Differential calculus on Hopf Group Coalgebra

A. S. Hegazi
Department of Mathematics, Faculty of Science,
Mansoura university, Mansoura, 35516, EGYPT.
hegazi@mans.edu.eg

W. Morsi
Department of Mathematics, Faculty of Science,
Mansoura university, Mansoura, 35516, EGYPT.
wulaa.mor@yahoo.com

M. Mansour
Department of Mathematics, Faculty of Science,
Mansoura university, Mansoura, 35516, EGYPT.
mansour@mans.edu.eg

Received: 09/11/2006; accepted: 30/01/2008.

Abstract. In this paper we construct the Differential calculus on the Hopf Group Coalgebra introduced by Turaev [10]. We proved that the concepts introduced by S.L.Woronowicz in constructing Differential calculus on Hopf Compact Matrix Pseudogroups (Quantum Groups) [5] can be adapted to serve again in our construction.

Keywords: Differential calculus, Hopf algebra, Hopf Group Coalgebra

MSC 2000 classification: primary 16D26, secondary 81R50

Introduction

Quantum groups, from a mathematical point of view, may be introduced by making emphasis on their $q$–deformed enveloping algebra aspects [2], [3] which leads to the quantized enveloping algebras, or by making emphasis in the $R$–matrix formalism that describes the deformed group algebra. Also, they are mathematically well defined in the framework of Hopf algebra [1]. Quantum groups provide an interesting example of non-commutative geometry [8]. Non-commutative differential calculus on quantum groups is a fundamental tool needed for many applications [7], [6].

S.L.Woronowicz [5] gave the general framework for bicovariant differential calculus on quantum groups following general ideas of A.Connes. Also, He showed that all important notions and formulae of classical Lie group theory admit a generalization to the quantum group case and he has restricted himself to compact matrix pseudogroups as introduced in [4]. In contrast to the classical
differential geometry on Lie groups, there is no functorial method to obtain a unique bicovariant differential calculus on a given quantum group [9].

Recently, Quasitriangular Hopf $\pi-$coalgebras are introduced by Turaev [10]. He has showed that they give rise to crossed $\pi-$categories. Virelizier [11] studied the algebraic properties of the Hopf $\pi-$coalgebras, also he has showed that the existence of integrals and trace for such coalgebras and has generalized the main properties of the quasitriangular Hopf algebras to the setting of Hopf $\pi-$coalgebra.

In this paper we will use the concepts introduced by S.L.Woronowicz [5] to construct the Differential calculus on the Hopf group coalgebra(introduced by Turaev [10]). We briefly describe the content of the paper.In section one we give the definition of Hopf group coalgebras [11]. In section two, we give the main definitions and theorems concerning first order differential calculus. Section three contains the construction of the $\pi-$graded Bicovariant bimodules. Finally, in section four we construct the first order differential calculus on the Hopf group coalgebra.

Now let us give some basic definitions about Hopf $\pi-$coalgebra where $\pi$ is a non-commutative discrete group.

1 Hopf Group Coalgebra

1 Definition. A $\pi-$coalgebra is a family $C = \{C_{\alpha}\}_{\alpha \in \pi}$ of $k-$linear spaces endowed with a family $\Delta = \{\Delta_{\alpha, \beta} : C_{\alpha, \beta} \to C_{\alpha} \otimes C_{\beta}\}_{\alpha, \beta \in \pi}$ of $k-$linear maps (the comultiplication) and a $k-$linear map $\varepsilon : C_1 \to k$ such that

- $\Delta$ is coassociative in the sense that for any $\alpha, \beta, \gamma \in \pi$,
  $(\Delta_{\alpha, \beta} \otimes \text{id})\Delta_{\alpha, \beta, \gamma} = (\text{id} \otimes \Delta_{\beta, \gamma})\Delta_{\alpha, \beta, \gamma},$

- for all $\alpha \in \pi$,
  $(\text{id} \otimes \varepsilon)\Delta_{\alpha, 1} = (\varepsilon \otimes \text{id})\Delta_{1, \alpha}.$

2 Notation. [Sweedler’s notation] In the case of Hopf group coalgebra Sweedler’s notations have been extended by Turaev [10] and Virelizier [11] in the following way: for any $\alpha, \beta \in \pi$ and $c \in C_{\alpha, \beta}$, they defined

$$\Delta_{\alpha, \beta}(c) = \sum_{(c)} c_{(1, \alpha)} \otimes c_{(2, \beta)} \in C_{\alpha} \otimes C_{\beta}.$$ or shortly, if we have the summation implicit

$$\Delta_{\alpha, \beta}(c) = c_{(1, \alpha)} \otimes c_{(2, \beta)}.$$
The coassociativity axiom gives that, for any $\alpha, \beta, \gamma \in \pi$ and $c \in C_{\alpha\beta\gamma}$

$$C_{(1,\alpha\beta)(1,\alpha)} \otimes C_{(1,\alpha\beta)(2,\beta)} \otimes C_{(2,\gamma)} = C_{(1,\alpha)} \otimes C_{(2,\beta\gamma)(1,\beta)} \otimes C_{(2,\beta\gamma)(2,\gamma)}.$$  

Let $C = \{C_\alpha\}_{\alpha \in \pi}, \Delta, \varepsilon$ be a $\pi-$coalgebra and $A$ be an algebra with multiplication $m$ and unit element $1_A$. The family $\Delta$ and the map $m$ induce a map

$*$ : $\text{conv}(C, A) \otimes \text{conv}(C, A) \to \text{conv}(C, A)$

defined by the composition

$$\text{Hom}(C_\alpha, A) \otimes \text{Hom}(C_\beta, A) \xrightarrow{\rho} \text{Hom}(C_\alpha \otimes C_\beta, A \otimes A) \xrightarrow{\text{Hom}(\Delta_{\alpha\beta\gamma}, m)} \text{Hom}(C_{\alpha\beta\gamma}, A)$$

where $\rho$ is the natural injection of $\text{Hom}(C_\alpha, A) \otimes \text{Hom}(C_\beta, A)$ into $\text{Hom}(C_\alpha \otimes C_\beta, A \otimes A)$

The map $*$ is called convolution product of $f, g$

Also, the maps

$$\varepsilon : C_1 \to k \quad \text{and} \quad \eta : k \to A$$

induce a map

$$\eta_{\text{Conv}(C, A)} : k \to \text{Conv}(C, A)$$

defined by

$$(\eta_{\text{Conv}(C, A)}(\lambda))(c) = \varepsilon(c)\eta(\lambda)$$

for all $c \in C_1$.

**3 Lemma.** The $k$-space

$$\text{Conv}(C, A) = \bigoplus_{\alpha \in \pi} \text{Hom}(C_\alpha, A)$$

endowed with the convolution product $*$ and the unit element $\varepsilon 1_A$ is a $\pi-$graded algebra called the convolution algebra.

**4 Remark.** If we put $A = k$ in the above lemma the $\pi-$graded algebra $\text{Conv}(C, k) = \bigoplus_{\alpha \in \pi} C_\alpha^*$ is called dual to $C$ and denoted by $C^*$.

**5 Definition.** A Hopf $\pi-$coalgebra is a $\pi-$coalgebra $H = \{H_\alpha\}_{\alpha \in \pi}, \Delta, \varepsilon$ endowed with a family

$$S = \{S_\alpha : H_\alpha \to H_{\alpha^{-1}}\}_{\alpha \in \pi}$$

of $k-$linear maps called the antipode such that

1. Each $H_\alpha$ is an algebra with multiplication $m_\alpha$ and unit element $1_\alpha \in H_\alpha$, 

(2) The linear maps
\[ \Delta_{\alpha,\beta} : H_{\alpha\beta} \to H_{\alpha} \otimes H_{\beta}, \]
\[ \varepsilon : H_1 \to k. \]
are algebra maps for all \( \alpha, \beta \in A \).

(3) For any \( \alpha \in \pi \)
\[ m_\alpha(S_{\alpha^{-1}})\Delta_{\alpha^{-1},\alpha} = m_\alpha(id \otimes S_{\alpha^{-1}})\Delta_{\alpha,\alpha^{-1}}. \]

6 Remark. If \( H = (\{H_\alpha\}_{\alpha \in \pi}, \Delta, \varepsilon, S) \) is a Hopf \( \pi \)-coalgebra then axiom (3) says that \( S_\alpha \) is the inverse of \( I_{H_{\alpha^{-1}}} \) in the convolution algebra \( \text{Conv}(H, H_{\alpha^{-1}}) \).

7 Remark. \((H_1, \Delta_{1,1}, \varepsilon, S_1)\) is a classical Hopf algebra

8 Lemma. Let \( H = (\{H_\alpha\}_{\alpha \in \pi}, \Delta, \varepsilon, S) \) be a Hopf \( \pi \)-coalgebra. Then

(1) \( \Delta_{\beta^{-1},\alpha^{-1}}S_{\alpha\beta} = \sigma_{H_{\alpha^{-1}},H_{\beta^{-1}}} (S_\alpha \otimes S_\beta) \Delta_{\alpha,\beta} \) for any \( \alpha, \beta \in \pi \),

(2) \( \varepsilon(S_1) = \varepsilon \),

(3) \( S_\alpha(ab) = S_\alpha(b)S_\alpha(a) \) for any \( \alpha \in \pi \) and \( a, b \in A \),

(4) \( S_{1_\alpha} = 1_{\alpha^{-1}} \) for any \( \alpha \in \pi \).

9 Definition. Let \( C = (\{C_\alpha\}_{\alpha \in \pi}, \Delta, \varepsilon) \) be a \( \pi \)-coalgebra. A right \( \pi \)-co-module over \( C \) is a family \( M = \{M_\alpha\}_{\alpha \in \pi} \) of \( k \)-linear spaces endowed with a family \( \rho = \{\rho_{\alpha,\beta} : M_{\alpha\beta} \to M_\alpha \otimes C_\beta\} \) of \( k \)-linear maps (the structure maps) such that

- For any \( \alpha, \beta, \gamma \in \pi \)
  \[ (\rho_{\alpha,\beta} \otimes \text{id})\rho_{\alpha,\beta,\gamma} = (\text{id} \otimes \Delta_{\beta,\gamma})\rho_{\alpha,\beta,\gamma} \]
  \[ \ast \]

- For any \( \alpha \in \pi \)
  \[ (\text{id} \otimes \varepsilon)\rho_{\alpha,1} = \text{id} \]
  \[ \ast \ast \]

10 Definition. A \( \pi \)-subcomodule of \( M \) is a family \( N = \{N_\alpha\}_{\alpha \in \pi} \) where \( N_\alpha \) is a \( k \)-linear subspace of \( M_\alpha \) such that for all \( \alpha, \beta \in \pi \)
\[ \rho_{\alpha,\beta}(N_{\alpha\beta}) \subset N_\alpha \otimes C_\beta \]
11 Definition. A $\pi$–comodule morphism between to right $\pi$–comodules $M$ and $M'$ over a $\pi$–coalgebra $C$ (with structure maps $\rho$ and $\rho'$, respectively) is a family $f = \{ f_\alpha : M_\alpha \rightarrow M'_\alpha \}$ of $k$–linear maps such that for all $\alpha, \beta \in \pi$

$$\rho'_{\alpha,\beta}(f_\alpha \otimes \text{id}) = (f_\alpha \otimes \text{id}) \rho_{\alpha,\beta}$$

12 Notation. [Sweedler’s notation] For any $\alpha, \beta \in \pi$ and $m \in M_{\alpha,\beta}$ we write

$$\rho_{\alpha,\beta}(m) = m_{(0,\alpha)} \otimes m_{(1,\beta)} \in M_\alpha \otimes C_\beta$$

also the axiom

$$(\rho_{\alpha,\beta} \otimes \text{id}) \rho_{\alpha,\beta,\gamma} = (\text{id} \otimes \Delta_{\beta,\gamma}) \rho_{\alpha,\beta,\gamma}$$

can be written as

$$m_{(0,\alpha\beta)(0,\alpha)} \otimes m_{(0,\alpha\beta)(1,\beta)} \otimes m_{(1,\gamma)} = m_{(0,\alpha)} \otimes m_{(1,\beta\gamma)(1,\beta)} \otimes m_{(1,\beta\gamma)(2,\gamma)}$$

This elements of $M_\alpha \otimes C_\beta \otimes C_\gamma$ is written as $m_{(0,\alpha)} \otimes m_{(1,\beta)} \otimes m_{(2,\gamma)}$

2 Basic Definitions of differential calculus

13 Definition. Let $A = \{ A_\alpha \}_{\alpha \in \pi}$ be a Hopf group coalgebra, $\Gamma = \{ \Gamma_\alpha \}_{\alpha \in \pi}$ be a $\pi$–graded bimodule over $A$, and

$$d = \{ d_\alpha : A_\alpha \rightarrow \Gamma_\alpha \} \quad \text{(2.1)}$$

be a family of linear maps. We say that $(\Gamma, d)$ is a $\pi$–graded first order differential calculus over $A$ if for any $\alpha \in \pi$

1. For any $a, b \in A_\alpha$

$$d_\alpha(ab) = d_\alpha(a)b + ad_\alpha(b) \quad \text{(2.2)}$$

2. Any element $\rho \in \Gamma_\alpha$ is of the form

$$\rho = \sum_{k=1}^{n} a_k d_\alpha b_k , \quad a_k, b_k \in A_\alpha$$

14 Definition. Two $\pi$–graded first order differential calculi are said to be isomorphic if there exists a family of bimodule isomorphisms $i = \{ i_\alpha : \Gamma_\alpha \rightarrow \Gamma'_\alpha \}$ such that

$$i_\alpha(d_\alpha a) = d'_\alpha a, \quad \text{for all } a \in A_\alpha, \alpha \in \pi.$$
Let \( A = \{A_\alpha\}_{\alpha \in \pi} \) be a Hopf group coalgebra, \( m_\alpha : A_\alpha \otimes A_\alpha \to A_\alpha \) be the multiplication defined on \( A_\alpha \) for each \( \alpha \). Define \( A^2 = \{A^2_\alpha\}_{\alpha \in \pi} \) such that

\[
A^2_\alpha = \{q \in A_\alpha \otimes A_\alpha, m_\alpha(q) = 0\} \quad (2.3)
\]

By definition \( A^2_\alpha \) is a linear subspace of \( A_\alpha \otimes A_\alpha \) for each \( \alpha \in \pi \). On \( A^2 \) define an \( A \)-bimodule structure as

For any \( \alpha \in \pi, c \in A_\alpha, \sum_k a_k \otimes b_k \in A^2_\alpha \)

\[
c \left( \sum_k a_k \otimes b_k \right) = \sum_k ca_k \otimes b_k \quad (2.4)
\]

\[\left( \sum_k a_k \otimes b_k \right) c = \sum_k a_k \otimes b_k c \quad (2.5)\]

Define \( D = \{D_\alpha\}_{\alpha \in \pi} \) by

\[D_\alpha(b) = 1_\alpha \otimes b - b \otimes 1_\alpha,\]

for all \( b \in A_\alpha, \alpha \in \pi \).

