_{\alpha,\beta}(\rho) &= \sum_k \Delta_{\alpha,\beta}(S_{\beta^{-1}\alpha^{-1}}(a_k))\Delta^l_{\alpha,\beta}(\xi_k)\\ &= \sum_{m,n} \Delta_{\alpha,\beta}(S_{\beta^{-1}\alpha^{-1}}(b_{mn}))(d_{mn} \otimes \rho_m)\\ &= \sum_m (1_\alpha \otimes S_{\beta^{-1}}(b_m))(1_\alpha \otimes \rho_m)\\ &= 1_\alpha \otimes P_\beta(\xi)\end{aligned}$$

But from lemma 3 $P_\beta(\xi) \in_{\text{inv}} \Gamma_\beta$ and hence the lemma is proved. QED

Let $A = (\{A_\alpha\}_{\alpha\in\pi}, \Delta, \varepsilon, S)$ be a hopf $\pi-$coalgebra. Throughout the next dealing we will consider that A is endowed with a family of linear maps $\Psi = \{\Psi_\alpha : A_\alpha \longrightarrow A_1\}$ of k-linear maps such that for each $\alpha \in \pi$,Ψ_α is an algebra map.For each $\alpha \in \pi$,define the map E_α to be the composition

$$A_\alpha \longrightarrow A_1 \longrightarrow k$$

i.e.

$$E_\alpha = \varepsilon\Psi_\alpha \tag{3.25}$$

Clearly, for each $\alpha \in \pi$ E_α is an algebra map for let $a, b \in A_\alpha$. Then

$$E_\alpha(ab) = \varepsilon(\Psi_\alpha(a)\ \Psi_\alpha(b))$$
$$= E_\alpha(a)E_\alpha(b)$$
$$E_\alpha(1_\alpha) = 1_k$$

Moreover, E_α is linear being the composition of two linear maps.

27 Theorem. *Let $\Gamma = (\{\Gamma_\alpha\}_{\alpha\in\pi}, \Delta^l)$ be a $\pi-$graded left covariant bimodule over A, $\{\omega_i^\alpha\}_{\alpha\in\pi}$be a basis of ${}_{\text{inv}}\Gamma_\alpha$, of all left invariant elements of Γ_α for each $\alpha \in \pi$. Then*

(1) For any $\alpha \in \pi$,any element $\rho \in \Gamma_\alpha$ is of the form

$$\rho = \sum_i a_i\omega_i \tag{3.26}$$

where a_i 's $\in A_\alpha$ are uniquely determined , ω_i's $\in_{\text{inv}} \Gamma_\alpha$,for any $\alpha \in \pi$.

(2) For any $\alpha \in \pi$,any element $\rho \in \Gamma_\alpha$ is of the form

$$\rho = \sum_i \omega_i b_i \tag{3.27}$$

where b_i 's $\in A_\alpha$ are uniquely determined , ω_i's $\in_{\text{inv}} \Gamma_\alpha$,for any $\alpha \in \pi$.

(3) There exists linear functionals $f_{ij} \in A' = \oplus_{\alpha\in\pi} A'_\alpha$ such that for any $\alpha \in \pi$

$$\omega_i b = \sum_j (f_{ij} * b)\omega_j \tag{3.28}$$

$$a\omega_i = \sum_j \omega_j((f_{ij}\circ S_1^{-1}) * a) \tag{3.29}$$

where $a, b \in A_\alpha, \omega_i$'s$, \omega_j$'s $\in_{\text{inv}} \Gamma_\alpha$. These functionals are uniquely determined by equation 3.28. They satisfy the following relations

$$f_{ij}(ab) = \sum_k f_{ik}(a)f_{kj}(b) \tag{3.30}$$

for any $i, j \in I$, $a, b \in A_\alpha$. Moreover

$$f_{ij}(1_\alpha) = \delta_{ij} \tag{3.31}$$

28 Remark. Any functional $f_{ij} \in A' = \oplus_{\alpha\in\pi} A'_\alpha$ is of the form $f_{ij} = \sum_\alpha f^\alpha_{ij}$ where

$$f^\alpha_{ij}(a) = 0 \quad if\ a \notin A_\alpha$$

Proof. To prove 1: For any $\alpha \in \pi$ let $\rho \in \Gamma_\alpha$. Using equation 3.13 we have that $\rho = \sum_i a_i \omega_i$, with $\omega_i^{,}s \in_{\text{inv}} \Gamma_\alpha$. To prove uniqueness assume that $\rho = \sum_i a_i \omega_i$. Then, using equations 3.1 and 3.21

$$\Delta^l_{\alpha,1}(\rho) = \sum_i a_{i(1,\alpha)} \otimes a_{i(2,1)} \xi_i$$

Applying $(\mathrm{id} \otimes P_1)$ to both sides of the above equation ,we get

$$(\mathrm{id} \otimes P_1)\Delta^l_{\alpha,1}(\rho) = \sum_i a_i \otimes \xi_i$$

since $P_1(\xi_i) = \xi_i$ for any $\xi_i \in_{\text{inv}} \Gamma_1$. Since $\omega_i^{,}s$,$i \in I$ are linearly independent , then by linearity of $\Delta^l_{\alpha,1}$, $\xi_i^{,}s$ are also linearly independent and so the coefficients $a_i^{,}s$ are uniquely determined, and this proves the uniqueness of the decomposition 3.26. To prove *(3)*: For any $\alpha \in \pi$, let $b \in A_\alpha, \omega_j \in_{\text{inv}} \Gamma_\alpha, j \in I$. Then $\omega_j b$ admits a decomposition in the of the form 3.26. Let $F^\alpha_{ji}(b)$ be the coefficients preceding ω_i in the decomposition 3.26 i.e.

$$\omega_j b = \sum_i F^\alpha_{ji}(b)\omega_i \tag{3.32}$$

Clearly, $F^\alpha_{ji}(b)$ are linear mappings acting on A_α. For any $a, b \in A_\alpha$,and any $j \in I$ we have

$$\begin{aligned}\sum_i F^\alpha_{ji}(ab)\omega_i &= \omega_j ab \\ &= \sum_{h,i} F^\alpha_{jh}(a)F^\alpha_{hi}(b)\omega_i\end{aligned}$$

using the uniqueness of the decomposition 3.26 we have

$$F^\alpha_{ji}(ab) = \sum_h F^\alpha_{jh}(a)F^\alpha_{hi}(b) \tag{3.33}$$

for all $i, j \in I, \alpha \in \pi, a, b \in A_\alpha$. Let f^α_{ji} be linear functionals defined on A_α introduced by the formula

$$f^\alpha_{ji}\,(a) = E_\alpha(F^\alpha_{ji}(a)) = \varepsilon(\Psi_\alpha(F^\alpha_{ji}(a))) \tag{3.34}$$

Define $f_{ji} \in A'$ by

$$f_{ji} = \sum_{\alpha\in\pi} f^\alpha_{ji}$$

where for any $\beta \in \pi, a \in A_\beta$

$$f_{ji}(a) = \sum_{\alpha\in\pi} f_{ji}^{\alpha}(a) = f_{ji}^{\beta}(a) \tag{3.35}$$

Applying E_α to both sides of equation 3.33 and using equations 3.34 and 3.35 we have

$$f_{ji}(ab) = \sum_h f_{jh}(a) f_{hi}(b)$$

for any $a, b \in A_\alpha$, and hence equation 3.30 is proven. From equation 3.30 we get

$$f_{ji} m_\alpha(a \otimes b) = \sum_h (f_{jh} \otimes f_{hi})(a \otimes b)$$

i.e.

$$f_{ji} m_\alpha = \sum_h (f_{jh} \otimes f_{hi}) \tag{3.36}$$

Inserting $b = 1_\alpha$ in equation 3.32 we get

$$\omega_j = \sum_i F_{ji}^{\alpha}(1_\alpha)\omega_i$$

i.e.

$$F_{ji}^{\alpha}(1_\alpha) = \delta_{ji} 1_\alpha$$

Applying E_α to both sides of the above equation ,and summing over α we get

$$f_{ji}(1_\alpha) = \delta_{ji}$$

and hence 3.31 is proven. To prove 3.28 Recall that from equation 3.32 for any $\alpha \in \pi, \omega_j \in_{\text{inv}} \Gamma_\alpha$, $b \in A_\alpha$

$$\omega_j b = \sum_i F_{ji}^{\alpha}(b)\omega_i$$

Applying $\Delta_{\alpha,1}^{l}$ to both sides of the above equation we obtain

$$\Delta_{\alpha,1}^{l}(\omega_j b) = \Delta_{\alpha,1}^{l}(\sum_i F_{ji}^{\alpha}(b)\omega_i)$$

$$(1_\alpha \otimes \xi_j)\Delta_{\alpha,1}(b) = \sum_i \Delta_{\alpha,1}(F_{ji}^{\alpha}(b))(1_\alpha \otimes \xi_i)$$

where $\xi_j, \xi_i \in_{\text{inv}} \Gamma_1$, $i, j \in I$. On the other hand using equation 3.32

$$(1_\alpha \otimes \xi_j)\Delta_{\alpha,1}(b) = \sum_i (\text{id} \otimes F_{ji}^{1})\Delta_{\alpha,1}(b)(1_\alpha \otimes \xi_i)$$

Comparing the last two equations we get

$$\Delta_{\alpha,1}(F^{\alpha}_{ji}(b)) = (\mathrm{id} \otimes F^{1}_{ji})\Delta_{\alpha,1}(b)$$

Applying $(\mathrm{id} \otimes \varepsilon)$ to both sides of the above equation , using equation 3.35 we get

$$\begin{aligned} F^{\alpha}_{ji}(b) &= (\mathrm{id} \otimes f_{ji})\Delta_{\alpha,1}(b) \\ &= f_{ji} * b \end{aligned}$$

