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Structure space of tensor products of Fréchet $*$-algebras

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Abstract. In the case of a non-commutative m^*-convex algebra, the topological spectrum (Gel'fand space) of the commutative case is replaced by the structure space consisting of equivalent classes of continuous topologically irreducible $*$-representations. We prove that the structure space of a completed tensor product of two Fréchet $*$-algebras is homeomorphic to the cartesian product of the structure spaces of the Fréchet $*$-algebras involved.

Keywords: m*-convex algebra, Fréchet $*$-algebra, structure space, inductive limit, inverse-limit preserving tensorial topology.

MSC 2000 classification: 46K10, 46M05, 46M40.

Dedicated to the memory of Professor Klaus Floret

Introduction

The topological spectrum plays an important rôle for the study of commutative topological algebras. In the non-commutative case, assuming an extra structure, that of involution, the rôle of the continuous characters is played by the continuous topologically irreducible $*$-representations. It is known (see [24, p. 193]) that practically all the operators that appear in Physics are unbounded operators, like e.g. the position and momentum operators, or the Schrödinger operator. This naturally leads to the study of the unbounded $*$-representations, when non-normed topological $*$-algebras are involved (cf., for instance, [1,17,27]). Nevertheless, any symmetric operator defined on the entirety of a Hilbert space is bounded [24, p. 195, Theorem 2.10]; moreover, every closed unbounded topologically irreducible $*$-representation of a locally C^*-algebra is always algebraically irreducible [4, Theorem 4.7, (3)]; this finally leads to the boundedness of the considered $*$-representation, since every closed algebraically irreducible $*$-representation of a symmetric algebra is bounded [2, Theorem 1]. All locally

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C^*-algebras are symmetric and of course there are several examples of symmetric algebras that are not locally C^*-algebras [13, Chapter II, Section 7]. So the study of bounded $*$-representations in the frame of non-normed topological $*$-algebras is not without interest.

In this paper we are concerned with the structure space (consisting of equivalent classes of continuous topologically irreducible $*$-representations) of tensor products of Fréchet $*$-algebras. In fact, we revisit some previous results of ours in [11] and we approach them from another point of view motivated, on the one hand by some new results on tensor product topologies and tensor products of enveloping locally C^*-algebras appeared in [14, Sections 4 and 5], and on the other hand by the results of [21, p. 165, Subsection 5.(1)]. The present results improve the corresponding ones in [11, Section 5] and give considerably simpler and more elegant proofs.

In the context of Banach $*$-algebras such studies started by M.A. Wulfsohn [28] in 1963 and continued by H.A. Smith [26] in 1968 and K.B. Laursen [20] in 1969.

We refer to [6,18,21] for topological tensor products and to [21,22], respectively [12–14], for the general theory of topological algebras, respectively topological $*$-algebras.

1 Preliminaries, definitions

All algebras we deal with are complex and the topological spaces are supposed to be Hausdorff.

Let A be a $*$-algebra. A seminorm p on A is called *m^*-seminorm*, resp. *C^*-seminorm*, if

$$p(xy) \le p(x)p(y) \;\text{ and }\; p(x^*) = p(x),\ \forall x, y \in A,\ \text{resp. } p(x^*x) = p(x)^2,\ \forall x \in A.$$

A C^*-seminorm p on A is automatically an m^*-seminorm [25]. Let now A be a $*$-algebra, endowed with a directed family $\Gamma = \{p\}$ of m^*- respectively C^*-seminorms. Let τ_Γ be the locally convex $*$-topology induced by Γ on A. Then $A[\tau_\Gamma]$ is called *m^*-convex* respectively *C^*-convex algebra*. When A has no involution, $A[\tau_\Gamma]$ is simply called *m-convex* or equivalently *locally m-convex* algebra (cf. [21]). A complete C^*-convex algebra $A[\tau_\Gamma]$ is called *locally C^*-algebra* [16] ($\Leftrightarrow$ *pro-C^*-algebra* [23]). A metrizable complete m^*-convex algebra is called *Fréchet $*$-algebra*. Let $A[\tau_\Gamma]$ be an m^*-convex algebra and $p \in \Gamma$. Put $N_p \equiv \{x \in A : p(x) = 0\}$; N_p is a closed $*$-ideal of A and A/N_p equipped with the $*$-norm $\|\cdot\|_p$ induced by p, is a normed $*$-algebra. The completion of

$(A/N_p, \|\cdot\|_p)$ denoted by A_p is a Banach $*$-algebra and [21,22]

$$A[\tau_\Gamma] \hookrightarrow \varprojlim_{p\in\Gamma} A_p, \text{ resp. } A[\tau_\Gamma] = \varprojlim_{p\in\Gamma} A_p, \text{ when } A[\tau_\Gamma] \text{ is complete.} \tag{1}$$

The embedding in (1) means *topological $*$-monomorphism* (i.e. injective bicontinuous $*$-morphism) and the equality *topological $*$-isomorphism* (i.e. surjective topological $*$-monomorphism). In the first case the family $\{A_p\}_{p\in\Gamma}$ is called *Arens-Michael Analysis* of $A[\tau_\Gamma]$ and in the second case *Arens-Michael decomposition* of $A[\tau_\Gamma]$ [21].