It is clear that \( m_\alpha(D_\alpha(b)) = 0 \), i.e. \( D_\alpha(b) \in A^2_\alpha \). Moreover

\[D_\alpha(ab) = D_\alpha(a)b + aD_\alpha(b)\]

15 Proposition. Let \( N = \{N_\alpha\}_{\alpha \in \pi} \) be a \( \pi \)-graded sub-bimodule of \( A^2 \), \( \Gamma = A^2/N \), \( \pi = \{\pi_\alpha : A^2_\alpha \to \Gamma_\alpha\} \) be the family of canonical epimorphisms, and \( d = \{d_\alpha = \pi_\alpha \circ D_\alpha\}_{\alpha \in \pi} \) be a first order differential calculus over \( A \). Any other \( \pi \)-graded first order differential calculus over \( A \) can be obtained in this way.

Proof. By definition of \( \Gamma = \{\Gamma_\alpha\}_{\alpha \in \pi} \), \( \Gamma \) is a \( \pi \)-graded bimodule over \( A \). Moreover, by definition of \( d = \{d_\alpha = \pi_\alpha \circ D_\alpha\}_{\alpha \in \pi} \) we find that \( \Gamma = \{(\Gamma_\alpha\}_{\alpha \in \pi}, d\) is a \( \pi \)-graded first order differential calculus over \( A \). It remains to show that any \( \pi \)-graded first order differential calculus over \( A \) can be obtained in this way.

Let \( \Gamma = \{(\Gamma_\alpha\}_{\alpha \in \pi}, d\) be any other \( \pi \)-graded first order differential calculus over \( A \). We have for each \( \alpha \in \pi, \sum_k a_k \otimes b_k \in A^2_\alpha, c \in A_\alpha \)

\[\sum_k ca_k d_\alpha b_k = c \left( \sum_k a_k d_\alpha b_k \right)\]

and

\[\sum_k a_k d_\alpha (b_k c) = \left( \sum_k a_k d_\alpha b_k \right) c\]
i.e. the family \( \pi = \{ \pi_{\alpha} : \mathcal{A}_\alpha^2 \rightarrow \Gamma_{\alpha} \} \) defined by the formula

\[
\pi_{\alpha} \left( \sum_k a_k \otimes b_k \right) = \sum_k a_k d_{\alpha} b_k
\]

is a bimodule morphism. We will show that \( \pi_{\alpha} \) is surjective for each \( \alpha \in \pi \).

Let \( \rho \in \Gamma_{\alpha} \) such that

\[
\rho = \sum_k a_k d_{\alpha} b_k, \quad a_k, b_k \in \mathcal{A}_{\alpha}
\]

Define an element \( q \in \mathcal{A}_{\alpha} \otimes \mathcal{A}_{\alpha} \) by

\[
q = \sum_k a_k \otimes b_k - a_k b_k \otimes 1_{\alpha}
\]

It is clear that \( m_{\alpha} q = 0 \), i.e. \( q \in \mathcal{A}_\alpha^2 \). Moreover,

\[
\pi_{\alpha}(q) = \rho
\]

therefore \( \pi_{\alpha} \) is surjective for each \( \alpha \in \pi \).

\[
\ker \pi = \{ \ker \pi_{\alpha} \}_{\alpha \in \pi} = \left\{ \sum_k a_k \otimes b_k \in \mathcal{A}_\alpha^2, \sum_k a_k d_{\alpha} b_k = 0 \right\}_{\alpha \in \pi}
\]

Taking

\[
N = \{ N_{\alpha} = \ker \pi_{\alpha} = \{ \sum_k a_k \otimes b_k \in \mathcal{A}_\alpha^2, \sum_k a_k d_{\alpha} b_k = 0 \} \}_{\alpha \in \pi}
\]

then \( \Gamma \) can be identified by \( \mathcal{A}_\alpha^2/N \) and for any \( b \in \mathcal{A}_{\alpha} \)

\[
\pi_{\alpha} D_{\alpha}(b) = d_{\alpha} b.
\]

16 Definition. Let \( \Gamma = (\{ \Gamma_{\alpha} \}_{\alpha \in \pi}, d) \) be a \( \pi \)-graded first order differential calculus over \( \mathcal{A} \). We say that \( \Gamma = (\{ \Gamma_{\alpha} \}_{\alpha \in \pi}, d) \) is left covariant if for any \( \alpha, \beta \in \pi \)

\[
\sum_k a_k d_{\alpha \beta} b_k = 0 \implies \sum_k \Delta_{\alpha \beta}(a_k)(\text{id} \otimes d_{\beta})\Delta_{\alpha \beta}(b_k) = 0
\]

for any \( a_k, b_k \in \mathcal{A}_{\alpha \beta}, k = 1, 2, \ldots, n \).
17 Proposition. Let $\Gamma = \{ \Gamma_\alpha \}_{\alpha \in \pi}, d$ be a left covariant $\pi$-graded first order differential calculus over $A$. Then there exists a family of linear mappings
\[ \Delta^l = \left\{ \Delta^l_{\alpha,\beta} : \Gamma_{\alpha\beta} \to A_\alpha \otimes \Gamma_\beta \right\} \quad (2.9) \]
such that
(1) For any $a \in A_{\alpha\beta}, \rho \in \Gamma_{\alpha\beta}$
\[ \Delta^l_{\alpha,\beta}(\rho a) = \Delta^l_{\alpha,\beta}(\rho) \Delta^l_{\alpha,\beta}(a) \quad (2.10) \]
(2) For any $\alpha, \beta, \gamma \in \pi$
\[ (\Delta^l_{\alpha,\beta} \otimes \text{id}) \Delta^l_{\alpha\beta\gamma} = (\text{id} \otimes \Delta^l_{\beta\gamma}) \Delta^l_{\alpha,\beta\gamma} \quad (2.12) \]
(3) For any $\rho \in \Gamma_\alpha$
\[ (\varepsilon \otimes \text{id}) \Delta^l_{1,\alpha}(\rho) = \rho \quad (2.13) \]
(4) For any $\alpha, \beta \in \pi$
\[ \Delta^l_{\alpha,\beta}d_{\alpha\beta} = (\text{id} \otimes d_\beta) \Delta^l_{\alpha,\beta}(a) \]

PROOF. Let $\Delta^l = \left\{ \Delta^l_{\alpha,\beta} \right\}_{\alpha,\beta \in \pi}$ where $\Delta^l_{\alpha,\beta} : \Gamma_{\alpha\beta} \to A_\alpha \otimes \Gamma_\beta$ is defined by
\[ \Delta^l_{\alpha,\beta} \left( \sum_{k=1}^n a_k d_{\alpha\beta} b_k \right) = \sum_{k=1}^n \Delta_{\alpha,\beta}(a_k)(\text{id} \otimes d_\beta) \Delta_{\alpha,\beta}(b_k) \]
where $a_k, b_k \in A_{\alpha\beta}, \alpha, \beta \in \pi$. Then by definition for each $\alpha, \beta \in \pi$ $\Delta^l_{\alpha,\beta}$ is a well defined linear map.

(1) Let $a \in A_{\alpha\beta}, \rho \in \Gamma_{\alpha\beta}$,
\[ \rho = \sum_{k=1}^n a_k d_{\alpha\beta} b_k, a_k, b_k \in A_{\alpha\beta} \]
\[ \Delta^l_{\alpha,\beta}(\rho a) = \Delta^l_{\alpha,\beta} \left( \sum_k a_k d_{\alpha\beta}(b_k a) - \sum_k a_k b_k d_{\alpha\beta} a \right) \]
\[ = \sum_k \Delta_{\alpha,\beta}(a_k)(b_k(1,\alpha)a_{(1,\alpha)} \otimes d_\beta b_k(2,\beta)a_{(2,\beta)}) \]
\[ = \left( \sum_k \Delta_{\alpha,\beta}(a_k)(\text{id} \otimes d_\beta) \Delta_{\alpha,\beta}(b_k) \right) \Delta_{\alpha,\beta}(a) \]
\[ = \Delta^l_{\alpha,\beta}(\rho) \Delta_{\alpha,\beta}(a) \]
Similarly,
\[ \Delta^l_{\alpha,\beta}(a\rho) = \Delta_{\alpha,\beta}(a)\Delta^l_{\alpha,\beta}(\rho) \]

(2) Let \( \text{ad}_{\alpha\beta\gamma} b \in \Gamma_{\alpha\beta\gamma} \), with \( a, b \in A_{\alpha\beta\gamma} \), then we have
\[
(\Delta_{\alpha,\beta} \otimes \text{id})\Delta^l_{\alpha\beta,\gamma}(\text{ad}_{\alpha\beta\gamma} b) = (\Delta_{\alpha,\beta} \otimes \text{id})\Delta_{\alpha\beta,\gamma}(a)(\text{id} \otimes \text{id} \otimes d_{\gamma})(\Delta_{\alpha,\beta} \otimes \text{id})\Delta_{\alpha\beta,\gamma}(b)
\]

On the other hand
\[
(\text{id} \otimes \Delta^l_{\beta,\gamma})\Delta^l_{\alpha,\beta\gamma}(\text{ad}_{\alpha\beta\gamma} b) = (\text{id} \otimes \Delta_{\beta,\gamma})\Delta_{\alpha,\beta\gamma}(a)(\text{id} \otimes \text{id} \otimes d_{\gamma})(\text{id} \otimes \Delta_{\beta,\gamma})\Delta_{\alpha,\beta\gamma}(b)
\]

(3) For \( \alpha \in \pi \) let \( \text{ad}_\alpha b \in \Gamma_\alpha \), \( a, b \in A_\alpha \)
\[
(\varepsilon \otimes \text{id})\Delta^l_{1,\alpha}(\text{ad}_\alpha b) = \varepsilon(a_{(1,1)})a_{(2,\alpha)}\varepsilon(b_{(1,1)})d_\alpha b_{(2,\alpha)} = \text{ad}_\alpha b.
\]

(4) Let \( a \in A_{\alpha\beta} \)
\[
\Delta^l_{\alpha,\beta}d_{\alpha\beta}(a) = \Delta_{\alpha,\beta}(1_{\alpha\beta})(\text{id} \otimes d_\beta)\Delta_{\alpha,\beta}(a)
\]
\[ = (\text{id} \otimes d_\beta)\Delta_{\alpha,\beta}(a)
\]

18 Definition. Let \( \Gamma = (\{\Gamma_\alpha\}_{\alpha \in \pi}, d) \) be a \( \pi \)-graded first order differential calculus over \( A \). We say that \( \Gamma = (\{\Gamma_\alpha\}_{\alpha \in \pi}, d) \) is right covariant if for any \( \alpha, \beta \in \pi \)
\[
\sum_{k=1}^{n} a_k d_{\alpha\beta} b_k = 0 \Rightarrow \sum_{k=1}^{n} \Delta_{\alpha,\beta}(a_k)(d_\alpha \otimes \text{id})\Delta_{\alpha,\beta}(b_k) = 0 \quad (2.14)
\]

We say that \( \Gamma = (\{\Gamma_\alpha\}_{\alpha \in \pi}, d) \) is bicovariant if it is left and right covariant.

19 Proposition. Let \( \Gamma = (\{\Gamma_\alpha\}_{\alpha \in \pi}, d) \) be a right covariant \( \pi \)-graded first order differential calculus over \( A \). Then there exists a family of linear mappings
\[
\Delta^r = \{\Delta^r_{\alpha,\beta} : \Gamma_\alpha \rightarrow \Gamma_\alpha \otimes A_\beta\}
\]

such that
(1) For any $a \in A_{\alpha \beta}, \rho \in \Gamma_{\alpha \beta}$

$$
\Delta^r_{\alpha, \beta}(a \rho) = \Delta^r_{\alpha, \beta}(a) \Delta^r_{\alpha, \beta}(\rho) \\
\Delta^r_{\alpha, \beta}(\rho a) = \Delta^r_{\alpha, \beta}(\rho) \Delta^r_{\alpha, \beta}(a)
$$

(2.16)

(2) for any $\alpha, \beta, \gamma \in \pi$

$$(\text{id} \otimes \Delta_{\beta, \gamma}) \Delta^r_{\alpha, \beta \gamma} = (\Delta^r_{\alpha, \beta} \otimes \text{id}) \Delta^r_{\alpha, \beta \gamma}$$

(2.17)

(3) For any $\rho \in \Gamma_{\alpha}$

$$(\text{id} \otimes \varepsilon) \Delta^r_{\alpha, 1}(\rho) = \rho$$

(2.18)

(4) for any $\alpha, \beta, \gamma \in \pi$

$$
\Delta^r_{\alpha, \beta \gamma} d_{\alpha, \beta} = (d_{\alpha} \otimes \text{id}) \Delta^r_{\alpha, \beta}
$$

PROOF. Similar to that of proposition 17, where for any $\alpha, \beta, \gamma \in \pi a_k, b_k \in A_{\alpha \beta}$

$$
\Delta^r_{\alpha, \beta} \left( \sum_k a_k d_{\alpha, \beta} b_k \right) = \sum_k \Delta^r_{\alpha, \beta}(a_k)(d_{\alpha} \otimes \text{id}) \Delta^r_{\alpha, \beta}(b_k)
$$

(2.19)

QED

20 Proposition. Let $\Gamma = (\{\Gamma_a\}_{a \in \pi}, d)$ be a bicovariant $\pi$-graded first order differential calculus over $A$, $\Delta^l, \Delta^r$ be the families of linear mappings introduced in proposition 17 and 19. Then we have

$$(\text{id} \otimes \Delta^r_{\beta, \gamma}) \Delta^l_{\alpha, \beta \gamma}(\text{ad}_{\alpha, \beta \gamma} b) = (\Delta^l_{\alpha, \beta} \otimes \text{id}) \Delta^r_{\alpha, \beta \gamma}(\text{ad}_{\alpha, \beta \gamma} b)$$

(2.20)

PROOF. Let $a, b \in A_{\alpha \beta \gamma}$

$$(\text{id} \otimes \Delta^r_{\beta, \gamma}) \Delta^l_{\alpha, \beta \gamma}(\text{ad}_{\alpha, \beta \gamma} b) = \\
(id \otimes \Delta^r_{\beta, \gamma}) \Delta^l_{\alpha, \beta \gamma}(a)(\text{id} \otimes d_{\beta} \otimes \text{id})(\text{id} \otimes \Delta^r_{\beta, \gamma}) \Delta^l_{\alpha, \beta \gamma}(b)$$

On the other hand

$$(\Delta^l_{\alpha, \beta} \otimes \text{id}) \Delta^r_{\alpha, \beta \gamma}(\text{ad}_{\alpha, \beta \gamma} b) = \\
(\Delta^l_{\alpha, \beta} \otimes \text{id}) \Delta^r_{\alpha, \beta \gamma}(a)(\text{id} \otimes d_{\beta} \otimes \text{id})(\Delta^l_{\alpha, \beta} \otimes \text{id}) \Delta^r_{\alpha, \beta \gamma}(b)$$

Using the coassociativity property we find that equation 2.20 holds. QED
3 \( \pi \)-graded Bicovariant bimodules

Throughout this section let \( A = \{ A_\alpha \}_{\alpha \in \pi} \) be a hopf group coalgebra.