Inserting this result into equation 3.32 we obtain equation 3.28. In order to prove equation 3.29 we have to show that

$$\sum_j (f_{ji} * f_{hj} \circ S_1^{-1}) = \delta_{ih}\varepsilon \tag{3.37}$$

Let $a \in A_1$. Then

$$\begin{aligned} \sum_j (f_{ji} * f_{hj} \circ S_1^{-1})(S_1(a)) &= \sum_j f_{hi}(\varepsilon(a)1_1) \\ &= \delta_{hi}\varepsilon(S_1(a)) \end{aligned}$$

i.e.

$$\sum_j f_{ji} * (f_{hj} \circ S_1^{-1}) = \delta_{ih}\varepsilon$$

Similarly, one can check that

$$\sum_j (f_{jh} \circ S_1^{-1}) * f_{ij} = \delta_{hi}\varepsilon \tag{3.38}$$

From equation 3.28 we have that for any $\alpha \in \pi$, $b \in A_\alpha, \omega_j \in_{\mathrm{inv}} \Gamma_\alpha$

$$\omega_j b = \sum_h (f_{jh} * b)\omega_h$$

Inserting in this equation $b = (f_{jh} \circ S_1^{-1}) * a$ for some $a \in A_\alpha$ and summing over j we obtain

$$\begin{aligned} \sum_j \omega_j (f_{jh} \circ S_1^{-1}) * a &= \sum_{j,h} ((f_{jh} * (f_{jh} \circ S_1^{-1})) * a)\omega_h \\ &= a\omega_i \end{aligned}$$

Recall that $\varepsilon * a = (\mathrm{id} \otimes \varepsilon)\Delta_{\alpha,1}(a) = a$, and hence equation 3.29 follows.

To prove *(2)*: For any $\alpha \in \pi, \rho \in \Gamma_\alpha$, we have from statement 1 and formula 3.29 that

$$\begin{aligned}\rho &= \sum_i a_i \omega_i, \qquad a_i \in A_\alpha,\ \omega_i \in_{\text{inv}} \Gamma_\alpha,\ i \in I \\ &= \sum_j \omega_j b_j,\end{aligned}$$

where

$$b_j = \sum_i (f_{ij} \circ S_1^{-1}) * a_i \in A_\alpha, \ \ \forall j \in I.$$

For uniqueness:

Assume that for some b_i ($i \in I$ only finite number of $b_i^{,}s$ are different from zero) we have:

$$\sum_i \omega_i b_i = 0$$

We have to show that all $b_i' s = 0 (i \in I\ )$. Using the uniqueness of decomposition 3.26 we have

$$\sum_i \omega_i b_i = 0$$

Then

$$\sum_{i,j} (f_{i,j} * b_i)\omega_j = 0$$

$$\sum_i (f_{i,j} * b_i) = 0 \qquad \forall j \in I$$

Computing the convolution product with $f_{jh} \circ S_1^{-1}$ summing over j and using equation 3.38

$$\begin{aligned}0 &= \sum_{i,j} ((f_{jh} \circ S_1^{-1}) * f_{i,j}) * b_i \\ &= b_i\end{aligned}$$

i.e. $b_i = 0$ for each $i \in I$. *QED*

Theorem 27 gives the complete description of left covariant $\pi-$graded bimodules. Using equations 3.28 and 3.1 we have

$$(\sum_i a_i \omega_i) b = \sum_i a_i(\omega_i b) = \sum_{i,.j} a_i (f_{ij} * b)\omega_j \qquad (3.39)$$

$$\Delta^l_{\alpha,\beta}(\sum_i a_i\omega_i) = \sum_i \Delta_{\alpha,\beta}(a_i)\Delta^l_{\alpha,\beta}(\omega_i) = \sum_i \Delta_{\alpha,\beta}(a_i)(1_\alpha \otimes \xi_i), \xi_i \in_{\text{inv}} \Gamma_\beta \tag{3.40}$$

If $(f_{ij})_{i,j\in I}$ is a family of linear functionals in $A' = \oplus_{\alpha\in\pi} A'_\alpha$ satisfying relations 3.30, 3.31, then considering the left module $\Gamma = \{\Gamma_\alpha\}_{\alpha\in\pi}$ generated by $\omega_i^\alpha, \alpha \in \pi, i \in I$, and using the above formulae to introduce the right multiplication by elements of A , and the left action of A we obtain a left covariant $\pi-$graded bimodule.

29 Definition. Let (Γ, Δ^r) be a right covariant π-graded bimodule over A. An element $\eta \in \Gamma_\alpha$ is said to be right invariant if

$$\Delta^r_{\alpha,1}(\eta) = \eta \otimes 1_1 \tag{3.41}$$

Denote by $\Gamma_{\text{inv}} = \{\Gamma^\alpha_{\text{inv}}\}$ the set of all left invariant elements of Γ. Clearly, $\Gamma^\alpha_{\text{inv}}$ is a linear subspace of Γ_α for each $\alpha \in \pi$.

30 Theorem. *Let $\Gamma = (\{\Gamma^\alpha\}_{\alpha\in\pi}, \Delta^r)$ be a right covariant $\pi-$graded bimodule over A,$\{\eta_i^\alpha\}_{\alpha\in\pi}$be a basis of $\Gamma^\alpha_{\text{inv}}$ of all right invariant elements of Γ_α for each $\alpha \in \pi$. Then*

(1) For any $\alpha \in \pi$,any element $\varrho \in \Gamma_\alpha$ is of the form

$$\varrho = \sum_i a_i\eta_i \tag{3.42}$$

where a_i 's $\in A_\alpha$ are uniquely determined , η_i's $\in \Gamma^\alpha_{\text{inv}}$,for any $\alpha \in \pi$.

(2) For any $\alpha \in \pi$, any element $\rho \in \Gamma_\alpha$ is of the form

$$\varrho = \sum_i \eta_i b_i \tag{3.43}$$

where b_i 's $\in A_\alpha$ are uniquely determined , η_i's $\in \Gamma^\alpha_{\text{inv}}$, for any $\alpha \in \pi$.

(3) There exists linear functionals $g_{ij} \in A' = \oplus_{\alpha\in\pi} A'_\alpha$ such that for any $\alpha \in \pi$

$$\eta_i b = \sum_j (b * g_{ij})\eta_j \tag{3.44}$$

$$a\eta_i = \sum_j \eta_j(a * (g_{ij} \circ S_1^{-1})) \tag{3.45}$$

where $a, b \in A_\alpha, \eta_i^{`}s, \eta_j^{`}s \in \Gamma_{\text{inv}}^\alpha$. *These functionals are uniquely determined by equation* 3.44. *They satisfy the following relations*

$$g_{ij}(ab) = \sum_k g_{ik}(a) g_{kj}(b) \tag{3.46}$$

for any $i, j \in I$, $a, b \in A_\alpha$. *Moreover*

$$g_{ij}(1_\alpha) = \delta_{ij} \tag{3.47}$$

The proof is similar to that of theorem 27.

31 Remark. Any functional $g_{ij} \in A' = \oplus_{\alpha\in\pi} A'_\alpha$ is of the form $g_{ij} = \sum_\alpha g_{ij}{}^\alpha$ where

$$g_{ij}^\alpha(a) = 0 \quad \text{\textit{if} } a \notin A_\alpha$$

32 Theorem. *Let* $\Gamma = (\{\Gamma_\alpha\}_{\alpha\in\pi}, \Delta^l, \Delta^r)$ *be a* $\pi-$*graded bicovariant bimodule over* A,$\{(\omega_i^\alpha)_{i\in\ I}\}_{\alpha\in\pi}$ *be a basis of* ${}_{\text{inv}}\Gamma = \{{}_{\text{inv}}\Gamma_\alpha\}_{\alpha\in\pi}$ *of all left invariant elements of* Γ. *Then*

(1) *For any* $i \in I, \alpha, \beta \in \pi, \omega_i^{\alpha\beta} \in \Gamma_{\alpha\beta}$

$$\Delta_{\alpha,\beta}^r(\omega_i^{\alpha\beta}) = \sum_j \omega_j^\alpha \otimes R_{ji} \tag{3.48}$$

where $i, j \in \pi, R_{ji} \in A_\beta$ *satisfy the following relation*

$$\Delta_{\alpha,\beta}(R_{ji}) = \sum_h R_{jh} \otimes R_{hi} \tag{3.49}$$

and for $R_{ji} \in A_1$

$$\varepsilon(R_{ji}) = \delta_{ji} \tag{3.50}$$

(2) *For each* $\alpha \in \pi$ *there exists a basis* $(\eta_i)_{i\in I}$ *of all right invariant elements of* Γ_α *such that for* $\omega_i \in \Gamma_\alpha$

$$\omega_i = \sum_j \eta_j R_{ji} \qquad \forall i \in I \tag{3.51}$$

(3) *For any* $j, h \in I$, $a \in A_\alpha$

$$R_{ij}(a * f_{ih}) = (g_{ji} * a) R_{hi}, \quad i, j \in I \tag{3.52}$$

where f_{ij}, g_{ij} *are functionals introduced in theorems 27 , 30*

Proof. Using equation 3.9 for any $\alpha, \beta, \gamma \in \pi$ we have

$$(\Delta^l_{\alpha,\beta} \otimes \mathrm{id})\Delta^r_{\alpha\beta,\gamma} = (\mathrm{id} \otimes \Delta^r_{\beta,\gamma})\Delta^l_{\alpha,\beta\gamma}$$