If $\mathcal{H}$ is a Hilbert space, $\mathcal{B}(\mathcal{H})$ stands for the C^*-algebra of all bounded linear operators on $\mathcal{H}$. If A is a $*$ algebra, a *$*$-representation* of A is a $*$-morphism $\mu : A \longrightarrow \mathcal{B}(\mathcal{H}_\mu)$; $\mathcal{H}_\mu$ is the Hilbert space on which μ acts. When $A[\tau_\Gamma]$ is an m^*-convex algebra, continuity of μ is always considered with respect to the norm-operator topology of $\mathcal{B}(\mathcal{H}_\mu)$. A $*$-representation μ of $A[\tau_\Gamma]$ is called *irreducible* if the only closed linear subspaces of $\mathcal{H}_\mu$ invariant under the operators $\mu(x)$, $x \in A$, are the trivial subspace $\{0\}$ and $\mathcal{H}_\mu$ itself. In the sequel, we shall use the following notation:

$$\begin{aligned} R(A) &\equiv \{\text{all continuous } *\text{-representations of } A\}, \\ R'(A) &\equiv \{\mu \in R(A) : \mu \text{ is irreducible}\}; \quad \text{in particular,} \\ R(A) &= \bigcup_p R_p(A), \quad \text{resp. } \ R'(A) = \bigcup_p R'_p(A), \end{aligned}$$

with $R_p(A) := \{\mu \in R(A) : \|\mu(x)\| \leq p(x), \forall x \in A\}$, $p \in \Gamma$; $R'_p(A)$ is similarly defined. It is clear that each $\mu \in R_p(A)$ defines well an element $\mu_p \in R(A/N_p)$ such that $\mu_p(x_p) := \mu(x)$, $\forall x \in A$, where $x_p \equiv x + N_p$; for the unique extension of μ_p on A_p we keep the same symbol. Particularly, we have

$$R_p(A) = R(A_p), \quad \text{resp. } \ R'_p(A) = R'(A_p), \ p \in \Gamma,$$

up to set-theoretical isomorphisms (see [10,11]).

Let now f be a continuous positive linear form of $A[\tau_\Gamma]$ and p an element of Γ describing the continuity of f. Then, the relation $f_p(x_p) := f(x),\ x \in A$, defines well a continuous positive linear form on A/N_p, extended uniquely on A_p. We retain the same symbol f_p for the extended positive linear form on A_p and we call it *associated to f*. Let A^* denote the topological dual of $A[\tau_\Gamma]$; i.e. A^* consists of all continuous linear forms of $A[\tau_\Gamma]$. A^*_s means A^* endowed with the weak*-topology. Let

$$\begin{aligned}\mathcal{P}(A) := \{&\text{all continuous positive linear forms } f \text{ of } A[\tau_\Gamma], \text{ with } \|f_p\| \leq 1 \ , \\ &\text{for the corresponding associated positive linear form } f_p \text{ of } f \ \} \text{ and}\\ \mathcal{B}(A) := \{&f \in \mathcal{P}(A) \text{ with } f \text{ pure and } \ \|f_p\| = 1\} \\ = \{&\text{non-zero extreme points of } \mathcal{P}(A)\}\end{aligned}$$

(for the last equality, see [12, Proposition 3.4]). Equip $\mathcal{P}(A)$, $\mathcal{B}(A)$ with the relative topology from A_s^*. Moreover, for any $p \in \Gamma$, consider the closed semiball $\mathcal{U}_p(1) \equiv \{x \in A : p(x) \leq 1\}$, and denote by $\mathcal{U}_p^0(1)$ the polar of $\mathcal{U}_p(1)$ in A^*. Let

$$\mathcal{P}_p(A) := \mathcal{P}(A) \cap \mathcal{U}_p^0(1) \ \text{ and } \ \mathcal{B}_p(A) := \mathcal{B}(A) \cap \mathcal{U}_p^0(1), \ p \in \Gamma.$$