21 Definition. Let \( \Gamma = \{ \Gamma_\alpha \}_{\alpha \in \pi} \) be a \( \pi \)-graded bimodule over \( A \), \( \Delta^l = \{ \Delta^l_{\alpha,\beta} : \Gamma_\alpha \to A_\alpha \otimes \Gamma_\beta \}_{\alpha,\beta \in \pi} \) be a family of linear maps. We say that \( \Gamma = (\{ \Gamma_\alpha \}_{\alpha \in \pi}, \Delta^l) \) is a left covariant \( \pi \)-graded bimodule over \( A \) if

1. For any \( a \in A_{\alpha\beta}, \rho \in \Gamma_{\alpha\beta} \), \( \alpha, \beta \in \pi \)
   \[
   \Delta^l_{\alpha,\beta}(a\rho) = \Delta_{\alpha,\beta}(a)\Delta^l_{\alpha,\beta}(\rho) \tag{3.1}
   \]
   \[
   \Delta^l_{\alpha,\beta}(\rho a) = \Delta^l_{\alpha,\beta}(\rho)\Delta_{\alpha,\beta}(a) \tag{3.2}
   \]

2. For all \( \alpha, \beta, \gamma \in \pi \)
   \[
   (\Delta_{\alpha,\beta} \otimes \text{id})\Delta^l_{\alpha\beta,\gamma} = (\text{id} \otimes \Delta^l_{\beta,\gamma})\Delta^l_{\alpha,\beta\gamma} \tag{3.3}
   \]

3. For any \( \rho \in \Gamma_{\alpha,\alpha} \), \( \alpha \in \pi \)
   \[
   (\varepsilon \otimes \text{id})\Delta^l_{1,\alpha}(\rho) = \rho \tag{3.4}
   \]

22 Definition. Let \( \Gamma = \{ \Gamma_\alpha \}_{\alpha \in \pi} \) be a \( \pi \)-graded bimodule over \( A \), \( \Delta^r = \{ \Delta^r_{\alpha,\beta} : \Gamma_\alpha \to \Gamma_\alpha \otimes A_\beta \}_{\alpha,\beta \in \pi} \) be a family of linear maps. We say that \( \Gamma = (\{ \Gamma_\alpha \}_{\alpha \in \pi}, \Delta^r) \) is a right covariant \( \pi \)-graded bimodule over \( A \) if

1. For any \( a \in A_{\alpha\beta}, \rho \in \Gamma_{\alpha\beta} \), \( \alpha, \beta \in \pi \)
   \[
   \Delta^r_{\alpha,\beta}(a\rho) = \Delta_{\alpha,\beta}(a)\Delta^r_{\alpha,\beta}(\rho) \tag{3.5}
   \]
   \[
   \Delta^r_{\alpha,\beta}(\rho a) = \Delta^r_{\alpha,\beta}(\rho)\Delta_{\alpha,\beta}(a) \tag{3.6}
   \]

2. For \( \alpha, \beta, \gamma \in \pi \)
   \[
   (\Delta_{\alpha,\beta} \otimes \text{id})\Delta^r_{\alpha\beta,\gamma} = (\text{id} \otimes \Delta^r_{\beta,\gamma})\Delta^r_{\alpha,\beta\gamma} \tag{3.7}
   \]

3. For any \( \rho \in \Gamma_{\alpha,\alpha} \), \( \alpha \in \pi \)
   \[
   (\text{id} \otimes \varepsilon)\Delta^r_{\alpha,1}(\rho) = \rho \tag{3.8}
   \]
23 Definition. Let \( \Gamma = \{ \Gamma_\alpha \}_{\alpha \in \pi} \) be a \( \pi \)-graded bimodule over \( A \), \( \Delta^l = \{ \Delta^l_{\alpha,\beta} : \Gamma_{\alpha,\beta} \to A_\alpha \otimes A_{\beta}, \Delta^r = \{ \Delta^r_{\alpha,\beta} : \Gamma_{\alpha,\beta} \to A_\alpha \otimes A_{\beta} \} \) be two families of linear maps. We say that \( \Gamma = (\{ \Gamma_\alpha \}_{\alpha \in \pi}, \Delta^l, \Delta^r) \) is a bicovariant \( \pi \)-graded bimodule over \( A \) if

1. \( \Gamma = (\{ \Gamma_\alpha \}_{\alpha \in \pi}, \Delta^l) \) is a left covariant \( \pi \)-graded bimodule over \( A \).
2. \( \Gamma = (\{ \Gamma_\alpha \}_{\alpha \in \pi}, \Delta^r) \) is a right covariant \( \pi \)-graded bimodule over \( A \).
3. For all \( \alpha, \beta, \gamma \in \pi \),

\[
(\Delta^l_{\alpha,\beta} \otimes \text{id}) \Delta^r_{\alpha,\beta,\gamma} = (\text{id} \otimes \Delta^r_{\beta,\gamma}) \Delta^l_{\alpha,\beta,\gamma}
\]

24 Definition. Let \( \Gamma = (\{ \Gamma_\alpha \}_{\alpha \in \pi}, \Delta^l) \) be a left covariant \( \pi \)-graded bimodule over \( A \). For any \( \alpha \in \pi \) an element \( \rho \in \Gamma_\alpha \) is said to be left invariant if

\[
\Delta^l_{1,\alpha}(\rho) = 1_1 \otimes \rho
\]

Denote by \( \text{inv} \Gamma = \{ \text{inv} \Gamma_\alpha \}_{\alpha \in \pi} \) the set of all left invariant elements of \( \Gamma \). Clearly \( \text{inv} \Gamma_\alpha \) is a linear subspace of \( \Gamma_\alpha \) for each \( \alpha \in \pi \).

25 Lemma. Let \( \Gamma = (\{ \Gamma_\alpha \}_{\alpha \in \pi}, \Delta^l) \) be a left covariant \( \pi \)-graded bimodule over \( A \), \( \Gamma = \{ \text{inv} \Gamma_\alpha \}_{\alpha \in \pi} \) be the linear subspace of all left invariant elements of \( \Gamma \). Then there exists a family

\[
P = \{ P_\alpha : \Gamma_1 \to \Gamma_\alpha \}_{\alpha \in \pi}
\]

of mappings such that

\[
P_\alpha(b \rho) = \varepsilon(b) P_\alpha(\rho)
\]

for any \( b \in A_1, \rho \in \Gamma_1, \alpha \in \pi \).

Moreover, for any \( \rho \in \Gamma_\alpha \), \( \alpha \in \pi \) we have

\[
\rho = \sum_k a_k P_\alpha(\rho_k)
\]

where \( a_k, \rho_k \) are elements of \( A_\alpha \), \( \Gamma_1 \) respectively such that

\[
\Delta^l_{\alpha,1}(\rho) = \sum_k a_k \otimes \rho_k
\]

and equation 3.13 can be uniquely written in this form.
Proof. For any $\alpha \in \pi$, $\rho \in \Gamma_1$ set

$$P_\alpha(\rho) = \sum_k S_{\alpha^{-1}}(a_k)\rho_k$$

(3.15)

where

$$\Delta^l_{\alpha^{-1},\alpha}(\rho) = \sum_{k=1}^n a_k \otimes \rho_k$$

Recall that for any $\alpha, \beta \in \pi$, $a \in A_{\beta^{-1}}$ where $\Delta_{\beta^{-1}\alpha^{-1},\alpha}(a) = a_{(1,\beta^{-1}\alpha^{-1})} \otimes a_{(2,\alpha)}$ we have

$$\Delta_{\alpha,\beta}(S_{\beta^{-1}\alpha^{-1}}(a_{(1,\beta^{-1}\alpha^{-1})}))(a_{(2,\alpha)} \otimes 1_\beta) = 1_\alpha \otimes S_{\beta^{-1}}(a)$$

(3.16)

For any $\rho \in \Gamma_1$, $\alpha \in \pi$ set

$$\Delta^l_{\alpha^{-1},\alpha}(\rho) = \sum_k a_k \otimes \rho_k$$

$$\Delta^l_{1,\alpha}(\rho_k) = \sum_{l} b_{kl} \otimes \rho_{kl}$$

$$\Delta_{\alpha^{-1},1}(a_k) = \sum_{m} c_{km} \otimes d_{km}$$

Using equation 3.3 we have

$$\sum_{k,l} a_k \otimes b_{kl} \otimes \rho_{kl} = \sum_{k,m} c_{km} \otimes d_{km} \otimes \rho_k$$

(3.17)

We compute

$$\Delta^l_{1,\alpha}(P_\alpha(\rho)) = \sum_k \Delta_{1,\alpha}(S_{\alpha^{-1}}(a_k))\Delta^l_{1,\alpha}(\rho_k)$$

$$= \sum_{k,l} \Delta_{1,\alpha}(S_{\alpha^{-1}}(c_{km}))(d_{km} \otimes \rho_k)$$

$$= \sum_k (1_1 \otimes S_{\alpha^{-1}}(a_k))(1_1 \otimes \rho_k)$$

$$= 1_1 \otimes P_\alpha(\rho)$$

This shows that $P_\alpha(\rho)$ is left invariant element in $\Gamma_\alpha$ for each $\alpha \in \pi$. 
To prove Equation 3.12, let \( b \in A_1, \rho \in \Gamma_1 \), set
\[
\Delta_{\alpha^{-1}, \alpha}(b) = \sum_k b_k \otimes d_k
\]
\[
\Delta_{\alpha^{-1}, \alpha}(\rho) = \sum_l c_l \otimes \rho_l
\]
\[
\Delta_{\alpha^{-1}, \alpha}(bp) = \Delta_{\alpha^{-1}, \alpha}(b) \Delta_{\alpha^{-1}, \alpha}(\rho) = \sum_{k,l} b_k c_l \otimes d_k \rho_l
\]
Then
\[
P_\alpha(bp) = \sum_{k,l} S_{\alpha^{-1}}(c_l) S_{\alpha^{-1}}(b_k) d_k \rho_l
\]
\[
= \varepsilon(b) P_\alpha(\rho)
\]
To prove equation 3.13. Let \( \alpha \in \pi, \rho \in \Gamma_\alpha \). Set
\[
\Delta_{1, \alpha}(\rho) = \sum_m d_m \otimes \varrho_m
\]
\[
\Delta_{\alpha^{-1}, \alpha}(\rho_k) = \sum_n b_{kn} \otimes \rho_{kn}
\]
\[
\Delta_{\alpha, \alpha^{-1}}(d_m) = \sum_l d_{ml} \otimes c_{ml}
\]
where
\[
\Delta_{\alpha, 1}(\rho) = \sum_k a_k \otimes \rho_k
\]
using equation 3.3 we have
\[
(\Delta_{\alpha, \alpha^{-1}} \otimes \text{id}) \Delta_{1, \alpha} = (\text{id} \otimes \Delta_{\alpha^{-1}, \alpha}) \Delta_{\alpha, 1}
\]
i.e.
\[
\sum_{m,l} d_{ml} \otimes c_{ml} \otimes \varrho_m = \sum_{k,n} a_k \otimes b_{kn} \otimes \rho_{kn}
\]
Then using equation 3.4 we have
\[
\rho = \sum_m \varepsilon(d_m) \varrho_m
\]
\[
= \sum_{k,n} a_k S_{\alpha^{-1}}(b_{kn}) \rho_{kn}
\]
\[
= \sum_m a_k P_\alpha(\rho_k)
\]
Finally, to prove the uniqueness of expression 3.13 let \( P' = \{ P'_\alpha : \Gamma_1 \rightarrow inv \Gamma_\alpha \} \) be another family of mappings satisfying that for \( \rho \in \Gamma_\alpha \)

\[
\rho = \sum_k a_k P'_\alpha(\rho_k) \tag{3.20}
\]

where \( a_k, \rho_k \) are elements of \( A_\alpha, \Gamma_1 \) respectively such that

\[
\Delta_{\alpha,1}(\rho) = \sum_k a_k \otimes \rho_k
\]

Let \( \rho \in \Gamma_\alpha \) such that \( \Delta_{\alpha,1}(\rho) = \sum_k a_k \otimes \rho_k \). Then using 3.13

\[
\rho = \sum_k a_k P_\alpha(\rho_k)
\]

But using equation 3.20 we have

\[
\rho = \sum_k a_k P'_\alpha(\rho_k)
\]

Subtracting the above two equations we obtain

\[
0 = \sum_k a_k (P_\alpha(\rho_k) - P'_\alpha(\rho_k))
\]

Assuming that all \( a_k \)'s all linearly independent we get

\[
P_\alpha(\rho_k) = P'_\alpha(\rho_k) \quad k = 1, 2, \ldots, n.
\]

which proves the uniqueness of the expression 3.13.

**26 Lemma.** Let \( \Gamma = (\{ \Gamma_\alpha \}_{\alpha \in \pi}, \Delta^l) \) be a left covariant \( \pi \)-graded bimodule over \( A \). Then, for any \( \alpha, \beta \in \pi, \rho \in inv \Gamma_{\alpha \beta} \) we have

\[
\Delta^l_{\alpha, \beta}(\rho) = 1_\alpha \otimes \varrho \tag{3.21}
\]

where \( \varrho \in inv \Gamma_\beta \).