Let $\omega_i^{\alpha\beta\gamma} \in \Gamma_{\alpha\beta\gamma}$

$$\begin{aligned}(\Delta^l_{\alpha,\beta} \otimes \mathrm{id})\Delta^r_{\alpha\beta,\gamma}(\omega_i^{\alpha\beta\gamma}) &= (\mathrm{id} \otimes \Delta^r_{\beta,\gamma})\Delta^l_{\alpha,\beta\gamma}(\omega_i^{\alpha\beta\gamma}) \\ &= 1_\alpha \otimes \Delta^r_{\beta,\gamma}(\omega_i^{\beta\gamma})\end{aligned}$$

i.e.

$$\Delta^r_{\alpha\beta,\gamma}(\omega_i) \in_{\mathrm{inv}} \Gamma_{\alpha\beta} \otimes A_\gamma$$

Then for $\omega_i^{\alpha\beta\gamma} \in \Gamma_{\alpha\beta\gamma}$

$$\Delta^r_{\alpha,\beta\gamma}(\omega_i^{\alpha\beta\gamma}) = \sum_j \omega_j^\alpha \otimes R_{ji}$$

Applying $(\mathrm{id} \otimes \Delta_{\beta,\gamma})$ to both sides of the above equation

$$\begin{aligned}\sum_j \omega_j^\alpha \otimes \Delta_{\beta,\gamma}(R_{ji}) &= (\Delta^r_{\alpha,\beta} \otimes \mathrm{id})\Delta^r_{\alpha\beta,\gamma}(\omega_i^{\alpha\beta\gamma}) \\ &= \sum_{j,h} \omega_j^\alpha \otimes R_{jh} \otimes R_{hi}\end{aligned}$$

Comparing both sides of the above equation

$$\Delta_{\beta,\gamma}(R_{ji}) = \sum_h R_{jh} \otimes R_{hi}$$

and hence equation 3.49 is proven. Let $\omega_i^\alpha \in \Gamma_\alpha$

$$\Delta_{\alpha,1}(\omega_i^\alpha) = \sum_j \omega_j^\alpha \otimes R_{ji} \quad , R_{ji} \in A_1$$

Applying $(\mathrm{id} \otimes \varepsilon)$ to both sides of the above equation

$$\begin{aligned}(\mathrm{id} \otimes \varepsilon)\Delta_{\alpha,1}(\omega_i^\alpha) &= \omega_i^\alpha \\ &= \sum_j \omega_j^\alpha \otimes \varepsilon(R_{ji})\end{aligned}$$

$\Longrightarrow$

$$\varepsilon(R_{ji}) = \delta_{ji}$$

To prove statement 2: First we have that for $R_{ij} \in A_1, \alpha \in \pi$

$$m_\alpha(\mathrm{id} \otimes S_{\alpha^{-1}})\Delta_{\alpha,\alpha^{-1}} = m_\alpha(S_{\alpha^{-1}} \otimes \mathrm{id})\Delta_{\alpha^{-1},\alpha} = \varepsilon 1_\alpha$$

By using equations 3.49 and 3.50 we obtain

$$\sum_h S_{\alpha^{-1}}(R_{ih})R_{hj} = \delta_{ij}1_\alpha \tag{3.53}$$

$$\sum_h R_{ih}S_{\alpha^{-1}}(R_{hj}) = \delta_{ij}1_\alpha \tag{3.54}$$

For any $\alpha \in \pi, j \in I$, let

$$\eta_j = \sum_i \omega_i S_{\alpha^{-1}}(R_{ij}) \tag{3.55}$$

Multiplying both sides of equation 3.55 by R_{ji} and summing over j then using equation 3.53 we obtain

$$\begin{aligned}\sum_j \eta_j R_{ji} &= \sum_{i,j} \omega_i S_{\alpha^{-1}}(R_{ij})R_{ji} \\ &= \omega_i\end{aligned}$$

and equation 3.51 follows. It remains to show that η_j defined in equation 3.55 is right invariant

Let $\eta_j \in \Gamma_\alpha$, $\eta_j = \sum_i \omega_i S_{\alpha^{-1}}(R_{ij}), \omega_i \in_{\text{inv}} \Gamma_\alpha, R_{ij} \in A_{\alpha^{-1}}$.

$$\begin{aligned}\Delta^r_{\alpha,1}(\eta_j) &= \sum_i \Delta^r_{\alpha,1}(\omega_i)\Delta_{\alpha,1}(S_{\alpha^{-1}}(R_{ij})) \\ &= \eta_j \otimes 1_1\end{aligned}$$

For any $\alpha \in \pi$,let $\eta \in \Gamma_\alpha$ be a right invariant element. According to theorem 30

$$\begin{aligned}\eta &= \sum_i \omega_i c_i \quad , c_i \in A_\alpha \\ &= \sum_{i,j} \eta_j R_{ji} c_i \quad , R_{ji} \in A_\alpha\end{aligned}$$

then

$$\eta = \sum_j \eta_j b_j, \quad b_j \in A_\alpha \tag{3.56}$$

If $\eta = \sum_i \omega_i S_{\alpha^{-1}}(R_{ij})$, then using equation 3.55 we have

$$\sum_{i,j} \omega_i S_{\alpha^{-1}}(R_{ij})b_j = 0$$

Using theorem 27 we have

$$\sum_j S_{\alpha^{-1}}(R_{ij})b_j = 0 \quad for\ each\ i \in I$$

Multiplying both sides of the above equation by R_{ji} we obtain

$$b_j = 0$$

for any $j \in I$.

This means that the decomposition 3.56 is unique.

Applying $\Delta^r_{1,\alpha}$ to both sides of decomposition 3.56

$$\Delta^r_{1,\alpha}(\eta) = \Delta^r_{1,\alpha}(\sum_j \eta_j b_j)$$
$$\xi^1 \otimes 1_\alpha = \sum_j (\xi^1_j \otimes 1_\alpha)\Delta_{1,\alpha}(b_j)$$

Comparing this formula with decomposition 3.56 we get that

$$\Delta_{1,\alpha}(b_j) = b_{j(1,1)} \otimes 1_\alpha$$

Applying $\varepsilon \otimes \mathrm{id}$ we get that $b_j = \varepsilon(b_{j(1,1)})1_\alpha$. This way we proved that for any $\alpha \in \pi$, any $\eta \in \Gamma^\alpha_{\mathrm{inv}}$ is unique linear combination of $\eta_j (j \in I)$. Therefore, $(\eta_j)_{j\in I}$ is a basis in $\Gamma^{\mathrm{inv}}_\alpha$ and statement 2 is proven.

To prove statement *(3)*:

Using equation 3.45 we have for any $\alpha \in \pi, a \in A_\alpha, \eta_j \in \Gamma^\alpha_{\mathrm{inv}}$

$$a\eta_j = \sum_i \eta_i(a * (g_{ji} \circ S_1^{-1}))$$

Using equation 3.55 we get

$$\sum_i a\omega_i S_{\alpha^{-1}}(R_{ij}) = \sum_{i,h} \omega_h S_{\alpha^{-1}}(R_{hi})(a * (g_{ji} \circ S_1^{-1}))$$

Using equation 3.31 we get

$$\sum_{i,h} \omega_h((f_{ih} \circ S_1^{-1}) * a)S_{\alpha^{-1}}(R_{ij}) = \sum_{i,h} \omega_h S_{\alpha^{-1}}(R_{hi})(a * (g_{ji} \circ S_1^{-1}))$$

Using theorem 27 we get

$$\sum_i ((f_{ih} \circ S_1^{-1}) * a)S_{\alpha^{-1}}(R_{ij}) = \sum_i S_{\alpha^{-1}}(R_{hi})(a * (g_{ji} \circ S_1^{-1}))$$

Applying $S_\alpha (S_\alpha = S^{-1}_{\alpha^{-1}})$ to both sides of this equation, using that S_α is antimultiplicative we get:

$$\sum_i R_{ij} S_\alpha((f_{ih} \circ S_1^{-1}) * a) = \sum_i S_\alpha(a * (g_{ji} \circ S_1^{-1})) R_{hi} \qquad (3.57)$$

We compute

$$\begin{aligned} S_\alpha((f_{ih} \circ S_1^{-1}) * a) &= (\mathrm{id} \otimes f_{ih})(S^{-1}_{\alpha^{-1}} \otimes S_1^{-1})\Delta_{\alpha,1}(a) \\ &= S^{-1}_{\alpha^{-1}}(a) * f_{ih} \end{aligned}$$

Similarly, we have

$$S_\alpha(a * (g_{ji} \circ \ S_1^{-1})) = g_{ji} * S^{-1}_{\alpha^{-1}}(a)$$

i.e.

$$\sum_i R_{ij}(S^{-1}_{\alpha^{-1}}(a) * f_{ih}) = \sum_i (g_{ji} * S^{-1}_{\alpha^{-1}}(a)) R_{hi}$$

Replacing a by $S^{-1}_{\alpha^{-1}}(a)$ we obtain

$$\sum_i R_{ij}(a * f_{ih}) = \sum_i (g_{ji} * a) R_{hi}$$

And equation 3.52 follows. Note that if $a, R_{ij}, R_{hi} \in A_1$, then applying ε to both sides of equation 3.52 and using equation 3.50 we obtain:

$$\begin{aligned} \varepsilon(\sum_i R_{ij}(a * f_{ih})) &= \varepsilon(\sum_i (g_{ji} * a) R_{hi}) \\ \sum_i \varepsilon(R_{ij})\varepsilon(a * f_{ih}) &= \sum_i \varepsilon(g_{ji} * a)\varepsilon(R_{hi}) \\ \sum_i \delta_{ij}\varepsilon(a * f_{ih}) &= \sum_i \varepsilon(g_{ji} * a)\delta_{hi} \end{aligned}$$

But

$$\varepsilon(a * f_{ih}) = f_{ih}(a)$$

Similarly: $\varepsilon(g_{ji} * a) = g_{ji}(a)$ i.e. $f_{ij}(a) = g_{ij}(a)$, for any $a \in A_1$. From which we get that

$$\sum_i R_{ij}(a * f_{ih}) = \sum_i (f_{ji} * a) R_{hi} \qquad (3.58)$$

QED

.