Then, $\mathcal{P}_p(A) = \mathcal{P}(A_p)$ resp. $\mathcal{B}_p(A) = \mathcal{B}(A_p)$, $p \in \Gamma$, with respect to homeomorphisms [10,12]. In particular,

$$\mathcal{P}(A) = \bigcup_p \mathcal{P}_p(A) \ \text{ and } \ \mathcal{B}(A) = \bigcup_p \mathcal{B}_p(A). \tag{2}$$

Now, for each $p \in \Gamma$, define $r_p(x) := \sup\{\|\mu(x)\| : \mu \in R'_p(A)\}, \ x \in A$. Then r_p, $p \in \Gamma$, is a C^*-seminorm on A with $r_p(x) \leq p(x)$, $x \in A$. The closed 2-sided $*$-ideal $\cap\{\ker r_p : p \in \Gamma\}$ of A, clearly coincides with the $*$-radical $R_A^* := \cap\{\ker \mu : \mu \in R'(A)\}$ of A. Let $\Gamma_\varepsilon \equiv \{r_p, p \in \Gamma\}$. The Hausdorff completion of $A[\tau_{\Gamma_\varepsilon}]$ is called *enveloping locally C^*-algebra* of $A[\tau_\Gamma]$ and is denoted by $\mathcal{E}(A)$ [10,14].

Let $\mu, \mu' \in R'(A)$. We say that μ, μ' are equivalent and we write $\mu \sim \mu'$, if there is a surjective isometric isomorphism $\mathcal{U} : \mathcal{H}_\mu \to \mathcal{H}_{\mu'}$, such that $\mu'(x) \circ \mathcal{U} = \mathcal{U} \circ \mu(x)$, $\forall x \in A$; "$\sim$" is an equivalent relation and

$$\|\mu(x)\| = \|\mu'(x)\|, \ x \in A, \ \forall \mu, \mu' \in R'(A) \ \text{ with } \ \mu \sim \mu'.$$

The quotient

$$\mathcal{R}(A) := R'(A)/\sim$$

is called *structure space* of A and its elements are denoted by $[\mu]$, $\mu \in R'(A)$. If moreover,

$$\mathcal{R}_p(A) := R'_p(A)/\sim, \ p \in \Gamma,$$

one has $\mathcal{R}_p(A) = \mathcal{R}(A_p)$, $p \in \Gamma$, up to a well defined surjection [12,14] and

$$\mathcal{R}(A) = \bigcup_p \mathcal{R}_p(A) = \bigcup_p \mathcal{R}(A_p). \tag{3}$$

2 Topologization of the structure space $\mathcal{R}(A)$

Let $A[\tau_\Gamma]$ be an m^*-convex algebra with a *bounded approximate identity* (abbreviated to bai). Using the GNS-construction [12, pp. 14–16], the map

$$\delta : \mathcal{B}(A) \longrightarrow \mathcal{R}(A) : f \longmapsto [\mu_f], \tag{4}$$

with μ_f the *GNS*-$*$-representation corresponding to f, is well defined and surjective and it is called *GNS-map*. We have mentioned that $\mathcal{B}(A)$ carries the relative weak*-topology from A_s^*. Equip $\mathcal{R}(A)$ with *the final topology* τ_δ induced on it by δ.

Let now $p, q \in \Gamma$ with $p \leq q$ and let $\{A_p\}_{p\in\Gamma}$ be the Arens-Michael analysis of A. Denote by $\tau_{pq} : A_q \longrightarrow A_p : x_q \longmapsto x_p$, the (continuous) *connecting $*$-morphism* between A_q, A_p. For each $[\mu_p] \in \mathcal{R}(A_p)$, the composition $\mu_q := \mu_p \circ \tau_{pq}$ defines an element $[\mu_q] \in \mathcal{R}(A_q)$ and the correspondence

$$R_{qp} : \mathcal{R}(A_p) \longrightarrow \mathcal{R}(A_q) : [\mu_p] \longmapsto [\mu_q]$$

is a well-defined continuous map, where $\mathcal{R}(A_i)$ carries the final topology τ_{δ_i} through the respective *GNS*-map

$$\delta_i : \mathcal{B}(A_i) \longrightarrow \mathcal{R}(A_i) : f_i \longmapsto [\mu_{f_i}],\ i = p, q.$$