**Proof.** Let \( \alpha, \beta \in \pi, \rho \in inv \Gamma_{\alpha \beta} \), then using lemma 3, and since the mappings \( P_\alpha \) are onto for each \( \alpha \in \pi \) then there exists an element \( \xi \in \Gamma_1 \) such that

\[
\rho = P_{\alpha \beta}(\xi) \tag{3.22}
\]
Set

\[
\Delta^l_{\beta^{-1} \alpha^{-1}, \alpha \beta}(\xi) = \sum_k a_k \otimes \xi_k
\]
\[
\Delta^l_{\alpha \beta}(\xi_k) = \sum_l c_{kl} \otimes \xi_{kl}
\]
\[
\Delta^l_{\beta^{-1}, \beta}(\xi) = \sum_m b_m \otimes \rho_m
\]

and

\[
\Delta^l_{\beta^{-1} \alpha^{-1}, \alpha}(b_m) = \sum_n b_{mn} \otimes d_{mn}
\] (3.23)

Using equation 3.3

\[
\sum_{k,l} a_k \otimes c_{kl} \otimes \xi_{kl} = \sum_{m,n} b_{mn} \otimes d_{mn} \otimes \rho_m
\] (3.24)

Applying \(\Delta^l_{\alpha \beta}\) to both sides of equation 3.22, using equations 3.24 and 3.16, we get

\[
\Delta^l_{\alpha \beta}(\rho) = \sum_k \Delta^l_{\alpha \beta}(S_{\beta^{-1} \alpha^{-1}}(a_k))\Delta^l_{\alpha \beta}(\xi_k)
\]
\[
= \sum_{m,n} \Delta^l_{\alpha \beta}(S_{\beta^{-1} \alpha^{-1}}(b_{mn}))(d_{mn} \otimes \rho_m)
\]
\[
= \sum_m (1_\alpha \otimes S_{\beta^{-1}}(b_m))(1_\alpha \otimes \rho_m)
\]
\[
= 1_\alpha \otimes P_{\beta}(\xi)
\]

But from lemma 3 \(P_{\beta}(\xi) \in \text{inv } \Gamma_{\beta}\) and hence the lemma is proved. \(\blacksquare\)

Let \(A = (\{A_\alpha\}_{\alpha \in \pi}, \Delta, \varepsilon, S)\) be a hopf \(\pi\)–coalgebra. Throughout the next dealing we will consider that \(A\) is endowed with a family of linear maps \(\Psi = \{\Psi_\alpha : A_\alpha \rightarrow A_1\}\) of \(k\)-linear maps such that for each \(\alpha \in \pi\), \(\Psi_\alpha\) is an algebra map. For each \(\alpha \in \pi\), define the map \(E_\alpha\) to be the composition

\[
A_\alpha \rightarrow A_1 \rightarrow k
\]

i.e.

\[
E_\alpha = \varepsilon \Psi_\alpha
\] (3.25)

Clearly, for each \(\alpha \in \pi\) \(E_\alpha\) is an algebra map for let \(a, b \in A_\alpha\). Then
\[ E_\alpha(ab) = \varepsilon(\Psi_\alpha(a) \Psi_\alpha(b)) \]
\[ = E_\alpha(a)E_\alpha(b) \]
\[ E_\alpha(1_\alpha) = 1_k \]

Moreover, \( E_\alpha \) is linear being the composition of two linear maps.

**27 Theorem.** Let \( \Gamma = (\{\Gamma_\alpha\}_{\alpha \in \pi}, \Delta^1) \) be a \( \pi \)-graded left covariant bimodule over \( A \), \( \{\omega^\alpha_i\}_{\alpha \in \pi} \) be a basis of \( \text{inv} \Gamma_\alpha \), of all left invariant elements of \( \Gamma_\alpha \) for each \( \alpha \in \pi \). Then

1. For any \( \alpha \in \pi \), any element \( \rho \in \Gamma_\alpha \) is of the form
   \[ \rho = \sum_i a_i \omega_i \]  
   (3.26)
   where \( a_i \)'s \( \in A_\alpha \) are uniquely determined, \( \omega_i \)'s \( \in \text{inv} \Gamma_\alpha \) for any \( \alpha \in \pi \).

2. For any \( \alpha \in \pi \), any element \( \rho \in \Gamma_\alpha \) is of the form
   \[ \rho = \sum_i \omega_i b_i \]  
   (3.27)
   where \( b_i \)'s \( \in A_\alpha \) are uniquely determined, \( \omega_i \)'s \( \in \text{inv} \Gamma_\alpha \) for any \( \alpha \in \pi \).

3. There exists linear functionals \( f_{ij} \in A' = \bigoplus_{\alpha \in \pi} A'_\alpha \) such that for any \( \alpha \in \pi \)
   \[ \omega_i b = \sum_j (f_{ij} \ast b)\omega_j \]  
   (3.28)
   \[ a \omega_i = \sum_j \omega_j ((f_{ij} \circ S^{-1}_1) \ast a) \]  
   (3.29)
   where \( a, b \in A_\alpha, \omega_i, \omega_j \)'s \( \in \text{inv} \Gamma_\alpha \). These functionals are uniquely determined by equation 3.28. They satisfy the following relations
   \[ f_{ij}(ab) = \sum_k f_{ik}(a)f_{kj}(b) \]  
   (3.30)
   for any \( i, j \in I \), \( a, b \in A_\alpha \). Moreover
   \[ f_{ij}(1_\alpha) = \delta_{ij} \]  
   (3.31)

**28 Remark.** Any functional \( f_{ij} \in A' = \bigoplus_{\alpha \in \pi} A'_\alpha \) is of the form \( f_{ij} = \sum_\alpha f_{ij}^\alpha \) where
\[ f_{ij}^\alpha(a) = 0 \quad \text{if} \ a \notin A_\alpha \]
Proof. To prove 1: For any $\alpha \in \pi$ let $\rho \in \Gamma_\alpha$. Using equation 3.13 we have that $\rho = \sum_i a_i \omega_i$, with $\omega_i \in \text{inv} \Gamma_\alpha$. To prove uniqueness assume that $\rho = \sum_i a_i \omega_i$. Then, using equations 3.1 and 3.21

$$\Delta_{\alpha,1}^{\ell}(\rho) = \sum_i a_{i(1,\alpha)} \otimes a_{i(2,1)} \xi_i$$

Applying $(\text{id} \otimes P_1)$ to both sides of the above equation, we get

$$(\text{id} \otimes P_1)\Delta_{\alpha,1}^{\ell}(\rho) = \sum_i a_i \otimes \xi_i$$

since $P_1(\xi_i) = \xi_i$ for any $\xi_i \in \text{inv} \Gamma_1$. Since $\omega_i$s, $i \in I$ are linearly independent, then by linearity of $\Delta_{h,1}$, $\xi_i$s are also linearly independent and so the coefficients $a_i$s are uniquely determined, and this proves the uniqueness of the decomposition 3.26. To prove (3): For any $\alpha \in \pi$, let $b \in A_\alpha$, $\omega_j \in \text{inv} \Gamma_\alpha$, $j \in I$. Then $\omega_j b$ admits a decomposition in the form 3.26. Let $F_{ji}^\alpha(b)$ be the coefficients preceding $\omega_i$ in the decomposition 3.26 i.e.

$$\omega_j b = \sum_i F_{ji}^\alpha(b) \omega_i \quad (3.32)$$

Clearly, $F_{ji}^\alpha(b)$ are linear mappings acting on $A_\alpha$. For any $a, b \in A_\alpha$, and any $j \in I$ we have

$$\sum_i F_{ji}^\alpha(ab) \omega_i = \omega_j ab$$

$$= \sum_{h,i} F_{jh}^\alpha(a) F_{hi}^\alpha(b) \omega_i$$

using the uniqueness of the decomposition 3.26 we have

$$F_{ji}^\alpha(ab) = \sum_h F_{jh}^\alpha(a) F_{hi}^\alpha(b) \quad (3.33)$$

for all $i, j \in I, \alpha \in \pi, a, b \in A_\alpha$. Let $f_{ji}^\alpha$ be linear functionals defined on $A_\alpha$ introduced by the formula

$$f_{ji}^\alpha(a) = E_\alpha(F_{ji}^\alpha(a)) = \varepsilon(\Psi_\alpha(F_{ji}^\alpha(a))) \quad (3.34)$$

Define $f_{ji} \in A'$ by

$$f_{ji} = \sum_{\alpha \in \pi} f_{ji}^\alpha$$
where for any $\beta \in \pi, a \in A_\beta$

$$f_{ji}(a) = \sum_{\alpha \in \pi} f_{ji}^\alpha(a) = f_{ji}^\beta(a) \tag{3.35}$$

Applying $E_\alpha$ to both sides of equation 3.33 and using equations 3.34 and 3.35 we have

$$f_{ji}(ab) = \sum_h f_{jh}(a)f_{hi}(b)$$

for any $a, b \in A_\alpha$, and hence equation 3.30 is proven. From equation 3.30 we get

$$f_{ji}m_\alpha(a \otimes b) = \sum_h (f_{jh} \otimes f_{hi})(a \otimes b)$$

i.e.

$$f_{ji}m_\alpha = \sum_h (f_{jh} \otimes f_{hi}) \tag{3.36}$$

Inserting $b = 1_\alpha$ in equation 3.32 we get

$$\omega_j = \sum_i F_{ji}^\alpha(1_\alpha)\omega_i$$

i.e.

$$F_{ji}^\alpha(1_\alpha) = \delta_{ji}1_\alpha$$

Applying $E_\alpha$ to both sides of the above equation, and summing over $\alpha$ we get

$$f_{ji}(1_\alpha) = \delta_{ji}$$

and hence 3.31 is proven. To prove 3.28 Recall that from equation 3.32 for any $\alpha \in \pi, \omega_j \in_{inv} \Gamma_\alpha, b \in A_\alpha$

$$\omega_j b = \sum_i F_{ji}^\alpha(b)\omega_i$$

Applying $\Delta^l_{\alpha,1}$ to both sides of the above equation we obtain

$$\Delta^l_{\alpha,1}(\omega_j b) = \Delta^l_{\alpha,1}(\sum_i F_{ji}^\alpha(b)\omega_i)$$

$$(1_\alpha \otimes \xi_j)\Delta_{\alpha,1}(b) = \sum_i \Delta_{\alpha,1}(F_{ji}^\alpha(b))(1_\alpha \otimes \xi_i)$$

where $\xi_j, \xi_i \in_{inv} \Gamma_1, i, j \in I$. On the other hand using equation 3.32

$$(1_\alpha \otimes \xi_j)\Delta_{\alpha,1}(b) = \sum_i (id \otimes F_{ji}^1)\Delta_{\alpha,1}(b)(1_\alpha \otimes \xi_i)$$
Comparing the last two equations we get

\[ \Delta_{\alpha,1}(F_{ji}^n(b)) = (\text{id} \otimes F_{ji}^1)\Delta_{\alpha,1}(b) \]

Applying \((\text{id} \otimes \varepsilon)\) to both sides of the above equation, using equation 3.35 we get

\[ F_{ji}^n(b) = (\text{id} \otimes f_{ji})\Delta_{\alpha,1}(b) = f_{ji} \ast b \]

Inserting this result into equation 3.32 we obtain equation 3.28. In order to prove equation 3.29 we have to show that

\[ \sum_j (f_{ji} \ast f_{kj} \circ S_1^{-1}) = \delta_{ih} \varepsilon \quad \text{(3.37)} \]

Let \(a \in A_1\). Then

\[ \sum_j (f_{ji} \ast f_{kj} \circ S_1^{-1})(S_1(a)) = \sum_j f_{hi}(\varepsilon(a)1_1) = \delta_{hi}\varepsilon(S_1(a)) \]

i.e.

\[ \sum_j f_{ji} \ast (f_{kj} \circ S_1^{-1}) = \delta_{ih} \varepsilon \quad \text{(3.38)} \]

Similarly, one can check that

\[ \sum_j (f_{jh} \circ S_1^{-1}) \ast f_{ij} = \delta_{hi} \varepsilon \quad \text{(3.38)} \]

From equation 3.28 we have that for any \(\alpha \in \pi\), \(b \in A_\alpha, \omega_j \in_{\text{inv}} \Gamma_\alpha\)

\[ \omega_j b = \sum_h (f_{jh} \ast b)\omega_h \]

Inserting in this equation \(b = (f_{jh} \circ S_1^{-1}) \ast a\) for some \(a \in A_\alpha\) and summing over \(j\) we obtain

\[ \sum_j \omega_j(f_{jh} \circ S_1^{-1}) \ast a = \sum_{j,h}((f_{jh} \ast (f_{jh} \circ S_1^{-1}) \ast a)\omega_h = a \omega_i \]

Recall that \(\varepsilon \ast a = (\text{id} \otimes \varepsilon)\Delta_{\alpha,1}(a) = a\), and hence equation 3.29 follows.
To prove (2): For any $\alpha \in \pi, \rho \in \Gamma_{\alpha}$, we have from statement 1 and formula 3.29 that

$$\rho = \sum_i a_i \omega_i, \quad a_i \in A_{\alpha}, \quad \omega_i \in \text{inv } \Gamma_{\alpha}, \quad i \in I$$

$$= \sum_j \omega_j b_j,$$

where

$$b_j = \sum_i (f_{ij} \circ S^{-1}_1) * a_i \in A_{\alpha}, \quad \forall j \in I.$$  

For uniqueness:

Assume that for some $b_i$ ($i \in I$ only finite number of $b_i$s are different from zero) we have:

$$\sum_i \omega_i b_i = 0$$

We have to show that all $b_i$'s $= 0 (i \in I)$. Using the uniqueness of decomposition 3.26 we have

$$\sum_i \omega_i b_i = 0$$

Then

$$\sum_{i,j} (f_{i,j} * b_i) \omega_j = 0$$

$$\sum_i (f_{i,j} * b_i) = 0 \quad \forall j \in I$$

Computing the convolution product with $f_{j,h} \circ S^{-1}_1$ summing over $j$ and using equation 3.38

$$0 = \sum_{i,j} ((f_{j,h} \circ S^{-1}_1) * f_{i,j}) * b_i$$

i.e. $b_i = 0$ for each $i \in I$.  

Theorem 27 gives the complete description of left covariant $\pi$–graded bimodules. Using equations 3.28 and 3.1 we have

$$(\sum_i a_i \omega_i) b = \sum_i a_i (\omega_i b) = \sum_i a_i (f_{ij} * b) \omega_j \quad (3.39)$$
\[ \Delta^l_{\alpha,\beta}(\sum_i a_i \omega_i) = \sum_i \Delta_{\alpha,\beta}(a_i) \Delta^l_{\alpha,\beta}(\omega_i) = \sum_i \Delta_{\alpha,\beta}(a_i)(1_\alpha \otimes \xi_i), \xi_i \in \text{inv} \Gamma_{\beta} \]  

(3.40)

If \((f_{ij})_{i,j \in I}\) is a family of linear functionals in \(A' = \bigoplus_{\alpha \in \pi} A'_\alpha\) satisfying relations 3.30, 3.31, then considering the left module \(\Gamma = \{\Gamma_{\alpha}\}_{\alpha \in \pi}\) generated by \(\omega^\alpha_i, \alpha \in \pi, i \in I\), and using the above formulae to introduce the right multiplication by elements of \(A\), and the left action of \(A\) we obtain a left covariant \(\pi\)-graded bimodule.