For any $\alpha, \beta \in \pi, \eta_j \in \Gamma_{\text{inv}}^{\alpha\beta}$, applying $\Delta^l_{\alpha,\beta}$ to both sides of equation 3.55 we obtain:

$$\begin{aligned}\Delta^l_{\alpha,\beta}(\eta_j) &= \sum_h \Delta^l_{\alpha,\beta}(\omega_h)\Delta_{\alpha,\beta}(S_{(\alpha\beta)^{-1}}(R_{hj}))\\ &= \sum_{i,h}(S_{\alpha^{-1}}(R_{ij}) \otimes \omega_h^\beta S_{\beta^{-1}}(R_{hi}))\end{aligned}$$

i.e.

$$\Delta^l_{\alpha,\beta}(\eta_j) = \sum_i S_{\alpha^{-1}}(R_{ij}) \otimes \eta_i^\beta \tag{3.59}$$

Using equations 3.5 and 3.48

$$\Delta^r_{\alpha,\beta}(\sum_i a_i\omega_i) = \sum_i \Delta_{\alpha,\beta}(a_i)(\omega_j^\alpha \otimes \ R_{ij}) \tag{3.60}$$

33 Theorem. *Let $(f_{ij})_{i,j\in I}$ be the family of functionals defined on A satisfying relations 3.30, 3.31, $(R^\alpha_{ij})_{i,j\in I}$be a family of elements of $A = \{A_\alpha\}_{\alpha\in\pi}$ satisfying relations 3.49, 3.50, 3.58 for each $\alpha \in \pi$. Consider the left module $\Gamma = \{\Gamma_\alpha\}_{\alpha\in\pi}$ over $A = \{A_\alpha\}_{\alpha\in\pi}$ generated by ω_i^α , $i \in I$, $\alpha \in \pi$ for each $\alpha \in \pi$, and using formulae 3.39, 3.40, 3.60 to introduce right multiplication by elements of A, left and right actions of A on Γ then $\Gamma = (\{\Gamma_\alpha\}_{\alpha\in\pi}, \Delta^l, \Delta^r)$ is a $\pi-$graded bicovariant bimodule over A.*

Proof. Using formula 3.39 to introduce right multiplication by elements of A, one can easily check that Γ is also a $\pi-$graded right module over A., i.e.$\Gamma = \{\Gamma_\alpha\}_{\alpha\in\pi}$ is a $\pi-$graded bimodule over A.

Using 3.40 to define a left action of A on Γ ,taking into consideration theorem 27 we find that equations 3.1 and 3.2 are satisfied. For $\rho \in \Gamma_{\alpha\beta}, b \in A_{\alpha\beta}$ and $\alpha, \beta \in \pi$, using theorem 27 so $\rho = \sum_i a_i\omega_i^{\alpha\beta}$, $a_i \in A_{\alpha\beta}, \omega_i^{\alpha\beta} \in_{\text{inv}} \Gamma_{\alpha\beta}$

$$\begin{aligned}\Delta^l_{\alpha,\beta}(b\rho) &= \Delta^l_{\alpha,\beta}(\sum_i (ba_i)\omega_i^{\alpha\beta})\\ &= \Delta_{\alpha,\beta}(b)\sum_i \Delta_{\alpha,\beta}(a_i)\Delta^l_{\alpha,\beta}(\omega_i^{\alpha\beta})\\ &= \Delta_{\alpha,\beta}(b)\Delta^l_{\alpha,\beta}(\rho)\end{aligned}$$

Similarly,

$$\Delta^l_{\alpha,\beta}(\rho b) = \Delta^l_{\alpha,\beta}(\rho)\Delta_{\alpha,\beta}(b)$$

Moreover, using theorem 27 for any α , β , $\gamma \in \pi$, $\rho \in \Gamma_{\alpha\beta\gamma}$ and $\rho = \sum_i a_i\omega_i$ where $a_i \in A_{\alpha\beta\gamma}$, $\omega_i \in_{\text{inv}} \Gamma_{\alpha\beta\gamma}$ we have:

$$\begin{aligned}(\Delta_{\alpha,\beta} \otimes \text{id})\Delta^l_{\alpha\beta,\gamma}(\rho) &= \sum_i (\Delta_{\alpha,\beta} \otimes \text{id})\Delta_{\alpha\beta,\gamma}(a_i)\Delta^l_{\alpha\beta,\gamma}(\omega_i^{\alpha\beta\gamma}) \\ &= \sum_i ((\Delta_{\alpha,\beta} \otimes \text{id})\Delta_{\alpha\beta,\gamma}(a_i))(1_\alpha \otimes 1_\beta \otimes \omega_i^\gamma)\end{aligned}$$

Similarly, we have

$$(\text{id} \otimes \Delta^l_{\beta,\gamma})\Delta^l_{\alpha,\beta\gamma}(\rho) = \sum_i (\Delta_{\alpha,\beta} \otimes \text{id})\Delta_{\alpha\beta,\gamma}(a_i)(1_\alpha \otimes 1_\beta \otimes \omega_i^\gamma)$$

i.e.

$$(\Delta_{\alpha,\beta} \otimes \text{id})\Delta^l_{\alpha\beta,\gamma} = (\text{id} \otimes \Delta^l_{\beta,\gamma})\Delta^l_{\alpha,\beta\gamma}$$

which means that equation 3.3 of definition 10 is satisfied.

Finally, for any $\alpha \in \pi$, letting $\rho \in \Gamma_\alpha$, then using 3.26 $\rho = \sum_i a_i\omega_i^\alpha, a_i \in A_\alpha, \omega_i \in_{\text{inv}} \Gamma_\alpha$

$$\begin{aligned}(\varepsilon \otimes \text{id})\Delta^l_{1,\alpha}(\rho) &= \sum_i (\varepsilon \otimes \text{id})\Delta_{1,\alpha}(a_i)\Delta^l_{1,\alpha}(\omega_i) \\ &= \sum_i (\varepsilon \otimes \text{id})(\Delta_{1,\alpha}(a_i))(\varepsilon \otimes \text{id})(\Delta^l_{1,\alpha}(\omega_i)) \\ &= \rho\end{aligned}$$

i.e. 3.4 is satisfied.

This means that $\Gamma = (\{\Gamma_\alpha\}_{\alpha\in\pi}, \Delta^l)$ is a $\pi-$graded left covariant bimodule over A.

Using formula 3.60 to introduce right action of A on Γ one can easily check that $\Gamma = (\{\Gamma_\alpha\}_{\alpha\in\pi}, \Delta^r)$ is a $\pi-$graded right covariant bimodule over A, for let $\rho \in \Gamma_{\alpha\beta}, b \in A_{\alpha\beta}, \alpha, \beta \in \pi$, using theorem 27, for $\rho = \sum_i a_i\omega_i$, $a_i \in A_{\alpha\beta}, \omega_i \in_{\text{inv}} \Gamma_{\alpha\beta}$

$$\begin{aligned}\Delta^r_{\alpha,\beta}(b\rho) &= \sum_i \Delta_{\alpha,\beta}(ba_i)\Delta^r_{\alpha,\beta}(\omega_i) \\ &= \Delta_{\alpha,\beta}(b)\sum_i \Delta_{\alpha,\beta}(a_i)\Delta^r_{\alpha,\beta}(\omega_i) \\ &= \Delta_{\alpha,\beta}(b)\Delta^r_{\alpha,\beta}(\rho)\end{aligned}$$

Similarly, for $\rho \in \Gamma_{\alpha\beta}, b \in A_{\alpha\beta}, \alpha, \beta \in \pi$, using theorem 27 $\rho = \sum_i a_i\omega_i, a_i \in A_{\alpha\beta}, \omega_i \in_{\text{inv}} \Gamma_{\alpha\beta}$. Using 3.44 we get

$$\Delta^r_{\alpha,\beta}(\rho b) = \Delta^r_{\alpha,\beta}(\rho)\Delta_{\alpha,\beta}(b)$$

and thus equations 3.5 and 3.6 are satisfied. Moreover, using theorem 27 for any $\alpha,\beta,\gamma\in\pi,\rho\in\Gamma_{\alpha\beta\gamma},\rho=\sum_i a_i\omega_i^{\alpha\beta\gamma}$ where $a_i\in A_{\alpha\beta\gamma},\omega_i^{\alpha\beta\gamma}\in_{\rm inv}\Gamma_{\alpha\beta\gamma}$ we have:

$$
\begin{aligned}
(\Delta^r_{\alpha,\beta}\otimes{\rm id})\Delta^r_{\alpha\beta,\gamma}(\rho) &= \sum_i(\Delta^r_{\alpha,\beta}\otimes{\rm id})(\Delta_{\alpha\beta,\gamma}(a_i)\Delta^r_{\alpha\beta,\gamma}(\omega_i))\\
&= \sum_{i,j,k}(\Delta_{\alpha,\beta}\otimes{\rm id})(\Delta_{\alpha\beta,\gamma}(a_i))(\omega_k^\alpha\otimes R_{kj}\otimes R_{ji})
\end{aligned}
$$