To check the continuity of R_{qp} consider the diagram

$$\begin{tikzcd} \mathcal{B}(A_p) \arrow[r, "\beta_{qp}"] \arrow[d, "\delta_p"'] & \mathcal{B}(A_q) \arrow[d, "\delta_q"] & \\ \mathcal{R}(A_p) \arrow[r, "R_{qp}"'] & \mathcal{R}(A_q) & \end{tikzcd} \qquad p \leq q$$

where for $f_p \in \mathcal{B}(A_p)$, $\beta_{qp}(f_p) := f_q$ with $f_q = f_p \circ \tau_{pq}$. The map β_{qp} is a well-defined continuous injection and the preceding diagram is commutative. So

$$R_{qp} \circ \delta_p = \delta_q \circ \beta_{qp},$$

where $\delta_q \circ \beta_{qp}$ is continuous. Hence, R_{qp} is continuous according to the definition of τ_{δ_p}. Moreover, for $p \leq q \leq r$ in Γ, one has $R_{rq} \circ R_{qp} = R_{rp}$. So the family $(\mathcal{R}(A_p), R_{qp})_{p\leq q}$ forms an inductive system of topological spaces, therefore in view of (3) we get

$$\mathcal{R}(A) = \varinjlim_p \mathcal{R}(A_p), \tag{5}$$

set-theoretically. We shall show (see Theorem 4) that in some cases *the inductive limit topology* $\tau_{\underrightarrow{\lim}}$ *on* $\mathcal{R}(A)$ coincides with *the final topology* τ_δ *on* $\mathcal{R}(A)$ mentioned above.

It is easily seen that $(\mathcal{B}(A_p), \beta_{qp})_{p\leq q}$ is also an inductive system of topological spaces, so that because of (2) we get set-theoretically the equalities

$$\mathcal{B}(A) = \underrightarrow{\lim}_p \mathcal{B}(A_p); \text{ and similarly } \mathcal{P}(A) = \underrightarrow{\lim}_p \mathcal{P}(A_p). \tag{6}$$

Denote by $\tau_{\mathcal{B}}$ *the relative topology on* $\mathcal{B}(A)$ *from* A_s^* and by $\tau_{\underrightarrow{\lim}}^{\mathcal{B}}$ *the inductive limit topology on* $\mathcal{B}(A)$ according to (6).

We shall show that under certain conditions $\tau_{\mathcal{B}} = \tau_{\underrightarrow{\lim}}^{\mathcal{B}}$; for this reason we need the following.

1 Lemma. *Let A be a unital Banach $*$-algebra with identity e. Then, the continuous* (natural) *injection* $\mathcal{B}(A) \longrightarrow \mathcal{P}(A) : f \longmapsto f$, *is closed.*

PROOF. Let $V \subseteq \mathcal{B}(A)$ be closed. We show that V is also closed in $\mathcal{P}(A)$. Let $\{f_\nu\}$ be a net in V, such that $f_\nu \longrightarrow h$, with $h \in \mathcal{P}(A)$. We prove that $h \in V$. Since, $f_\nu \in \mathcal{B}(A)$, we get

$$h(e) = \lim_\nu f_\nu(e) = \lim_\nu \|f_\nu\| = 1, \text{ whence it follows that } \|h\| = 1. \tag{7}$$

Let now $g \in \mathcal{P}(A)$ with $g \leq h$. Then, for each $z \in A$, we have $g(z^*z) \leq h(z^*z) = \lim_\nu f_\nu(z^*z)$. Hence, there is an index ν_0, such that

$$g(z^*z) \leq f_\nu(z^*z^*), \ \forall z \in A \ \text{ and } \ \nu \geq \nu_0.$$

Since each f_ν is pure, there are numbers

$$\alpha_\nu \in [0,1] \text{ with } g = \alpha_\nu f_\nu, \ \forall \nu \geq \nu_0.$$

Applying in the previous equality a similar argument to that of (7), we get that $\lim_\nu \alpha_\nu = g(e)$. Thus, $g = g(e)h$, which together with (7) implies that h is pure, therefore $h \in V$. $\boxed{QED}$

An m-convex algebra $A[\tau_\Gamma]$ is called *barrelled* [15], if the underlying locally convex space of $A[\tau_\Gamma]$ is barrelled, in the sense that every barrel (i.e. a closed absolutely convex and absorbing subset) V of $A[\tau_\Gamma]$ is a 0-neighborhood. Every Fréchet algebra is barrelled. Given a topological space X, a family $\{S_\alpha\}_{\alpha\in I}$ of compact subspaces of X, is called a *k-covering family* for X, if for each compact subset K of X there is an index $\alpha \in I$ such that $K \subseteq S_\alpha$ [21, p. 165, Definition 5.1].