**29 Definition.** Let \((\Gamma, \Delta^r)\) be a right covariant \(\pi\)-graded bimodule over \(A\). An element \(\eta \in \Gamma_{\alpha}\) is said to be right invariant if

\[ \Delta^r_{\alpha,1}(\eta) = \eta \otimes 1_1 \]  

(3.41)

Denote by \(\Gamma_{\text{inv}} = \{\Gamma^\alpha_{\text{inv}}\}\) the set of all left invariant elements of \(\Gamma\). Clearly, \(\Gamma^\alpha_{\text{inv}}\) is a linear subspace of \(\Gamma_{\alpha}\) for each \(\alpha \in \pi\).

**30 Theorem.** Let \(\Gamma = (\{\Gamma^\alpha\}_{\alpha \in \pi}, \Delta^r)\) be a right covariant \(\pi\)-graded bimodule over \(A\), \(\{\eta^\alpha_i\}_{\alpha \in \pi}\) be a basis of \(\Gamma^\alpha_{\text{inv}}\) of all right invariant elements of \(\Gamma_{\alpha}\) for each \(\alpha \in \pi\). Then

(1) For any \(\alpha \in \pi\), any element \(\varrho \in \Gamma_{\alpha}\) is of the form

\[ \varrho = \sum_i a_i \eta_i \]  

(3.42)

where \(a_i\)'s \in \(A_{\alpha}\) are uniquely determined, \(\eta_i\)'s \in \(\Gamma^\alpha_{\text{inv}}\) for any \(\alpha \in \pi\).

(2) For any \(\alpha \in \pi\), any element \(\varphi \in \Gamma_{\alpha}\) is of the form

\[ \varphi = \sum_i \eta_i b_i \]  

(3.43)

where \(b_i\)'s \in \(A_{\alpha}\) are uniquely determined, \(\eta_i\)'s \in \(\Gamma^\alpha_{\text{inv}}\), for any \(\alpha \in \pi\).

(3) There exists linear functionals \(g_{ij} \in A' = \bigoplus_{\alpha \in \pi} A'_\alpha\) such that for any \(\alpha \in \pi\)

\[ \eta_i b = \sum_j (b \ast g_{ij}) \eta_j \]  

(3.44)

\[ a \eta_i = \sum_j \eta_j (a \ast (g_{ij} \circ S^{-1}_i)) \]  

(3.45)
where $a, b \in A_\alpha, \eta_i, s, \eta_j, s \in \Gamma^\alpha_{\text{inv}}$. These functionals are uniquely determined by equation 3.44. They satisfy the following relations

$$g_{ij}(ab) = \sum_k g_{ik}(a)g_{kj}(b)$$

(3.46)

for any $i, j \in I$, $a, b \in A_\alpha$. Moreover

$$g_{ij}(1_\alpha) = \delta_{ij}$$

(3.47)

The proof is similar to that of theorem 27.

31 Remark. Any functional $g_{ij} \in A' = \bigoplus_{\alpha \in \pi} A'_\alpha$ is of the form $g_{ij} = \sum_{\alpha} g_{ij}^\alpha$ where

$$g_{ij}^\alpha(a) = 0 \quad \text{if } a \notin A_\alpha$$

32 Theorem. Let $\Gamma = \{\Gamma_\alpha\}_{\alpha \in \pi}, \Delta^l, \Delta^r$ be a $\pi$–graded bicovariant bimodule over $A, \{(\omega^\alpha_i)_{i \in I}\}_{\alpha \in \pi}$ be a basis of $\text{inv} \Gamma = \{\text{inv} \Gamma_\alpha\}_{\alpha \in \pi}$ of all left invariant elements of $\Gamma$. Then

1. For any $i \in I, \alpha, \beta \in \pi, \omega^\alpha_i \in \Gamma_{\alpha\beta}$

$$\Delta^l_{\alpha,\beta}(\omega_i^\alpha) = \sum_j \omega_j^\alpha \otimes R_{ji}$$

(3.48)

where $i, j \in \pi, R_{ji} \in A_{\beta}$ satisfy the following relation

$$\Delta_{\alpha,\beta}(R_{ji}) = \sum_h R_{jh} \otimes R_{hi}$$

(3.49)

and for $R_{ji} \in A_1$

$$\varepsilon(R_{ji}) = \delta_{ji}$$

(3.50)

2. For each $\alpha \in \pi$ there exists a basis $(\eta_i)_{i \in I}$ of all right invariant elements of $\Gamma_\alpha$ such that for $\omega_i \in \Gamma_\alpha$

$$\omega_i = \sum_j \eta_j R_{ji} \quad \forall i \in I$$

(3.51)

3. For any $j, h \in I, a \in A_\alpha$

$$R_{ij}(a * f_{jh}) = (g_{ji} * a)R_{hi}, \quad i, j \in I$$

(3.52)

where $f_{ij}, g_{ij}$ are functionals introduced in theorems 27, 30.
**Proof.** Using equation 3.9 for any $\alpha, \beta, \gamma \in \pi$ we have

$$(\Delta^l_{\alpha,\beta} \otimes \text{id}) \Delta^r_{\alpha,\beta,\gamma} = (\text{id} \otimes \Delta^r_{\beta,\gamma}) \Delta^l_{\alpha,\beta,\gamma}$$

Let $\omega^\alpha_{\beta\gamma} \in \Gamma_{\alpha\beta\gamma}$

$$(\Delta^l_{\alpha,\beta} \otimes \text{id}) \Delta^r_{\alpha,\beta,\gamma}(\omega^\alpha_{\beta\gamma}) = (\text{id} \otimes \Delta^r_{\beta,\gamma}) \Delta^l_{\alpha,\beta,\gamma}(\omega^\alpha_{\beta\gamma}) = 1_{\alpha} \otimes \Delta^r_{\beta,\gamma}(\omega^\beta\gamma_i)$$

i.e.

$$\Delta^r_{\alpha,\beta,\gamma}(\omega^\alpha_{\beta\gamma}) \in \text{inv} \Gamma_{\alpha\beta} \otimes A_{\gamma}$$

Then for $\omega^\alpha_{\beta\gamma} \in \Gamma_{\alpha\beta\gamma}$

$$\Delta^r_{\alpha,\beta,\gamma}(\omega^\alpha_{\beta\gamma}) = \sum_j \omega^\alpha_j \otimes R_{ji}$$

Applying $(\text{id} \otimes \Delta_{\beta,\gamma})$ to both sides of the above equation

$$\sum_j \omega^\alpha_j \otimes \Delta_{\beta,\gamma}(R_{ji}) = (\Delta^r_{\alpha,\beta} \otimes \text{id}) \Delta^r_{\alpha,\beta,\gamma}(\omega^\alpha_{\beta\gamma})$$

$$= \sum_{j,h} \omega^\alpha_j \otimes R_{jh} \otimes R_{hi}$$

Comparing both sides of the above equation

$$\Delta_{\beta,\gamma}(R_{ji}) = \sum_h R_{jh} \otimes R_{hi}$$

and hence equation 3.49 is proven. Let $\omega^\alpha_i \in \Gamma_{\alpha}$

$$\Delta_{\alpha,1}(\omega^\alpha_i) = \sum_j \omega^\alpha_j \otimes R_{ji}, R_{ji} \in A_1$$

Applying $(\text{id} \otimes \varepsilon)$ to both sides of the above equation

$$(\text{id} \otimes \varepsilon) \Delta_{\alpha,1}(\omega^\alpha_i) = \omega^\alpha_i$$

$$= \sum_j \omega^\alpha_j \otimes \varepsilon(R_{ji})$$

$$\implies$$

$$\varepsilon(R_{ji}) = \delta_{ji}$$

To prove statement 2: First we have that for $R_{ij} \in A_1, \alpha \in \pi$

$$m_{\alpha}(\text{id} \otimes S_{\alpha-1}) \Delta_{\alpha,\alpha-1} = m_{\alpha}(S_{\alpha-1} \otimes \text{id}) \Delta_{\alpha-1,\alpha} = \varepsilon_{1_{\alpha}}$$
By using equations 3.49 and 3.50 we obtain
\[ \sum_h S_{\alpha^{-1}}(R_{ih})R_{hj} = \delta_{ij}1_\alpha \] (3.53)
\[ \sum_h R_{ih}S_{\alpha^{-1}}(R_{hj}) = \delta_{ij}1_\alpha \] (3.54)

For any \( \alpha \in \pi, j \in I \), let
\[ \eta_j = \sum_i \omega_i S_{\alpha^{-1}}(R_{ij}) \] (3.55)

Multiplying both sides of equation 3.55 by \( R_{ji} \) and summing over \( j \) then using equation 3.53 we obtain
\[ \sum_j \eta_j R_{ji} = \sum_{i,j} \omega_i S_{\alpha^{-1}}(R_{ij})R_{ji} \]
\[ = \omega_i \]

and equation 3.51 follows. It remains to show that \( \eta_j \) defined in equation 3.55 is right invariant.

Let \( \eta_j \in \Gamma_\alpha, \eta_j = \sum_i \omega_i S_{\alpha^{-1}}(R_{ij}), \omega_i \in \text{inv} \Gamma_\alpha, R_{ij} \in A_{\alpha^{-1}}. \)
\[ \Delta_{\alpha,1}(\eta_j) = \sum_i \Delta_{\alpha,1}(\omega_i) \Delta_{\alpha,1}(S_{\alpha^{-1}}(R_{ij})) \]
\[ = \eta_j \otimes 1_1 \]

For any \( \alpha \in \pi \), let \( \eta \in \Gamma_\alpha \) be a right invariant element. According to theorem 30
\[ \eta = \sum_i \omega_i c_i, \quad c_i \in A_\alpha \]
\[ = \sum_{i,j} \eta_j R_{ji} c_i, \quad R_{ji} \in A_\alpha \]
then
\[ \eta = \sum_j \eta_j b_j, \quad b_j \in A_\alpha \] (3.56)

If \( \eta = \sum_i \omega_i S_{\alpha^{-1}}(R_{ij}) \), then using equation 3.55 we have
\[ \sum_{i,j} \omega_i S_{\alpha^{-1}}(R_{ij})b_j = 0 \]
Using theorem 27 we have
\[ \sum_j S_{\alpha^{-1}}(R_{ij})b_j = 0 \quad \text{for each } i \in I \]

Multiplying both sides of the above equation by \( R_{ji} \) we obtain
\[ b_j = 0 \]
for any \( j \in I \).

This means that the decomposition 3.56 is unique.

Applying \( \Delta_{1,\alpha}^r \) to both sides of decomposition 3.56
\[ \Delta_{1,\alpha}^r(\eta) = \Delta_{1,\alpha}^r(\sum_j \eta_j b_j) \]

\[ \xi^1 \otimes 1_\alpha = \sum_j (\xi_j^1 \otimes 1_\alpha) \Delta_{1,\alpha}(b_j) \]

Comparing this formula with decomposition 3.56 we get that
\[ \Delta_{1,\alpha}(b_j) = b_{j(1,1)} \otimes 1_\alpha \]

Applying \( \varepsilon \otimes \text{id} \) we get that \( b_j = \varepsilon(b_{j(1,1)})1_\alpha \). This way we proved that for any \( \alpha \in \pi \), any \( \eta \in \Gamma_{\text{inv}}^\alpha \) is unique linear combination of \( \eta_j(j \in I) \). Therefore, \( (\eta_j)_{j \in I} \) is a basis in \( \Gamma_{\text{inv}}^\alpha \) and statement 2 is proven.

To prove statement (3):
Using equation 3.45 we have for any \( \alpha \in \pi, a \in A_\alpha, \eta_j \in \Gamma_{\text{inv}}^\alpha \)
\[ an_j = \sum_i \eta_i (a \ast (g_{ji} \circ S_1^{-1})) \]

Using equation 3.55 we get
\[ \sum_i a_{\omega_i} S_{\alpha^{-1}}(R_{ij}) = \sum_{i,h} \omega_h S_{\alpha^{-1}}(R_{jh})(a \ast (g_{ji} \circ S_1^{-1})) \]

Using equation 3.31 we get
\[ \sum_{i,h} \omega_h ((f_{ih} \circ S_1^{-1}) \ast a) S_{\alpha^{-1}}(R_{ij}) = \sum_{i,h} \omega_h S_{\alpha^{-1}}(R_{jh})(a \ast (g_{ji} \circ S_1^{-1})) \]

Using theorem 27 we get
\[ \sum_i ((f_{ih} \circ S_1^{-1}) \ast a) S_{\alpha^{-1}}(R_{ij}) = \sum_i S_{\alpha^{-1}}(R_{jh})(a \ast (g_{ji} \circ S_1^{-1})) \]
Applying $S_\alpha(S_\alpha = S_{\alpha-1}^{-1})$ to both sides of this equation, using that $S_\alpha$ is antimultiplicative we get:

$$\sum_i R_{ij} S_\alpha((f_{ih} \circ S_1^{-1}) \ast a) = \sum_i S_\alpha(a \ast (g_{ji} \circ S_1^{-1})) R_{hi}$$ \hspace{1cm} (3.57)

We compute

$$S_\alpha((f_{ih} \circ S_1^{-1}) \ast a) = (\text{id} \otimes f_{ih})(S_{\alpha^{-1}} \otimes S_1^{-1}) \Delta_{\alpha,1}(a) = S_{\alpha^{-1}}(a) \ast f_{ih}$$

Similarly, we have

$$S_\alpha(a \ast (g_{ji} \circ S_1^{-1})) = g_{ji} \ast S_{\alpha^{-1}}^{-1}(a)$$

i.e.

$$\sum_i R_{ij}(S_{\alpha^{-1}}^{-1}(a) \ast f_{ih}) = \sum_i (g_{ji} \ast S_{\alpha^{-1}}^{-1}(a)) R_{hi}$$

Replacing $a$ by $S_{\alpha^{-1}}^{-1}(a)$ we obtain

$$\sum_i R_{ij}(a \ast f_{ih}) = \sum_i (g_{ji} \ast a) R_{hi}$$

And equation 3.52 follows. Note that if $a, R_{ij}, R_{hi} \in A_1$, then applying $\varepsilon$ to both sides of equation 3.52 and using equation 3.50 we obtain:

$$\varepsilon(\sum_i R_{ij}(a \ast f_{ih})) = \varepsilon(\sum_i (g_{ji} \ast a) R_{hi})$$

$$\sum_i \varepsilon(R_{ij}) \varepsilon(a \ast f_{ih}) = \sum_i \varepsilon(g_{ji} \ast a) \varepsilon(R_{hi})$$

$$\sum_i \delta_{ij} \varepsilon(a \ast f_{ih}) = \sum_i \varepsilon(g_{ji} \ast a) \delta_{hi}$$