Similarly,

$$
({\rm id}\otimes\Delta_{\beta,\gamma})\Delta^r_{\alpha,\beta\gamma}(\rho)=\sum_{i,j,k}({\rm id}\otimes\Delta_{\beta,\gamma})(\Delta_{\alpha,\beta\gamma}(a_i))(\omega_k^\alpha\otimes R_{kj}\otimes R_{ji})
$$

i.e. $(\Delta^r_{\alpha,\beta}\otimes{\rm id})\Delta^r_{\alpha\beta,\gamma}=({\rm id}\otimes\Delta_{\beta,\gamma})\Delta^r_{\alpha,\beta\gamma}$ which means that equation 3.7 is satisfied. Finally, for any $\alpha\in\pi$, letting $\rho\in\Gamma_\alpha$, then using 3.26 $\rho\in\Gamma_\alpha,\rho=\sum_i a_i\omega_i$ where $a_i\in A_\alpha,\omega_i\in_{\rm inv}\Gamma_\alpha$

$$
\begin{aligned}
({\rm id}\otimes\varepsilon)\Delta^r_{\alpha,1}(\rho) &= \sum_i a_i\omega_i\\
&= \rho
\end{aligned}
$$

i.e. 3.8 is satisfied. This means that $\Gamma=(\{\Gamma_\alpha\}_{\alpha\in\pi},\Delta^r)$ is a $\pi-$graded right covariant bimodule over A. To prove the bicovariance conditions , for any $\alpha,\beta,\gamma\in\pi$,$\rho\in\Gamma_{\alpha\beta\gamma}$, using theorem 27, for $\rho=\sum_i a_i\omega_i^{\alpha\beta\gamma}$, $a_i\in A_{\alpha\beta\gamma}$, $\omega_i^{\alpha\beta\gamma}\in_{\rm inv}\Gamma_{\alpha\beta\gamma}$ we compute

$$
\begin{aligned}
({\rm id}\otimes\Delta^r_{\beta,\gamma})\Delta^l_{\alpha,\beta\gamma}(\rho) &= \sum_i({\rm id}\otimes\Delta^r_{\beta,\gamma})(\Delta_{\alpha,\beta\gamma}(a_i)\Delta^l_{\alpha,\beta\gamma}(\omega_i^{\alpha\beta\gamma}))\\
&= \sum_{i,j}({\rm id}\otimes\Delta_{\beta,\gamma})(\Delta_{\alpha,\beta\gamma}(a_i))(1_\alpha\otimes\omega_j^\beta\otimes R_{ji})
\end{aligned}
$$

Similarly,

$$
(\Delta^l_{\alpha,\beta}\otimes{\rm id})\Delta^r_{\alpha\beta,\gamma}(\rho)=\sum_{i,j}(\Delta_{\alpha,\beta}\otimes{\rm id})(\Delta_{\alpha\beta,\gamma}(a_i))(1_\alpha\otimes\omega_j^\beta\otimes R_{ji})
$$

and hence equation 3.9 is proved and $\Gamma=(\{\Gamma_\alpha\}_{\alpha\in\pi},\Delta^l,\Delta^r)$is a $\pi-$graded bicovariant bimodule over A. QED

4 First order differential calculus on Hopf Group Coalgebras

Let $A^2=\{A^2_\alpha\}_{\alpha\in\pi}$ be the $\pi-$graded bimodule introduced in section 2. We introduce left and right actions of A on A^2. For any $\alpha,\beta\in\pi$ let $q\in$

$A_{\alpha\beta} \otimes A_{\alpha\beta}$,and $(\Delta_{\alpha,\beta} \otimes \Delta_{\alpha,\beta})(q) = \sum_k a_k \otimes b_k \otimes c_k \otimes d_k$, where $a_k, c_k \in A_\alpha$, $b_k, d_k \in A_\beta$, $k = 1, 2, \ldots, n$. We set

$$\Phi^l_{\alpha,\beta}(q) = \sum_k a_k c_k \otimes b_k \otimes d_k \tag{4.1}$$

$$\Phi^r_{\alpha,\beta}(q) = \sum_k a_k \otimes c_k \otimes b_k d_k \tag{4.2}$$

We compute

$$\begin{aligned}(\mathrm{id} \otimes m_\beta)(\Phi^l_{\alpha,\beta}(q)) &= \Delta_{\alpha,\beta}(m_{\alpha\beta}(q)) \\ &= 0\end{aligned}$$

Similarly we have

$$(m_\alpha \otimes \mathrm{id})(\Phi^r_{\alpha,\beta}(q)) = 0$$

Therefore,

$$\Phi^l_{\alpha,\beta} : A^2_{\alpha\beta} \longrightarrow A_\alpha \otimes A^2_\beta \tag{4.3}$$

and

$$\Phi^r_{\alpha,\beta} : A^2_{\alpha\beta} \longrightarrow A^2_\alpha \otimes A_\beta \tag{4.4}$$

Clearly, both are linear map. We will show that $A^2 = (\{A^2_\alpha\}_{\alpha\in\pi}, \Phi^l, \Phi^r)$ is a $\pi-$ graded bicovariant bimodule over A.

First, we will prove that $A^2 = (\{A^2_\alpha\}_{\alpha\in\pi}, \Phi^l, \Phi^r)$ is a $\pi-$ graded left covariant bimodule over A.

Let $\alpha, \beta \in \pi, q \in A^2_{\alpha\beta}, q = b \otimes c$, then

$$\begin{aligned}\Phi^l_{\alpha,\beta}(aq) &= (a_{(1,\alpha)} \otimes a_{(2,\beta)}) \cdot (b_{(1,\alpha)} c_{(1,\alpha)} \otimes b_{(2,\beta)} \otimes c_{(2,\beta)}) \\ &= \Delta_{\alpha,\beta}(a) \cdot \Phi^l_{\alpha,\beta}(q)\end{aligned}$$

Similarly

$$\Phi^l_{\alpha,\beta}(qa) = \Phi^l_{\alpha,\beta}(q) \cdot \Delta_{\alpha,\beta}(a)$$

Moreover, for any $\alpha, \beta, \gamma \in \pi, q \in A^2_{\alpha\beta\gamma}, q = a \otimes b$ we compute

$$\begin{aligned}(\Delta_{\alpha,\beta} \otimes \mathrm{id})\Phi^l_{\alpha\beta,\gamma}(q) &= (\Delta_{\alpha,\beta} \otimes \mathrm{id})\Phi^l_{\alpha\beta,\gamma}(a \otimes b) \\ &= (\Delta_{\alpha,\beta} \otimes \mathrm{id})(a_{(1,\alpha\beta)} b_{(1,\alpha\beta)} \otimes a_{(2,\gamma)} \otimes b_{(2,\gamma)}) \\ &= a_{(1,\alpha)} b_{(1,\alpha)} \otimes a_{(2,\beta)} b_{(2,\beta)} \otimes a_{(3,\gamma)} \otimes b_{(3,\gamma)}\end{aligned}$$

Similarly,

$$(\mathrm{id} \otimes \Phi^l_{\beta,\gamma})\Phi^l_{\alpha,\beta\gamma}(q) = a_{(1,\alpha)} b_{(1,\alpha)} \otimes a_{(2,\beta)} b_{(2,\beta)} \otimes a_{(3,\gamma)} \otimes b_{(3,\gamma)}$$

i.e.

$$(\Delta_{\alpha,\beta} \otimes \mathrm{id})\Phi^l_{\alpha,\beta\gamma} = (\mathrm{id} \otimes \Phi^l_{\beta,\gamma})\Phi^l_{\alpha\beta,\gamma}$$

Finally, for any $\alpha \in \pi, q \in A^2_\alpha, q = a \otimes b$

$$\begin{aligned}(\varepsilon \otimes \mathrm{id})\Phi^l_{1,\alpha}(q) &= \varepsilon(a_{(1,1)}b_{(1,1)})a_{(2,\alpha)} \otimes b_{(2,\alpha)} \\ &= a \otimes b \\ &= q\end{aligned}$$

and thus the conditions of definition 10 are fulfilled and $A^2 = (\{A^2_\alpha\}_{\alpha\in\pi}, \Phi^l)$ is a $\pi-$ graded left covariant bimodule over A. Similarly, one can check that $A^2 = (\{A^2_\alpha\}_{\alpha\in\pi}, \Phi^r)$ is a $\pi-$ graded right covariant bimodule over A. Finally, we check the bicovariance condition

For any $\alpha, \beta, \gamma \in \pi, q \in A^2_{\alpha\beta\gamma}, q = a \otimes b$ we compute

$$\begin{aligned}(\mathrm{id} \otimes \Phi^r_{\beta,\gamma})\Phi^l_{\alpha,\beta\gamma}(q) &= (\mathrm{id} \otimes \Phi^r_{\beta,\gamma})(a_{(1,\alpha)}b_{(1,\alpha)} \otimes a_{(2,\beta\gamma)} \otimes b_{(2,\beta\gamma)}) \\ &= a_{(1,\alpha)}b_{(1,\alpha)} \otimes a_{(2,\beta)} \otimes b_{(2,\beta)} \otimes a_{(3,\gamma)}b_{(3,\gamma)}\end{aligned}$$

Similarly,

$$(\Phi^l_{\alpha,\beta} \otimes \mathrm{id})\Phi^r_{\alpha\beta,\gamma}(q) = a_{(1,\alpha)}b_{(1,\alpha)} \otimes a_{(2,\beta)} \otimes b_{(2,\beta)} \otimes a_{(3,\gamma)}b_{(3,\gamma)}$$

which proves that $A^2 = (\{A^2_\alpha\}_{\alpha\in\pi}, \Phi^l, \Phi^r)$ is a $\pi-$ graded bicovariant bimodule over A.