But

$$\varepsilon(a \ast f_{ih}) = f_{ih}(a)$$

Similarly: $\varepsilon(g_{ji} \ast a) = g_{ji}(a)$ i.e. $f_{ij}(a) = g_{ij}(a)$, for any $a \in A_1$. From which we get that

$$\sum_i R_{ij}(a \ast f_{ih}) = \sum_i (f_{ji} \ast a) R_{hi}$$ \hspace{1cm} (3.58)
For any $\alpha, \beta \in \pi, \eta_j \in \Gamma_{\text{inv}}$, applying $\Delta^l_{\alpha,\beta}$ to both sides of equation 3.55 we obtain:

$$\Delta^l_{\alpha,\beta}(\eta_j) = \sum_h \Delta^l_{\alpha,\beta}(\omega_h)\Delta_{\alpha,\beta}(S(\alpha\beta)^{-1}(R_{hj}))$$

$$= \sum_{i,h} (S_{\alpha^{-1}}(R_{ij}) \otimes \omega_h^\beta S_{\beta^{-1}}(R_{hi}))$$

i.e.

$$\Delta^l_{\alpha,\beta}(\eta_j) = \sum_i S_{\alpha^{-1}}(R_{ij}) \otimes \eta_i^\beta \quad (3.59)$$

Using equations 3.5 and 3.48

$$\Delta^r_{\alpha,\beta}(\sum_i a_i \omega_i) = \sum_i \Delta_{\alpha,\beta}(a_i)(\omega_i^\alpha \otimes R_{ij}) \quad (3.60)$$

33 Theorem. Let $(f_{ij})_{i,j \in I}$ be the family of functionals defined on $A$ satisfying relations 3.30, 3.31, $(R^\alpha_{ij})_{i,j \in I}$ be a family of elements of $A = \{A_\alpha\}_{\alpha \in \pi}$ satisfying relations 3.49, 3.50, 3.58 for each $\alpha \in \pi$. Consider the left module $\Gamma = \{\Gamma_\alpha\}_{\alpha \in \pi}$ over $A = \{A_\alpha\}_{\alpha \in \pi}$ generated by $\omega_i^\alpha$, $i \in I$, $\alpha \in \pi$ for each $\alpha \in \pi$, and using formulae 3.39, 3.40, 3.60 to introduce right multiplication by elements of $A$, left and right actions of $A$ on $\Gamma$ then $\Gamma = \{(\Gamma_\alpha)_{\alpha \in \pi}, \Delta^l, \Delta^r\}$ is a $\pi$-graded bicovariant bimodule over $A$.

Proof. Using formula 3.39 to introduce right multiplication by elements of $A$, one can easily check that $\Gamma$ is also a $\pi$-graded right module over $A$, i.e. $\Gamma = \{\Gamma_\alpha\}_{\alpha \in \pi}$ is a $\pi$-graded bimodule over $A$.

Using 3.40 to define a left action of $A$ on $\Gamma$, taking into consideration theorem 27 we find that equations 3.1 and 3.2 are satisfied. For $\rho \in \Gamma_{\alpha,\beta}, b \in A_{\alpha,\beta}$ and $\alpha, \beta \in \pi$, using theorem 27 so $\rho = \sum a_i \omega_i^{\alpha,\beta}, a_i \in A_{\alpha,\beta}, \omega_i^{\alpha,\beta} \in \text{inv} \Gamma_{\alpha,\beta}$

$$\Delta^l_{\alpha,\beta}(bp) = \Delta^l_{\alpha,\beta}(\sum_i (ba_i)\omega_i^{\alpha,\beta})$$

$$= \Delta_{\alpha,\beta}(b) \sum_i \Delta_{\alpha,\beta}(a_i) \Delta^l_{\alpha,\beta}(\omega_i^{\alpha,\beta})$$

$$= \Delta_{\alpha,\beta}(b) \Delta^l_{\alpha,\beta}(\rho)$$

Similarly,

$$\Delta^l_{\alpha,\beta}(b\rho) = \Delta^l_{\alpha,\beta}(\rho) \Delta_{\alpha,\beta}(b)$$
Moreover, using theorem 27 for any $\alpha, \beta, \gamma \in \pi$, $\rho \in \Gamma_{\alpha\beta\gamma}$ and $\rho = \sum_i a_i \omega_i$, where $a_i \in A_{\alpha\beta\gamma}$, $\omega_i \in \text{inv} \Gamma_{\alpha\beta\gamma}$ we have:

$$(\Delta_{\alpha,\beta} \otimes \text{id})\Delta^l_{\alpha\beta,\gamma}(\rho) = \sum_i (\Delta_{\alpha,\beta} \otimes \text{id})\Delta_{\alpha\beta,\gamma}(a_i) \Delta^l_{\alpha\beta,\gamma}(\omega_i^{\alpha\beta\gamma})$$

$$= \sum_i ((\Delta_{\alpha,\beta} \otimes \text{id})\Delta_{\alpha\beta,\gamma}(a_i))(1_{\alpha} \otimes 1_{\beta} \otimes \omega_i^{\alpha\beta\gamma})$$

Similarly, we have

$$(\text{id} \otimes \Delta^l_{\beta,\gamma})\Delta^l_{\alpha\beta,\gamma}(\rho) = \sum_i ((\Delta_{\alpha,\beta} \otimes \text{id})\Delta_{\alpha\beta,\gamma}(a_i))(1_{\alpha} \otimes 1_{\beta} \otimes \omega_i^{\alpha\beta\gamma})$$

i.e.

$$(\Delta_{\alpha,\beta} \otimes \text{id})\Delta^l_{\alpha\beta,\gamma} = (\text{id} \otimes \Delta^l_{\beta,\gamma})\Delta^l_{\alpha\beta,\gamma}$$

which means that equation 3.3 of definition 10 is satisfied.

Finally, for any $\alpha \in \pi$, letting $\rho \in \Gamma_{\alpha}$, then using 3.26 $\rho = \sum_i a_i \omega_i^\alpha$, $a_i \in A_{\alpha}$, $\omega_i \in \text{inv} \Gamma_\alpha$

$$(\varepsilon \otimes \text{id})\Delta^l_{1,\alpha}(\rho) = \sum_i (\varepsilon \otimes \text{id})\Delta_{1,\alpha}(a_i) \Delta^l_{1,\alpha}(\omega_i)$$

$$= \sum_i (\varepsilon \otimes \text{id})(\Delta_{1,\alpha}(a_i))(\varepsilon \otimes \text{id})(\Delta^l_{1,\alpha}(\omega_i))$$

$$= \rho$$

i.e. 3.4 is satisfied.

This means that $\Gamma = \{ \Gamma_\alpha \}_{\alpha \in \pi}, \Delta^l$ is a $\pi$–graded left covariant bimodule over $A$.

Using formula 3.60 to introduce right action of $A$ on $\Gamma$ one can easily check that $\Gamma = \{ \Gamma_\alpha \}_{\alpha \in \pi}, \Delta^r$ is a $\pi$–graded right covariant bimodule over $A$, for let $\rho \in \Gamma_{\alpha\beta}, b \in A_{\alpha\beta}, \alpha, \beta \in \pi$, using theorem 27, for $\rho = \sum_i a_i \omega_i$, $a_i \in A_{\alpha\beta}, \omega_i \in \text{inv} \Gamma_{\alpha\beta}$

$$\Delta^r_{\alpha,\beta}(b \rho) = \sum_i \Delta_{\alpha,\beta}(ba_i) \Delta^r_{\alpha,\beta}(\omega_i)$$

$$= \Delta_{\alpha,\beta}(b) \sum_i \Delta_{\alpha,\beta}(a_i) \Delta^r_{\alpha,\beta}(\omega_i)$$

$$= \Delta_{\alpha,\beta}(b) \Delta^r_{\alpha,\beta}(\rho)$$

Similarly, for $\rho \in \Gamma_{\alpha\beta}, b \in A_{\alpha\beta}, \alpha, \beta \in \pi$, using theorem 27 $\rho = \sum_i a_i \omega_i$, $a_i \in A_{\alpha\beta}, \omega_i \in \text{inv} \Gamma_{\alpha\beta}$. Using 3.44 we get

$$\Delta^r_{\alpha,\beta}(\rho b) = \Delta^r_{\alpha,\beta}(\rho) \Delta_{\alpha,\beta}(b)$$
and thus equations 3.5 and 3.6 are satisfied. Moreover, using theorem 27 for any $\alpha, \beta, \gamma \in \pi, \rho \in \Gamma_{\alpha \beta \gamma}$, $\rho = \sum_i a_i \omega_i^{\alpha \beta \gamma}$ where $a_i \in A_{\alpha \beta \gamma}$, $\omega_i^{\alpha \beta \gamma} \in \Inv \Gamma_{\alpha \beta \gamma}$ we have:

$$(\Delta'_{\alpha, \beta} \otimes \id) \Delta_{\alpha \beta, \gamma}(\rho) = \sum_i (\Delta'_{\alpha, \beta} \otimes \id)(\Delta_{\alpha \beta, \gamma}(a_i) \Delta'_{\alpha \beta, \gamma}(\omega_i))$$

$$= \sum_{i,j,k} (\Delta'_{\alpha, \beta} \otimes \id)(\Delta_{\alpha \beta, \gamma}(a_i))(\omega_k^\alpha \otimes R_{kj} \otimes R_{ji})$$

Similarly,

$$(\id \otimes \Delta'_{\beta, \gamma}) \Delta_{\alpha \beta, \gamma}(\rho) = \sum_{i,j,k} (\id \otimes \Delta'_{\beta, \gamma})(\Delta_{\alpha \beta, \gamma}(a_i)) (\omega_k^\beta \otimes R_{kj} \otimes R_{ji})$$

i.e. $\Delta'_{\alpha, \beta} \otimes \id) \Delta'_{\alpha \beta, \gamma} = (\id \otimes \Delta'_{\beta, \gamma}) \Delta'_{\alpha \beta, \gamma}$ which means that equation 3.7 is satisfied. Finally, for any $\alpha \in \pi$, letting $\rho \in \Gamma_{\alpha}$, then using 3.26 $\rho \in \Gamma_{\alpha}$, \rho = \sum_i a_i \omega_i$ where $a_i \in A_{\alpha}$, $\omega_i \in \Inv \Gamma_{\alpha}$ we compute

$$(\id \otimes \epsilon) \Delta'_{\alpha, 1}(\rho) = \sum_i a_i \omega_i$$

$$= \rho$$

i.e. 3.8 is satisfied. This means that $\Gamma = \{ \{ \Gamma_{\alpha} \}_{\alpha \in \pi}, \Delta' \}$ is a $\pi$–graded right covariant bimodule over $A$. To prove the bicovariance conditions , for any $\alpha, \beta, \gamma \in \pi, \rho \in \Gamma_{\alpha \beta \gamma}$, using theorem 27, for $\rho = \sum_i a_i \omega_i^{\alpha \beta \gamma}$, $a_i \in A_{\alpha \beta \gamma}$, $\omega_i^{\alpha \beta \gamma} \in \Inv \Gamma_{\alpha \beta \gamma}$ we compute

$$(\id \otimes \Delta'_{\beta, \gamma}) \Delta^l_{\alpha \beta, \gamma}(\rho) = \sum_i (\id \otimes \Delta'_{\beta, \gamma})(\Delta_{\alpha \beta, \gamma}(a_i) \Delta^l_{\alpha \beta, \gamma}(\omega_i^{\alpha \beta \gamma}))$$

$$= \sum_{i,j} (\id \otimes \Delta'_{\beta, \gamma})(\Delta_{\alpha \beta, \gamma}(a_i))(1_{\alpha} \otimes \omega_j^\beta \otimes R_{ji})$$

Similarly,

$$(\Delta^l_{\alpha, \beta} \otimes \id) \Delta^r_{\alpha \beta, \gamma}(\rho) = \sum_{i,j} (\Delta^l_{\alpha, \beta} \otimes \id)(\Delta_{\alpha \beta, \gamma}(a_i))(1_{\alpha} \otimes \omega_j^\beta \otimes R_{ji})$$

and hence equation 3.9 is proved and $\Gamma = \{ \{ \Gamma_{\alpha} \}_{\alpha \in \pi}, \Delta^l, \Delta^r \}$ is a $\pi$–graded bicovariant bimodule over $A$.

**QED**

4 First order differential calculus on Hopf Group Coalgebras

Let $A^2 = \{ A^2_{\alpha} \}_{\alpha \in \pi}$ be the $\pi$–graded bimodule introduced in section 2. We introduce left and right actions of $A$ on $A^2$. For any $\alpha, \beta \in \pi$ let $q \in
Differential calculus on Hopf Group Coalgebra

\[ A_{\alpha\beta} \otimes A_{\alpha\beta} \text{ and } (\Delta_{\alpha\beta} \otimes \Delta_{\alpha\beta})(q) = \sum_k a_k \otimes b_k \otimes c_k \otimes d_k, \text{ where } a_k, c_k \in A_{\alpha}, b_k, d_k \in A_{\beta}, \ k = 1, 2, \ldots, n. \text{ We set} \]

\[ \Phi_{\alpha,\beta}^l(q) = \sum_k a_k c_k \otimes b_k \otimes d_k \tag{4.1} \]

\[ \Phi_{\alpha,\beta}^r(q) = \sum_k a_k \otimes c_k \otimes b_k d_k \tag{4.2} \]

We compute

\[ (\text{id} \otimes m_{\beta})(\Phi_{\alpha,\beta}^l(q)) = \Delta_{\alpha,\beta}(m_{\alpha\beta}(q)) = 0 \]

Similarly we have

\[ (m_{\alpha} \otimes \text{id})(\Phi_{\alpha,\beta}^r(q)) = 0 \]

Therefore,

\[ \Phi_{\alpha,\beta}^l : A^2_{\alpha\beta} \to A_\alpha \otimes A^2_{\beta} \tag{4.3} \]

and

\[ \Phi_{\alpha,\beta}^r : A^2_{\alpha\beta} \to A^2_\alpha \otimes A_{\beta} \tag{4.4} \]

Clearly, both are linear maps. We will show that \( A^2 = (\{ A^2_{\alpha} \}_{\alpha \in \pi}, \Phi^l, \Phi^r) \) is a \( \pi \)-graded bicovariant bimodule over \( A \).

First, we will prove that \( A^2 = (\{ A^2_{\alpha} \}_{\alpha \in \pi}, \Phi^l, \Phi^r) \) is a \( \pi \)-graded left covariant bimodule over \( A \).

Let \( \alpha, \beta \in \pi, q \in A^2_{\alpha\beta}, q = b \otimes c \), then

\[ \Phi_{\alpha,\beta}^l(aq) = (a_{1(\alpha)} \otimes a_{2(\beta)}) \cdot (b_{1(\alpha)}c_{1(\alpha)} \otimes b_{2(\beta)} \otimes c_{2(\beta)}) = \Delta_{\alpha,\beta}(a) \cdot \Phi_{\alpha,\beta}^l(q) \]

Similarly

\[ \Phi_{\alpha,\beta}^l(qa) = \Phi_{\alpha,\beta}^l(q) \cdot \Delta_{\alpha,\beta}(a) \]

Moreover, for any \( \alpha, \beta, \gamma \in \pi, q \in A^2_{\alpha\beta\gamma}, q = a \otimes b \) we compute

\[ (\Delta_{\alpha,\beta} \otimes \text{id})\Phi_{\alpha,\beta,\gamma}^l(q) = \Delta_{\alpha,\beta}(a_{1(\alpha)} \otimes b_{1(\alpha)} \otimes a_{2(\beta)} \otimes b_{2(\beta)} \otimes a_{3(\gamma)} \otimes b_{3(\gamma)}) \]

Similarly

\[ (\text{id} \otimes \Phi_{\beta,\gamma}^l)\Phi_{\alpha,\beta,\gamma}^l(q) = a_{1(\alpha)}b_{1(\alpha)} \otimes a_{2(\beta)}b_{2(\beta)} \otimes a_{3(\gamma)} \otimes b_{3(\gamma)} \]
\[(\Delta_{\alpha,\beta} \otimes \text{id})\Phi^I_{\alpha,\beta,\gamma} = (\text{id} \otimes \Phi^I_{\beta,\gamma})\Phi^I_{\alpha,\beta,\gamma}\]

Finally, for any \(\alpha \in \pi\), \(q \in A^2_{\alpha}\), \(q = a \otimes b\)

\[(\varepsilon \otimes \text{id})\Phi^I_{1,\alpha}(q) = \varepsilon(a(1,1)b(1,1))a(2,\alpha) \otimes b(2,\alpha) = a \otimes b = q\]

and thus the conditions of definition 10 are fulfilled and \(A^2 = \{A^2_\alpha\}_{\alpha \in \pi}, \Phi^I\) is a \(\pi\)-graded left covariant bimodule over \(A\). Similarly, one can check that \(A^2 = \{A^2_\alpha\}_{\alpha \in \pi}, \Phi^r\) is a \(\pi\)-graded right covariant bimodule over \(A\). Finally, we check the bicovariance condition

For any \(\alpha, \beta, \gamma \in \pi\), \(q \in A^2_{\alpha,\beta,\gamma}\), \(q = a \otimes b\) we compute

\[(\text{id} \otimes \Phi^r_{\beta,\gamma})\Phi^I_{\alpha,\beta,\gamma}(q) = (\text{id} \otimes \Phi^r_{\beta,\gamma})(a(1,\alpha)b(1,\alpha) \otimes a(2,\beta) \otimes b(2,\beta)) = a(1,\alpha)b(1,\alpha) \otimes a(2,\beta) \otimes b(2,\beta) \otimes a(3,\gamma)b(3,\gamma)\]

Similarly,

\[(\Phi^I_{\alpha,\beta} \otimes \text{id})\Phi^r_{\alpha,\beta,\gamma}(q) = a(1,\alpha)b(1,\alpha) \otimes a(2,\beta) \otimes b(2,\beta) \otimes a(3,\gamma)b(3,\gamma)\]

which proves that \(A^2 = \{A^2_\alpha\}_{\alpha \in \pi}, \Phi^I, \Phi^r\) is a \(\pi\)-graded bicovariant bimodule over \(A\).

On \(A \otimes A = \{A_\alpha \otimes A_\alpha\}_{\alpha \in \pi}\) we define two families of linear mappings

\(r = \{r_\alpha : A_\alpha \otimes A_\alpha \to A_\alpha \otimes A_1\}_{\alpha \in \pi}\)

\(t = \{t_\alpha : A_\alpha \otimes A_\alpha \to A_1 \otimes A_\alpha\}_{\alpha \in \pi}\)

For any \(\alpha \in \pi, a, b \in A_\alpha\) we set

\[r_\alpha(a \otimes b) = (a \otimes 1_1)\Delta_{\alpha,1}(b) \quad (4.5)\]

\[t_\alpha(a \otimes b) = (1_1 \otimes a)\Delta_{1,\alpha}(b) \quad (4.6)\]

It is clear that \(r_\alpha, t_\alpha\) are bijections for each \(\alpha \in \pi\) for example for \(a \in A_\alpha, b \in A_1\) the inverse of \(r_\alpha\) is given by

\[r^{-1}_\alpha(a \otimes b) = (a \otimes 1_\alpha)(S_{\alpha-1} \otimes \text{id})\Delta_{\alpha-1,\alpha}(b) \quad (4.7)\]

Similarly, for \(a \in A_1, b \in A_\alpha\) the inverse of \(t_\alpha\) is given by

\[t^{-1}_\alpha(a \otimes b) = (b \otimes 1_\alpha)(S_{\alpha-1} \otimes \text{id})\sigma_{A_{\alpha-1},A_\alpha}\Delta_{\alpha,\alpha-1}(b) \quad (4.8)\]
One can easily show that for each $\alpha \in \pi$, $r_{\alpha}(A^2_{\alpha}) = A_{\alpha} \otimes \ker \varepsilon$, for let $\alpha \in \pi$, $a \in A_{\alpha}, b \in \ker \varepsilon$

$$m_{\alpha}r_{\alpha}^{-1}(a \otimes b) = aS_{\alpha^{-1}}(b_{(1,\alpha^{-1})})b_{(2,\alpha)} = 0$$

From which we get $r_{\alpha}^{-1}(A_{\alpha} \otimes \ker \varepsilon) = A^2_{\alpha}$ i.e.

$$r_{\alpha}(A^2_{\alpha}) = A_{\alpha} \otimes \ker \varepsilon$$

Similarly, one can prove that

$$t_{\alpha}(A^2_{\alpha}) = \ker \varepsilon \otimes A_{\alpha}$$

**34 Proposition.** For any $\alpha, \beta, \gamma \in \pi$

$$(\Delta_{\alpha,\beta} \otimes \text{id})r_{\alpha\beta} = (\text{id} \otimes r_{\beta})\Phi^l_{\alpha,\beta}$$

$$(\text{id} \otimes \Delta_{\alpha,\beta})t_{\alpha\beta} = (t_{\alpha} \otimes \text{id})\Phi^r_{\alpha,\beta}$$

**PROOF.** We will prove that for any $\alpha \in \pi$

$$r_{\alpha} = (\text{id} \otimes \varepsilon \otimes \text{id})\Phi^l_{\alpha,1}$$

$$t_{\alpha} = (\varepsilon \otimes \text{id} \otimes \text{id})\Phi^r_{1,\alpha}$$

For any $\alpha \in \pi, a, b \in A_{\alpha}, a \otimes b \in A_{\alpha} \otimes A_{\alpha}$

$$(\text{id} \otimes \varepsilon \otimes \text{id})\Phi^l_{\alpha,1}(a \otimes b) = a_{(1,\alpha)}b_{(1,\alpha)}\varepsilon(a_{(2,1)}) \otimes b_{(2,1)}$$

$$= (a \otimes 1_1)\Delta_{\alpha,1}(b)$$

$$= r_{\alpha}(a \otimes b)$$

Similarly,

$$(\varepsilon \otimes \text{id} \otimes \text{id})\Phi^r_{1,\alpha}(a \otimes b) = t_{\alpha}(a \otimes b)$$

To prove equation 4.11

$$(\Delta_{\alpha,\beta} \otimes \text{id})r_{\alpha\beta} = (\text{id} \otimes \text{id} \otimes \varepsilon \otimes \text{id})(\text{id} \otimes \Phi^l_{\beta,1})\Phi^l_{\alpha,\beta}$$

$$= (\text{id} \otimes r_{\beta})\Phi^l_{\alpha,\beta}$$

Similarly, one can prove equation 4.12.
Proposition. For any $\alpha \in \pi$ an element of $A^2_\alpha$ is left- (right- respectively) invariant if and only if it is of the form $r^{-1}_\alpha(1_\alpha \otimes x)(t^{-1}_\alpha(y \otimes 1_\alpha)$ respectively where $x \in \ker \varepsilon$ (y $\in \ker \varepsilon$ respectively).

Proof. For any $\alpha \in \pi$, let $x \in \ker \varepsilon$. We compute

$$
\Phi^l_{1,\alpha}(r^{-1}_\alpha(1_\alpha \otimes x)) = 1_1 \otimes \varepsilon(x(2,1))S^{-1}_\alpha(x(1,\alpha^{-1})) \otimes x(3,\alpha) = 1_1 \otimes r^{-1}_\alpha(1_\alpha \otimes x)
$$

i.e. $r^{-1}_\alpha(1_\alpha \otimes x)$ is left -invariant element.

Conversely, if $r^{-1}_\alpha(1_\alpha \otimes x)$ is left -invariant element for some $\alpha \in \pi$, let $x \in \ker \varepsilon$,

$a \in A_\alpha$. Equation 4.11 implies that

$$(\text{id} \otimes r^{-1}_\alpha)\Phi^l_{1,\alpha}(r^{-1}_\alpha(a \otimes x)) = (\Delta_{1,\alpha} \otimes \text{id})r_\alpha(r^{-1}_\alpha(a \otimes x))$$

From which we obtain

$$1_1 \otimes a \otimes x = \Delta_{1,\alpha}(a) \otimes x$$

i.e.

$$\Delta_{1,\alpha}(a) = 1_1 \otimes a$$

From which we obtain

$$a = 1_\alpha$$

QED

Theorem. Let $R$ be a right ideal of $A_1$ contained in $\ker \varepsilon$, $N = \{N_\alpha\}_{\alpha \in \pi}$, where for each $\alpha \in \pi$, $N_\alpha = r^{-1}_\alpha(A_\alpha \otimes R)$ is a sub-bimodule of $A^2_\alpha$.

Moreover, let $\Gamma = \{\Gamma_\alpha\}_{\alpha \in \pi}, \Gamma_\alpha = A^2_\alpha/N_\alpha, \Pi = \{\Pi_\alpha : A^2_\alpha \rightarrow A^2_\alpha/N_\alpha\}$ be the family of canonical epimorphisms, $d = \{d_\alpha : d_\alpha = \Pi_\alpha \circ D_\alpha\}$. Then the $\pi$-graded first order differential calculus $\Gamma = (\{\Gamma_\alpha\}_{\alpha \in \pi}, d)$ is left covariant. Any $\pi$-graded left covariant first order differential calculus on $A$ can be obtained in this way.

Proof. For any $\alpha \in \pi$, let $R$ be a right ideal of $A_1$ contained in $\ker \varepsilon$. We shall prove that $r^{-1}_\alpha(A_\alpha \otimes R)$ is a sub-bimodule of $A^2_\alpha$. For any $\alpha \in \pi$, let $q \in r^{-1}_\alpha(A_\alpha \otimes R)$, i.e. $q = r^{-1}_\alpha(b \otimes c), b \in A_\alpha, c \in R$. For $a \in A_\alpha$

$$a \cdot q = r^{-1}_\alpha(ab \otimes c) \in r^{-1}_\alpha(A_\alpha \otimes R)$$

$$q \cdot a = r^{-1}_\alpha(ba(1,\alpha) \otimes ca(2,1)) \in r^{-1}_\alpha(A_\alpha \otimes R)$$

which proves that $N_\alpha = r^{-1}_\alpha(A_\alpha \otimes R)$ is a sub-bimodule of $A^2_\alpha$. 

QED
To prove that it is left covariant we have to prove that for any \( \alpha, \beta \in \pi \),
\[
\Phi^l_{\alpha, \beta}(N_{\alpha \beta}) \subset A_\alpha \otimes N_\beta.
\]
Using 4.11 we have
\[
\Phi^l_{\alpha, \beta} = (\text{id} \otimes r^{-1}_\beta)(\Delta_{\alpha, \beta} \otimes \text{id})r_{\alpha \beta}
\]
Now, for any \( \alpha, \beta \in \pi \), consider
\[
N_{\alpha \beta} = r^{-1}_\alpha(A_\alpha \otimes r^{-1}_\beta) \cdot x_i \in \text{ker} \varepsilon
\]
Denoting by \( R_\alpha \) the linear span of all \( x_i \)'s we obtain that
\[
N_\alpha = r^{-1}_\alpha(A_\alpha \otimes R_\alpha)
\]
Conversely, if \( N = \{N_\alpha\}_{\alpha \in \pi}, \Phi^l \) is a left covariant bimodule, then , using
theorem 27 and proposition 35 there exists a family \( (x_i)_{i \in I} \) of elements of \( \text{ker} \varepsilon \)
such that for any \( \alpha \in \pi, q \in N_\alpha \) can be written as
\[
q = \sum a_i \cdot r^{-1}_\alpha(1_\alpha \otimes x_i)
\]
Denoting by \( R_\alpha \) the linear span of all \( x_i \)'s we obtain that
\[
N_\alpha = r^{-1}_\alpha(A_\alpha \otimes R_\alpha)
\]
36 A. S. Hegazi, W. Morsi, M. Mansour

It remains to show that $R$ is a right ideal of $A_1$. Let $x \in R, a \in A_1$, then $r_1^{-1}(1 \otimes x) \in N_1$.

$$r_1^{-1}(1 \otimes x) \cdot a = r_1^{-1}((1 \otimes x)r_1(1 \otimes a))$$

$$\in N_1 = r_1^{-1}(A_1 \otimes R)$$

$(N_1$ is a bimodule)

i.e

$$(1 \otimes x)r_1(1 \otimes a) \in A_1 \otimes R$$

therefore

$$(1 \otimes x)r_1(1 \otimes a) = r_1(r_1^{-1}((1 \otimes x)\Delta_{1,1}(a)))$$

$$= (1 \otimes x)\Delta_{1,1}(a) \in A_1 \otimes R$$

and $(\varepsilon \otimes \text{id})((1 \otimes x)\Delta_{1,1}(a)) = xa \in R$ \hfill Q.E.D.

37 Theorem. Let $R$ be a right ideal of $A_1$ contained in $\ker \varepsilon, N = \{N_\alpha\}_{\alpha \in \pi}$, where for each $\alpha \in \pi, N_\alpha = t_\alpha^{-1}(A_\alpha \otimes R)$ is a sub-bimodule of $A^2 = \{A^2_\alpha\}_{\alpha \in \pi}$. Moreover, let $\Gamma = \{\Gamma_\alpha\}_{\alpha \in \pi}, \Gamma_\alpha = A^2_\alpha/N_\alpha, \Pi = \{\Pi_\alpha : A^2_\alpha \to A^2_\alpha/N_\alpha\}$ be the family of canonical epimorphisms, $d = \{d_\alpha : d_\alpha = \Pi_\alpha \circ D_\alpha\}$. Then the first order differential calculus $\Gamma = (\{\Gamma_\alpha\}_{\alpha \in \pi}, d)$ is right covariant. Any right covariant first order differential calculus on $A$ can be obtained in this way.