On $A \otimes A = \{A_\alpha \otimes A_\alpha\}_{\alpha\in\pi}$ we define two families of linear mappings
$r = \{r_\alpha : A_\alpha \otimes A_\alpha \longrightarrow A_\alpha \otimes A_1\}_{\alpha\in\pi}$
$t = \{t_\alpha : A_\alpha \otimes A_\alpha \longrightarrow A_1 \otimes A_\alpha\}_{\alpha\in\pi}$
For any $\alpha \in \pi, a, b \in A_\alpha$ we set

$$r_\alpha(a \otimes b) = (a \otimes 1_1)\Delta_{\alpha,1}(b) \tag{4.5}$$

$$t_\alpha(a \otimes b) = (1_1 \otimes a)\Delta_{1,\alpha}(b) \tag{4.6}$$

It is clear that r_α, t_α are bijections for each $\alpha \in \pi$ for example for $a \in A_\alpha, b \in A_1$ the inverse of r_α is given by

$$r^{-1}_\alpha(a \otimes b) = (a \otimes 1_\alpha)(S_{\alpha^{-1}} \otimes \mathrm{id})\Delta_{\alpha^{-1},\alpha}(b) \tag{4.7}$$

Similarly , for $a \in A_1, b \in A_\alpha$ the inverse of t_α is given by

$$t^{-1}_\alpha(a \otimes b) = (b \otimes 1_\alpha)(S_{\alpha^{-1}} \otimes \mathrm{id})\sigma_{A_{\alpha^{-1}},A_\alpha}\Delta_{\alpha,\alpha^{-1}}(b) \tag{4.8}$$

One can easily show that for each $\alpha \in \pi$, $r_\alpha(A^2_\alpha) = A_\alpha \otimes \ker \varepsilon$, for let $\alpha \in \pi$, $a \in A_\alpha, b \in \ker \varepsilon$

$$\begin{aligned} m_\alpha r_\alpha^{-1}(a \otimes b) &= aS_{\alpha^{-1}}(b_{(1,\alpha^{-1})})b_{(2,\alpha)} \\ &= 0 \end{aligned}$$

From which we get $r_\alpha^{-1}(A_\alpha \otimes \ker \varepsilon) = A^2_\alpha$ i.e.

$$r_\alpha(A^2_\alpha) = A_\alpha \otimes \ker \varepsilon \tag{4.9}$$

Similarly, one can prove that

$$t_\alpha(A^2_\alpha) = \ker \varepsilon \otimes A_\alpha \tag{4.10}$$

34 Proposition. *For any* $\alpha, \beta, \gamma \in \pi$

$$(\Delta_{\alpha,\beta} \otimes \mathrm{id})r_{\alpha\beta} = (\mathrm{id} \otimes r_\beta)\Phi^l_{\alpha,\beta} \tag{4.11}$$

$$(\mathrm{id} \otimes \Delta_{\alpha,\beta})t_{\alpha\beta} = (t_\alpha \otimes \mathrm{id})\Phi^r_{\alpha,\beta} \tag{4.12}$$

PROOF. We will prove that for any $\alpha \in \pi$

$$r_\alpha = (\mathrm{id} \otimes \varepsilon \otimes \mathrm{id})\Phi^l_{\alpha,1} \tag{4.13}$$

$$t_\alpha = (\varepsilon \otimes \mathrm{id} \otimes \mathrm{id})\Phi^r_{1,\alpha} \tag{4.14}$$

For any $\alpha \in \pi, a, b \in A_\alpha, a \otimes b \in A_\alpha \otimes A_\alpha$

$$\begin{aligned} (\mathrm{id} \otimes \varepsilon \otimes \mathrm{id})\Phi^l_{\alpha,1}(a \otimes b) &= a_{(1,\alpha)}b_{(1,\alpha)}\varepsilon(a_{(2,1)}) \otimes b_{(2,1)} \\ &= (a \otimes 1_1)\Delta_{\alpha,1}(b) \\ &= r_\alpha(a \otimes b) \end{aligned}$$

Similarly,

$$(\varepsilon \otimes \mathrm{id} \otimes \mathrm{id})\Phi^r_{1,\alpha}(a \otimes b) = t_\alpha(a \otimes b)$$

To prove equation 4.11

$$\begin{aligned} (\Delta_{\alpha,\beta} \otimes \mathrm{id})r_{\alpha\beta} &= (\mathrm{id} \otimes \mathrm{id} \otimes \varepsilon \otimes \mathrm{id})(\mathrm{id} \otimes \Phi^l_{\beta,1})\Phi^l_{\alpha,\beta} \\ &= (\mathrm{id} \otimes r_\beta)\Phi^l_{\alpha,\beta} \end{aligned}$$

Similarly, one can prove equation 4.12. QED

35 Proposition. *For any $\alpha \in \pi$ an element of A^2_α is left- (right- respectively) invariant if and only if it is of the form $r^{-1}_\alpha(1_\alpha \otimes x)(t^{-1}_\alpha(y \otimes 1_\alpha)$ respectively) where $x \in \ker\varepsilon$ ($y \in \ker\varepsilon$ respectively).*

Proof. For any $\alpha \in \pi$, let $x \in \ker\varepsilon$. We compute

$$\begin{aligned}\Phi^l_{1,\alpha}(r^{-1}_\alpha(1_\alpha \otimes x)) &= 1_1 \otimes \varepsilon(x_{(2,1)})S_{\alpha^{-1}}(x_{(1,\alpha^{-1})}) \otimes x_{(3,\alpha)}\\ &= 1_1 \otimes r^{-1}_\alpha(1_\alpha \otimes x)\end{aligned}$$

i.e. $r^{-1}_\alpha(1_\alpha \otimes x)$ is left -invariant element.

Conversely, if $r^{-1}_\alpha(1_\alpha \otimes x)$ is left -invariant element for some $\alpha \in \pi$, let $x \in \ker\varepsilon$,

$a \in A_\alpha$. Equation 4.11 implies that

$$(\mathrm{id} \otimes r_\alpha)\Phi^l_{1,\alpha}(r^{-1}_\alpha(a \otimes x)) = (\Delta_{1,\alpha} \otimes \mathrm{id})r_\alpha(r^{-1}_\alpha(a \otimes x))$$

From which we obtain

$$1_1 \otimes a \otimes x = \Delta_{1,\alpha}(a) \otimes x$$

i.e.

$$\Delta_{1,\alpha}(a) = 1_1 \otimes a$$

From which we obtain

$$a = 1_\alpha.$$

QED

36 Theorem. *Let R be a right ideal of A_1 contained in $ker\varepsilon$, $N = \{N_\alpha\}_{\alpha\in\pi}$, where for each $\alpha \in \pi$, $N_\alpha = r^{-1}_\alpha(A_\alpha \otimes R)$ is a sub-bimodule of $A^2 = \{A^2_\alpha\}_{\alpha\in\pi}$. Moreover, let $\Gamma = \{\Gamma_\alpha\}_{\alpha\in\pi}, \Gamma_\alpha = A^2_\alpha/N_\alpha, \Pi = \{\Pi_\alpha : A^2_\alpha \longrightarrow A^2_\alpha/N_\alpha\}$ be the family of canonical epimorphisms, $d = \{d_\alpha : d_\alpha = \Pi_\alpha \circ D_\alpha\}$. Then the $\pi-$graded first order differential calculus $\Gamma = (\{\Gamma_\alpha\}_{\alpha\in\pi}, d)$ is left covariant. Any $\pi-$graded left covariant first order differential calculus on A can be obtained in this way.*

Proof. For any $\alpha \in \pi$,let R be a right ideal of A_1 contained in $ker\varepsilon$. We shall prove that $r^{-1}_\alpha(A_\alpha \otimes R)$ is a sub-bimodule of A^2_α. For any $\alpha \in \pi$, let $q \in r^{-1}_\alpha(A_\alpha \otimes R)$, i.e. $q = r^{-1}_\alpha(b \otimes c), b \in A_\alpha, c \in R$. For $a \in A_\alpha$

$$\begin{aligned}a \cdot q &= r^{-1}_\alpha(ab \otimes c)\\ &\in r^{-1}_\alpha(A_\alpha \otimes R)\\ q \cdot a &= r^{-1}_\alpha(ba_{(1,\alpha)} \otimes ca_{(2,1)})\\ &\in r^{-1}_\alpha(A_\alpha \otimes R)\end{aligned}$$

which proves that $N_\alpha = r^{-1}_\alpha(A_\alpha \otimes R)$ is a sub-bimodule of A^2_α.

To prove that it is left covariant we have to prove that for any $\alpha, \beta \in \pi$, $\Phi^l_{\alpha,\beta}(N_{\alpha\beta}) \subset A_\alpha \otimes N_\beta$.

Using 4.11 we have

$$\Phi^l_{\alpha,\beta} = (\mathrm{id} \otimes r_\beta^{-1})(\Delta_{\alpha,\beta} \otimes \mathrm{id}) r_{\alpha\beta}$$

Now, for any $\alpha, \beta \in \pi$, consider $N_{\alpha\beta} = r_{\alpha\beta}^{-1}(A_{\alpha\beta} \otimes R)$

$$\begin{aligned}\Phi^l_{\alpha,\beta}(N_{\alpha\beta}) &= (\mathrm{id} \otimes r_\beta^{-1})(\Delta_{\alpha,\beta}(A_{\alpha\beta}) \otimes R)\\ &\subset (\mathrm{id} \otimes r_\beta^{-1})(A_\alpha \otimes A_\beta \otimes R)\\ &= A_\alpha \otimes N_\beta\end{aligned}$$

Conversely, if $N = (\{N_\alpha\}_{\alpha\in\pi}, \Phi^l)$ is a left covariant bimodule, then , using theorem 27 and proposition 35 there exists a family $(x_i)_{i\in I}$ of elements of $ker\varepsilon$ such that for any $\alpha \in \pi, q \in N_\alpha$ can be written as $q = \sum_i a_i \cdot r_\alpha^{-1}(1_\alpha \otimes x_i), a_i \in A_\alpha$. But for each $i \in I$ we have

$$\begin{aligned}a_i \cdot r_\alpha^{-1}(1_\alpha \otimes x_i) &= r_\alpha^{-1}(r_\alpha(a_i \otimes 1_\alpha)(1_\alpha \otimes x_i))\\ &= r_\alpha^{-1}(a_i \otimes x_i)\end{aligned}$$

Denoting by R_α the linear span of all $x_i^{,}s$ we obtain that $N_\alpha = r_\alpha^{-1}(A_\alpha \otimes R_\alpha)$

We shall show that all $R_\alpha^{,}s$ coincide with R_1. From proposition 35 we have

$${}_{\mathrm{inv}}N_\alpha = r_\alpha^{-1}(1_\alpha \otimes R_\alpha)$$

and since N_α is a left covariant bimodule we have

$$\begin{aligned}\Phi^l_{\alpha,1}({}_{\mathrm{inv}}N_\alpha) &= 1_\alpha \otimes_{\mathrm{inv}} N_1\\ &= 1_\alpha \otimes r_1^{-1}(1_1 \otimes R_1)\end{aligned}$$

Now let $r_\alpha^{-1}(1_\alpha \otimes x_i) \in_{\mathrm{inv}} N_\alpha$, $x_i \in R_\alpha$

$$\begin{aligned}\Phi^l_{\alpha,1}(r_\alpha^{-1}(1_\alpha \otimes x_i)) &= S_{\alpha^{-1}}(x_{i(2,\alpha^{-1})}) x_{i(3,\alpha)} \otimes S_{1^{-1}}(x_{i(1,1)}) \otimes x_{i(4,1)}\\ &= 1_\alpha \otimes S_{1^{-1}}(x_{i(1,1)}) \otimes x_{i(2,1)}\\ &= 1_\alpha \otimes r_1^{-1}(1_1 \otimes x_i)\end{aligned}$$

i.e.

$$x_i \in R_1 \Longrightarrow R_\alpha \subseteq R_1$$

Similarly we can show that $R_1 \subseteq R_\alpha$, and hence $R_\alpha = R_1$ for each $\alpha \in \pi$. Denote by R to any of the $R_\alpha^{,}s$, then

$$N_\alpha = r_\alpha^{-1}(A_\alpha \otimes R)$$

It remains to show that R is a right ideal of A_1. Let $x \in R, a \in A_1$, then $r_1^{-1}(1_1 \otimes x) \in N_1$.

$$\begin{aligned} r_1^{-1}(1_1 \otimes x) \cdot a &= r_1^{-1}((1_1 \otimes x) r_1(1_1 \otimes a)) \\ &\in N_1 = r_1^{-1}(A_1 \otimes R) \\ &\quad (N_1 \text{ is a bimodule}) \end{aligned}$$

i.e

$$(1_1 \otimes x) r_1(1_1 \otimes a) \in A_1 \otimes R$$

therefore

$$\begin{aligned} (1_1 \otimes x) r_1(1_1 \otimes a) &= r_1(r_1^{-1}((1_1 \otimes x)\Delta_{1,1}(a))) \\ &= (1_1 \otimes x)\Delta_{1,1}(a) \in A_1 \otimes R \end{aligned}$$

and $(\varepsilon \otimes \mathrm{id})((1_1 \otimes x)\Delta_{1,1}(a)) = xa \in R$ QED

37 Theorem. *Let R be a right ideal of A_1 contained in $ker\varepsilon, N = \{N_\alpha\}_{\alpha \in \pi}$, where for each $\alpha \in \pi, N_\alpha = t_\alpha^{-1}(A_\alpha \otimes R)$ is a sub-bimodule of $A^2 = \{A_\alpha^2\}_{\alpha \in \pi}$. Moreover, let $\Gamma = \{\Gamma_\alpha\}_{\alpha \in \pi}, \Gamma_\alpha = A_\alpha^2 / N_\alpha, \Pi = \{\Pi_\alpha : A_\alpha^2 \longrightarrow A_\alpha^2 / N_\alpha\}$ be the family of canonical epimorphisms , $d = \{d_\alpha : d_\alpha = \Pi_\alpha \circ D_\alpha\}$. Then the first order differential calculus $\Gamma = (\{\Gamma_\alpha\}_{\alpha \in \pi}, d)$ is right covariant. Any right covariant first order differential calculus on A can be obtained in this way.*

Proof. For any $\alpha \in \pi$, let R be a right ideal of A_1 contained in $ker\varepsilon$. We shall prove that $t_\alpha^{-1}(R \otimes A_\alpha)$ is a sub-bimodule of A_α^2. For any $\alpha \in \pi$, let $q \in t_\alpha^{-1}(R \otimes A_\alpha)$, i.e. $q = t_\alpha^{-1}(d \otimes e), d \in R, e \in A_\alpha$. For $a \in A_\alpha$

$$\begin{aligned} a \cdot q &= t_\alpha^{-1}(t_\alpha((a \otimes 1_\alpha) q)) \\ &= t_\alpha^{-1}((1_1 \otimes a) t_\alpha(q)) \\ &= t_\alpha^{-1}(d \otimes ae) \\ &\in t_\alpha^{-1}(R \otimes A_\alpha) \end{aligned}$$

Similarly,

$$q \cdot a \in t_\alpha^{-1}(R \otimes A_\alpha)$$

which proves that $N_\alpha = t_\alpha^{-1}(R \otimes A_\alpha)$ is a sub-bimodule of A_α^2. To prove that it is right covariant we have to prove that for any $\alpha, \beta \in \pi, \Phi_{\alpha,\beta}^l(N_{\alpha\beta}) \subset N_\alpha \otimes A_\beta$. Using equation 4.13 we have

$$\Phi_{\alpha,\beta}^r = (t_\alpha^{-1} \otimes \mathrm{id})(\mathrm{id} \otimes \Delta_{\alpha,\beta}) t_{\alpha\beta}$$

Now, for any $\alpha, \beta \in \pi$, consider $N_{\alpha\beta} = t_{\alpha\beta}^{-1}(R \otimes A_{\alpha\beta})$

$$\begin{aligned}\Phi_{\alpha,\beta}^{l}(N_{\alpha\beta}) &= (t_{\alpha}^{-1} \otimes \mathrm{id})(\mathrm{id} \otimes \Delta_{\alpha,\beta})t_{\alpha\beta}(t_{\alpha\beta}^{-1}(R \otimes A_{\alpha\beta})) \\ &\subset (t_{\alpha}^{-1} \otimes \mathrm{id})(R \otimes A_{\alpha} \otimes A_{\beta}) \\ &= N_{\alpha} \otimes A_{\beta}\end{aligned}$$

Conversely, if $N = (\{N_{\alpha}\}_{\alpha\in\pi}, \Phi^{r})$ is a right covariant bimodule then, using theorem 27 and proposition 35 there exists a family $(y_i)_{i\in I}$ of elements of $ker\varepsilon$ such that for any $\alpha \in \pi, q \in N_{\alpha}$ can be written as $q = \sum_i a_i \cdot\ t_{\alpha}^{-1}(y_i \otimes 1_{\alpha}), a_i \in A_{\alpha}$. But for each $i \in I$ we have

$$\begin{aligned}a_i \cdot r_{\alpha}^{-1}(y_i \otimes 1_{\alpha}) &= t_{\alpha}^{-1}(t_{\alpha}(1_{\alpha} \otimes a_i)(y_i \otimes 1_{\alpha})) \\ &= t_{\alpha}^{-1}(y_i \otimes a_i)\end{aligned}$$

Denoting by R_{α} the linear span of all $x_i^{,}s$ we obtain that

$$N_{\alpha} = t_{\alpha}^{-1}(R \otimes A_{\alpha})$$

We shall show that all $R_{\alpha}^{,}s$ coincide with R_1. From proposition 35 we have

$$N_{\alpha}^{\mathrm{inv}} = t_{\alpha}^{-1}(1_{\alpha} \otimes A_{\alpha})$$

and since N_{α} is a left covariant bimodule we have

$$\begin{aligned}\Phi_{1,\alpha}^{r}(N_{\alpha}^{\mathrm{inv}}) &= N_{1}^{\mathrm{inv}} \otimes 1_{\alpha} \\ &= t_{1}^{-1}(R \otimes A_1) \otimes 1_{\alpha}\end{aligned}$$

Now let $t_{\alpha}^{-1}(y_i \otimes 1_{\alpha}) \in N_{\alpha}^{\mathrm{inv}}$, $y_i \in R_{\alpha}$

$$\begin{aligned}\Phi_{1,\alpha}^{r}(t_{\alpha}^{-1}(y_i \otimes 1_{\alpha})) &= S_{1^{-1}}(y_{i(3,1)}) \otimes y_{i(1,1)} \otimes 1_{\alpha}\varepsilon(y_{i(2,1)}) \\ &= t_{1}^{-1}(y_i \otimes 1_1) \otimes 1_{\alpha}\end{aligned}$$

i.e.

$$y_i \in R_1 \Longrightarrow R_{\alpha} \subseteq R_1$$

Similarly we can show that $R_1 \subseteq R_{\alpha}$, and hence $R_{\alpha} = R_1$ for each $\alpha \in \pi$.
Denote by R to any of the $R_{\alpha}^{,}s$, then $N_{\alpha} = t_{\alpha}^{-1}(R \otimes A_{\alpha})$
It remains to show that R is a right ideal of A_1.
Let $y \in R, a \in A_1, A$ then $t_1^{-1}(y \otimes 1_1) \in N_1$.