Proof. For any $\alpha \in \pi$, let $R$ be a right ideal of $A_1$ contained in $\ker \varepsilon$. We shall prove that $t_\alpha^{-1}(R \otimes A_\alpha)$ is a sub-bimodule of $A^2_\alpha$. For any $\alpha \in \pi$, let $q \in t_\alpha^{-1}(R \otimes A_\alpha)$, i.e. $q = t_\alpha^{-1}(d \otimes e), d \in R, e \in A_\alpha$. For $a \in A_\alpha$

$$a \cdot q = t_\alpha^{-1}(t_\alpha((a \otimes 1_\alpha)q))$$

$$= t_\alpha^{-1}((1 \otimes a)t_\alpha(q))$$

$$= t_\alpha^{-1}(d \otimes ae)$$

$$\in t_\alpha^{-1}(R \otimes A_\alpha)$$

Similarly,

$$q \cdot a \in t_\alpha^{-1}(R \otimes A_\alpha)$$

which proves that $N_\alpha = t_\alpha^{-1}(R \otimes A_\alpha)$ is a sub-bimodule of $A^2_\alpha$. To prove that it is right covariant we have to prove that for any $\alpha, \beta \in \pi, \Phi_{\alpha,\beta}(N_\alpha) \subset N_\alpha \otimes A_\beta$.

Using equation 4.13 we have

$$\Phi_{\alpha,\beta} = (t_\alpha^{-1} \otimes \text{id})(\text{id} \otimes \Delta_{\alpha,\beta})t_{\alpha\beta}$$
Now, for any $\alpha, \beta \in \pi$, consider $N_{\alpha \beta} = t^{-1}_{\alpha \beta}(R \otimes A_{\alpha \beta})$

$$\Phi_{\alpha, \beta}^{(l)}(N_{\alpha \beta}) = (t^{-1}_{\alpha} \otimes \text{id})(\text{id} \otimes \Delta_{\alpha, \beta})t_{\alpha \beta}(t^{-1}_{\alpha \beta}(R \otimes A_{\alpha \beta}))$$

$$\subset (t^{-1}_{\alpha} \otimes \text{id})(R \otimes A_{\alpha} \otimes A_{\beta})$$

$$= N_{\alpha} \otimes A_{\beta}$$

Conversely, if $N = \{N_{\alpha}\}_{\alpha \in \pi}, \Phi^{(r)}$ is a right covariant bimodule then, using theorem 27 and proposition 35 there exists a family $(\{y_i\})_{i \in I}$ of elements of $\ker \varepsilon$ such that for any $\alpha \in \pi$, $q \in N_{\alpha}$ can be written as $q = \sum a_i \cdot t^{-1}_{\alpha}(y_i \otimes 1_{\alpha}), a_i \in A_{\alpha}$. But for each $i \in I$ we have

$$a_i \cdot r^{-1}_{\alpha}(y_i \otimes 1_{\alpha}) = t^{-1}_{\alpha}(t_{\alpha}(1_{\alpha} \otimes a_i)(y_i \otimes 1_{\alpha}))$$

Denoting by $R_{\alpha}$ the linear span of all $x_i$s we obtain that

$$N_{\alpha} = t^{-1}_{\alpha}(R \otimes A_{\alpha})$$

We shall show that all $R_{\alpha}$s coincide with $R_1$. From proposition 35 we have

$$N_{\alpha}^{inv} = t^{-1}_{\alpha}(1_{\alpha} \otimes A_{\alpha})$$

and since $N_{\alpha}$ is a left covariant bimodule we have

$$\Phi_{1, \alpha}^{(r)}(N_{\alpha}^{inv}) = N_{\alpha}^{inv} \otimes 1_{\alpha}$$

$$= t^{-1}_{1}(R \otimes A_1) \otimes 1_{\alpha}$$

Now let $t^{-1}_{\alpha}(y_i \otimes 1_{\alpha}) \in N_{\alpha}^{inv}, y_i \in R_{\alpha}$

$$\Phi_{1, \alpha}^{(r)}(t^{-1}_{\alpha}(y_i \otimes 1_{\alpha})) = S_1^{-1}(y_i(3,1)) \otimes y_i(1,1) \otimes 1_{\alpha} \varepsilon(y_i(2,1))$$

$$= t^{-1}_{1}(y_i \otimes 1_{1}) \otimes 1_{\alpha}$$

i.e.

$$y_i \in R_1 \implies R_{\alpha} \subseteq R_1$$

Similarly we can show that $R_1 \subseteq R_{\alpha}$, and hence $R_{\alpha} = R_1$ for each $\alpha \in \pi$.

Denote by $R$ to any of the $R_{\alpha}$s, then $N_{\alpha} = t^{-1}_{\alpha}(R \otimes A_{\alpha})$

It remains to show that $R$ is a right ideal of $A_1$.

Let $y \in R, a \in A_1, A$ then $t^{-1}_{1}(y \otimes 1_{1}) \in N_{1}$.

$$t^{-1}_{1}(y \otimes 1_{1}) \cdot a = t^{-1}_{1}((y \otimes 1_{1})t_{1}(1_{1} \otimes a)) \in N_{1}$$

$$= t^{-1}_{\alpha}(R \otimes A_{1})$$
i.e $$(y \otimes 1_1)t_1(1_1 \otimes a) \in R \otimes A_1$$

therefore

$$(y \otimes 1_1)t_1(1_1 \otimes a) = t_1(t_1^{-1}((y \otimes 1_1)\Delta_{1,1}(a)))$$

$$= (y \otimes 1_1)\Delta_{1,1}(a) \in A_1 \otimes R$$

and $$(\text{id} \otimes \varepsilon)((y \otimes 1_1)\Delta_{1,1}(a)) = ya \in R$$

We shall now formulate the concept of ad – invariance. Let

$$\text{ad}_\alpha : A_1 \rightarrow A_1 \otimes A_\alpha$$

be such that for any $a \in A_1$

$$\text{ad}_\alpha(a) = t_\alpha(r_\alpha^{-1}(1_\alpha \otimes a))$$

(4.15)

i.e.

$$\text{ad}_\alpha(a) = a_{(2,1)} \otimes S_{\alpha^{-1}}(a_{(1,\alpha^{-1})})a_{(3,\alpha)}$$

where

$$(\text{id} \otimes \Delta_{1,\alpha})\Delta_{\alpha^{-1},\alpha}(a) = a_{(1,\alpha^{-1})} \otimes a_{(2,1)} \otimes a_{(3,\alpha)}$$

(4.16)

such that

$$(\text{id} \otimes \Delta_{\alpha,\beta})\text{ad}_\alpha(a) = (\text{id} \otimes \Delta_{\alpha,\beta})\text{ad}_\alpha(a)$$

(4.17)

Using equation 4.15, and the standard properties of comultiplication and converse one can prove equation 4.17, for let $a \in A_1$. For any $\alpha, \beta \in \pi$

$$(\text{id} \otimes \Delta_{\alpha,\beta})\text{ad}_\alpha(a) = (\text{id} \otimes \Delta_{\alpha,\beta})(\text{ad}_\alpha(a))$$

$$= a_{(3,1)} \otimes S_{\alpha^{-1}}(a_{(2,\alpha^{-1})})a_{(4,\alpha)} \otimes S_{\beta^{-1}}(a_{(1,\beta^{-1})})a_{(5,\beta)}$$

Similarly,

$$(\text{id} \otimes \Delta_{\alpha,\beta})\text{ad}_\alpha(a) = a_{(3,1)} \otimes S_{\alpha^{-1}}(a_{(2,\alpha^{-1})})a_{(4,\alpha)} \otimes S_{\beta^{-1}}(a_{(1,\beta^{-1})})a_{(5,\beta)}$$

which proves equation 4.17.

A linear subset $T \subset A_1$ is $\pi$ – ad invariant if $\text{ad}_\alpha(T) \subset T \otimes A$ for any $\alpha \in \pi$.

**38 Lemma.** Let $T$ be $\pi$ – ad invariant subset of $A_1$, $R$ be a right ideal of $A_1$ generated by $T$. Then $R$ is $\pi$ – ad invariant.
Proof. Let \(a, b \in A_1\), we will prove that for any \(\alpha \in \pi\)
\[\text{ad}_\alpha(ab) = (1_1 \otimes S_{\alpha^{-1}}(b_{(1,\alpha^{-1})})) \text{ad}_\alpha(a) \Delta_{1,\alpha}(b_{(2,\alpha)})\]  
(4.18)
\[r_{\alpha}^{-1}(1_\alpha \otimes ab) = S_{\alpha^{-1}}(a_{(1,\alpha^{-1})}b_{(1,\alpha^{-1})}) \otimes a_{(2,\alpha)}b_{(2,\alpha)}\]
\[= (S_{\alpha^{-1}}(b_{(1,\alpha^{-1})}) \otimes 1_\alpha)r_{\alpha}^{-1}(a)(1_\alpha \otimes b_{(2,\alpha)})\]

Applying \(t_\alpha\) to both sides of the above equation we get
\[t_\alpha r_{\alpha}^{-1}(1_\alpha \otimes ab) = t_\alpha(S_{\alpha^{-1}}(b_{(1,\alpha^{-1})}) \otimes 1_\alpha)t_\alpha r_{\alpha}^{-1}(a)t_\alpha(1_\alpha \otimes b_{(2,\alpha)})\]  
(4.19)
\[\text{ad}_\alpha(ab) = (1_\alpha \otimes S_{\alpha^{-1}}(b_{(1,\alpha^{-1})})) \text{ad}_\alpha(a) \Delta_{1,\alpha}(b_{(2,\alpha)})\]  
(4.20)

Thus for \(a, b \in T, a, b, ab \in R, R\) being an ideal in \(A_1, T\) being \(\pi - \text{ad}\) invariant we find that
\[\text{ad}_\alpha(ab) \in R \otimes A_\alpha\]
i.e.
\[\text{ad}_\alpha(R) \subset R \otimes A_\alpha\]
which means that \(R\) is \(\pi - \text{ad}\) invariant.

Let \(A^2 = (A^2_\alpha)_{\alpha \in \pi}, \Phi^l, \Phi^r\) is a \(\pi - \text{graded} bimodule\) over \(A\). By virtue of condition 3 of definition 23 we have for any \(\alpha, \beta, \gamma \in \pi\) \((\Phi^l_{\alpha,1} \otimes \text{id})\Phi^r_{\alpha,\beta} = (\text{id} \otimes \Phi^r_{1,\beta})\Phi^l_{\alpha,\beta}\).

Applying \(\text{id} \otimes \varepsilon \otimes \text{id} \otimes \text{id}\) to both sides of the above equation and using equations 4.13 and 4.14 we get
\[\Phi^r_{\alpha,\beta} = (r_{\alpha}^{-1} \otimes \text{id})(\text{id} \otimes t_\beta)\Phi^l_{\alpha,\beta}\]

Now let \(x \in \ker \varepsilon\). From proposition 35 for any \(\alpha \in \pi\) we have \(r_{\alpha}^{-1}(1_\alpha \otimes x)\)
is a left invariant element then
\[\Phi^r_{\alpha,\beta}(r_{\alpha \beta}^{-1}(1_\alpha \beta \otimes x)) = (r_{\alpha}^{-1} \otimes \text{id})(\text{id} \otimes t_\beta)(1_\alpha \otimes r_{\beta}^{-1}(1_\beta \otimes x))\]
\[= (r_{\alpha}^{-1} \otimes \text{id})(1_\alpha \otimes \text{ad}_\beta(x))\]

39 Theorem. Let \(R\) be a right ideal of \(A_1\) contained in \(\ker \varepsilon\) and \(\Gamma = (\{\Gamma_\alpha\}_{\alpha \in \pi}, d)\) be the \(\pi - \text{graded} left covariant first order differential calculus described in theorem 5. Then \(\Gamma = (\{\Gamma_\alpha\}_{\alpha \in \pi}, d)\) is bicovariant if and only if \(R\) is \(\pi - \text{ad}\) invariant.
Proof. Let for any $\alpha \in \pi R$ be a right ideal of $A_1$ such that $R \subset \ker \varepsilon$ and $N_\alpha = r_\alpha^{-1}(A_\alpha \otimes R)$. Using theorem 35 we see that $N = \{N_\alpha, \alpha \in \pi, \Phi^{l}\}$ is a $\pi$–graded left covariant bimodule. Assume that $R$ is $\pi$–ad invariant, let $\text{inv}_N N_\alpha$ be the set of all left invariant elements of $N_\alpha$ for each $\alpha \in \pi$. Then formula 3.78 shows that for any $\alpha, \beta \in \pi$ $\Phi^{r}_{\alpha, \beta}(\text{inv}_N N_{\alpha \beta}) \subset \text{inv}_N N_\alpha \otimes A_\beta$.

Now decomposition 3.26 shows that $\Phi^{r}_{\alpha, \beta}(N_{\alpha \beta}) \subset N_\alpha \otimes A_\beta$, and this means that implication 2.14 holds.

Conversely, assume that $N = \{N_\alpha\}_{\alpha \in \pi}$ is a $\pi$–graded bicovariant bimodule. This means that 2.14 holds. Then (see proof 37) for each $\alpha \in \pi$, $N_\alpha = t_\alpha^{-1}(R' \otimes A_\alpha)$ where $R'$ be a right ideal of $A_1$ such that $R' \subset \ker \varepsilon$. In particular, $N_1 = t_1^{-1}(R' \otimes A_1)$. Using 3.78 and that $(\varepsilon \otimes \text{id}) t_1^{-1}(a \otimes b) = a \varepsilon(b)$, and $(\text{id} \otimes \varepsilon) r_1^{-1}(a \otimes b) = a S_1(b)$, one can easily checks that $R = R'$.

So we have for any $\alpha \in \pi$

$$r_\alpha^{-1}(A_\alpha \otimes R) = t_\alpha^{-1}(R \otimes A_\alpha)$$

$$t_\alpha r_\alpha^{-1}(A_\alpha \otimes R) = R \otimes A_\alpha$$

therefore $\text{ad}_\alpha(R) = t_\alpha r_\alpha^{-1}(1_\alpha \otimes R)$

$$\subset t_\alpha r_\alpha^{-1}(A_\alpha \otimes R) = R \otimes A_\alpha$$

therefore $R$ is $\pi$–ad invariant. \[QED\]

References