$$\begin{aligned}t_1^{-1}(y \otimes 1_1) \cdot a &= t_1^{-1}((y \otimes 1_1)t_1(1_1 \otimes a)) \in N_1 \\ &= t_{\alpha}^{-1}(R \otimes A_1)\end{aligned}$$

i.e $(y\otimes 1_1)t_1(1_1\otimes a)\in R\otimes A_1$

therefore

$$\begin{aligned}(y\otimes 1_1)t_1(1_1\otimes a) &= t_1(t_1^{-1}((y\otimes 1_1)\Delta_{1,1}(a)))\\ &= (y\otimes 1_1)\Delta_{1,1}(a)\in A_1\otimes R\end{aligned}$$

and $(\mathrm{id}\otimes\varepsilon)((y\otimes 1_1)\Delta_{1,1}(a)) = ya\in R$

QED

We shall now formulate the concept of ad −invariance. Let

$$\mathrm{ad}_\alpha : A_1 \longrightarrow A_1\otimes A_\alpha$$

be such that for any $a\in A_1$

$$\mathrm{ad}_\alpha(a) = t_\alpha(r_\alpha^{-1}(1_\alpha\otimes a)) \tag{4.15}$$

i.e.

$$\mathrm{ad}_\alpha(a) = a_{(2,1)}\otimes S_{\alpha^{-1}}(a_{(1,\alpha^{-1})})a_{(3,\alpha)}$$

where

$$(\mathrm{id}\otimes\Delta_{1,\alpha})\Delta_{\alpha^{-1},\alpha}(a) = a_{(1,\alpha^{-1})}\otimes a_{(2,1)}\otimes a_{(3,\alpha)} \tag{4.16}$$

such that

$$(\mathrm{ad}_\alpha\otimes\mathrm{id})\,\mathrm{ad}_\beta(a) = (\mathrm{id}\otimes\Delta_{\alpha,\beta})\,\mathrm{ad}_{\alpha\beta} \tag{4.17}$$

Using equation 4.15, and the standard properties of comultiplication and converse one can prove equation 4.17, for let $a\in A_1$. For any $\alpha,\beta\in\pi$

$$\begin{aligned}(\mathrm{ad}_\alpha\otimes\mathrm{id})\,\mathrm{ad}_\beta(a) &= (\mathrm{ad}_\alpha\otimes\mathrm{id})(a_{(2,1)}\otimes S_{\beta^{-1}}(a_{(1,\beta^{-1})})a_{(3,\beta)})\\ &= a_{(3,1)}\otimes S_{\alpha^{-1}}(a_{(2,\alpha^{-1})})a_{(4,\alpha)}\otimes S_{\beta^{-1}}(a_{(1,\beta^{-1})})a_{(5,\beta)}\end{aligned}$$

Similarly,

$$(\mathrm{id}\otimes\Delta_{\alpha,\beta})\,\mathrm{ad}_{\alpha\beta} = a_{(3,1)}\otimes S_{\alpha^{-1}}(a_{(2,\alpha^{-1})})a_{(4,\alpha)}\otimes S_{\beta^{-1}}(a_{(1,\beta^{-1})})a_{(5,\beta)}$$

which proves equation 4.17. QED

A linear subset $T\subset A_1$ is $\pi-\mathrm{ad}$ invariant if $\mathrm{ad}_\alpha(T)\subset T\otimes A$ for any $\alpha\in\pi$.

38 Lemma. *Let T be $\pi-\mathrm{ad}$ invariant subset of A_1, R be a right ideal of A_1 generated by T. Then R is $\pi-\mathrm{ad}$ invariant.*

Proof. Let $a, b \in A_1$, we will prove that for any $\alpha \in \pi$

$$\mathrm{ad}_\alpha(ab) = (1_1 \otimes S_{\alpha^{-1}}(b_{(1,\alpha^{-1})}))\,\mathrm{ad}_\alpha(a)\Delta_{1,\alpha}(b_{(2,\alpha)}) \tag{4.18}$$

$$\begin{aligned} r_\alpha^{-1}(1_\alpha \otimes ab) &= S_{\alpha^{-1}}(a_{(1,\alpha^{-1})}b_{(1,\alpha^{-1})}) \otimes a_{(2,\alpha)}b_{(2,\alpha)} \\ &= (S_{\alpha^{-1}}(b_{(1,\alpha^{-1})}) \otimes 1_\alpha) r_\alpha^{-1}(a)(1_\alpha \otimes b_{(2,\alpha)}) \end{aligned}$$

Applying t_α to both sides of the above equation we get

$$t_\alpha r_\alpha^{-1}(1_\alpha \otimes ab) = t_\alpha(S_{\alpha^{-1}}(b_{(1,\alpha^{-1})}) \otimes 1_\alpha) t_\alpha r_\alpha^{-1}(a) t_\alpha(1_\alpha \otimes b_{(2,\alpha)}) \tag{4.19}$$

$$\mathrm{ad}_\alpha(ab) = (1_\alpha \otimes S_{\alpha^{-1}}(b_{(1,\alpha^{-1})}))\,\mathrm{ad}_\alpha(a)\Delta_{1,\alpha}(b_{(2,\alpha)}) \tag{4.20}$$

Thus for $a, b \in T, a, b, ab \in R$, R being an ideal in A_1, T being $\pi-$ ad invariant we find that

$$\mathrm{ad}_\alpha(ab) \in R \otimes A_\alpha$$

i.e.

$$\mathrm{ad}_\alpha(R) \subset R \otimes A_\alpha$$

which means that R is $\pi-$ ad invariant. *QED*

Let $A^2 = (\{A^2_\alpha\}_{\alpha\in\pi}, \Phi^l, \Phi^r)$ is a $\pi-$ graded bicovariant bimodule over A. By virtue of condition 3 of definition 23 we have for any $\alpha, \beta, \gamma \in \pi$ $(\Phi^l_{\alpha,1} \otimes \mathrm{id})\Phi^r_{\alpha,\beta} = (\mathrm{id} \otimes \Phi^r_{1,\beta})\Phi^l_{\alpha,\beta}$

Applying $\mathrm{id} \otimes \varepsilon \otimes \mathrm{id} \otimes \mathrm{id}$ to both sides of the above equation and using equations 4.13 and 4.14 we get

$$\Phi^r_{\alpha,\beta} = (r_\alpha^{-1} \otimes \mathrm{id})(\mathrm{id} \otimes t_\beta)\Phi^l_{\alpha,\beta}$$

Now let $x \in \ker \varepsilon$. From proposition 35 for any $\alpha \in \pi$ we have $r_\alpha^{-1}(1_\alpha \otimes x)$ is a left invariant element then

$$\begin{aligned} \Phi^r_{\alpha,\beta}(r_{\alpha\beta}^{-1}(1_{\alpha\beta} \otimes x)) &= (r_\alpha^{-1} \otimes \mathrm{id})(\mathrm{id} \otimes t_\beta)(1_\alpha \otimes r_\beta^{-1}(1_\beta \otimes x)) \\ &= (r_\alpha^{-1} \otimes \mathrm{id})(1_\alpha \otimes \mathrm{ad}_\beta(x)) \end{aligned}$$

39 Theorem. *Let R be a right ideal of A_1 contained in $\ker \varepsilon$ and $\Gamma = (\{\Gamma_\alpha\}_{\alpha\in\pi}, d)$be the $\pi-$graded left covariant first order differential calculus described in theorem 5. Then $\Gamma = (\{\Gamma_\alpha\}_{\alpha\in\pi}, d)$ is bicovariant if and only if R is $\pi-$ad invariant.*

Proof. Let for any $\alpha \in \pi$ R be a right ideal of A_1such that $R \subset \ker \varepsilon$ and $N_\alpha = r_\alpha^{-1}(A_\alpha \otimes R)$. Using theorem 35 we see that $N = (\{N_\alpha\}_{\alpha\in\pi}, \Phi^l)$ is a $\pi-$graded left covariant bimodule. Assume that R is $\pi-$ad invariant , let ${}_{\text{inv}}N_\alpha$ be the set of all left invariant elements of N_α for each $\alpha \in \pi$. Then formula 3.78 shows that for any $\alpha, \beta \in \pi$

$$\Phi^r_{\alpha,\beta}({}_{\text{inv}}N_{\alpha\beta}) \subset_{\text{inv}} N_\alpha \otimes A_\beta$$

Now decomposition 3.26 shows that $\Phi^r_{\alpha,\beta}(N_{\alpha\beta}) \subset N_\alpha \otimes A_\beta$,and this means that implication 2.14 holds.

Conversely, assume that $N = \{N_\alpha\}_{\alpha\in\pi}$ is a $\pi-$graded bicovariant bimodule. This means that 2.14 holds.Then (see proof 37) for each $\alpha \in \pi$, $N_\alpha = t_\alpha^{-1}(R' \otimes A_\alpha)$ where R' be a right ideal of A_1 such that $R' \subset \ker \varepsilon$. In particular, $N_1 = t_1^{-1}(R' \otimes A_1)$. Using 3.78 and that $(\varepsilon \otimes \text{id})t_1^{-1}(a \otimes b) = a\varepsilon(b)$,and $(\text{id} \otimes \varepsilon)r_1^{-1}(a \otimes b) = aS_1(b)$. one can easily checks that $R = R'$.

So we have for any $\alpha \in \pi$

$$r_\alpha^{-1}(A_\alpha \otimes R) = t_\alpha^{-1}(R \otimes A_\alpha)$$

$$t_\alpha r_\alpha^{-1}(A_\alpha \otimes R) = R \otimes A_\alpha$$

therefore $\text{ad}_\alpha(R) = t_\alpha r_\alpha^{-1}(1_\alpha \otimes R)$
$\subset t_\alpha r_\alpha^{-1}(A_\alpha \otimes R) = R \otimes A_\alpha$
therefore R is $\pi-$ad invariant. QED

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