2 Lemma. *Let $A[\tau_\Gamma]$ be a unital barrelled m^*-convex algebra. Then, the family $\{\mathcal{B}(A_p)\}_{p\in\Gamma}$ is a k-covering for $\mathcal{B}(A)$.*

PROOF. From (2) $\mathcal{B}(A) = \bigcup_{p\in\Gamma} \mathcal{B}_p(A)$, where $\mathcal{B}_p(A)$ is homeomorphic to $\mathcal{B}(A_p)$ (see Section 2). From Lemma 1, $\mathcal{B}(A_p)$ is weakly*-closed in $\mathcal{P}(A_p)$ and since $\mathcal{P}(A_p)$ is weakly*-compact [8, p. 44], $\mathcal{B}(A_p)$ is also weakly*-compact. Let now K be an arbitrary weakly*-compact subset of $\mathcal{B}(A)$. Then, K is an equicontinuous subset of A_s^* [15, p. 212, Corollary], consequently there is a 0-neighborhood $\mathcal{U}_p(\varepsilon) \equiv \{x \in A : p(x) \le \varepsilon\}$, $0 < \varepsilon \le 1$, in $A[\tau]$, such that

$$K \subseteq \mathcal{U}_p^0(\varepsilon) = \mathcal{U}_{p_{\mathcal{U}}}^0(1),$$

with $\mathcal{U} \equiv \mathcal{U}_p(\varepsilon)$ and $p_{\mathcal{U}}$ the gauge function of $\mathcal{U}$. But $p(x) = \varepsilon p_{\mathcal{U}}(x)$, $\forall x \in A$, so that $N_p = N_{p_{\mathcal{U}}}$ and $A_p = A_{p_{\mathcal{U}}}$. Thus, there is $p \in \Gamma$, such that

$$K \subseteq \mathcal{B}(A) \cap \mathcal{U}_{p_{\mathcal{U}}}^0(1) = \mathcal{B}_{p_{\mathcal{U}}}(A) = \mathcal{B}(A_p)$$

(for the equalities, see Section 2). This completes the proof. $\boxed{QED}$

3 Theorem. *Let $A[\tau_\Gamma]$ be a barrelled m^*-convex algebra with a bai. Let $\mathcal{B}(A)$ be locally equicontinuous in A_s^*. Then, $\tau_{\mathcal{B}} = \tau_{\varinjlim}^{\mathcal{B}}$ on $\mathcal{B}(A)$, that is $\mathcal{B}(A) = \varinjlim \mathcal{B}(A_p)$, up to a homeomorphism.*

PROOF. Suppose that $A[\tau_\Gamma]$ is unital. Then, from Lemma 2, $\{\mathcal{B}(A_p)\}_{p\in\Gamma}$ is a k-covering family for $\mathcal{B}(A)$. Showing that $(\mathcal{B}(A), \tau_{\mathcal{B}})$ is a k-space [19, p. 230], we conclude from [21, pp. 166, 167, Lemma 5.2 and Corollary 5.1] that $\tau_{\mathcal{B}} = \tau_{\varinjlim}^{\mathcal{B}}$. It is sufficient to show that $(\mathcal{B}(A), \tau_{\mathcal{B}})$ is a locally compact space. Indeed, let $f \in \mathcal{B}(A)$. Since $\mathcal{B}(A)$ is locally equicontinuous, there is an equicontinuous neighborhood V of f. By Alaoglou-Bourbaki theorem, V is relatively weakly*-compact in A_s^*. Arguing now in a similar way as in the proof of [21, p. 143, Theorem 1.1] we obtain a compact neighborhood of f in $\mathcal{B}(A)$.

Let us now come to the given, non-unital case. Passing to the unitization $A_1[\tau_1]$ of $A[\tau_\Gamma]$, with $\tau_1 = \tau_{\Gamma_1}$, $\Gamma_1 = \{p_1\}$, where $p_1(x,\lambda) := p(x) + |\lambda|$, for any $(x,\lambda) \in A_1 \equiv A \oplus \mathbb{C}$ and $p \in \Gamma$, we have that $A_1[\tau_1]$ is a barrelled m^*-convex algebra (see [15, p. 215, Corollary, (b)]). Involution on A_1 is defined by $(x,\lambda)^* := (x^*, \overline{\lambda})$, for every $(x,\lambda) \in A_1$. Moreover, defining the function $f_0(x,\lambda) := \lambda$ for every $(x,\lambda) \in A_1$, we conclude that $f_0 \in \mathcal{B}(A_1)$ and $\mathcal{B}(A) = \mathcal{B}(A_1) \setminus \{f_0\}$, up to a homeomorphism sending an element $f \in \mathcal{B}(A)$ to an element $f_1 \in \mathcal{B}(A_1)$ such that for any $(x,\lambda) \in A_1$, $f_1(x,\lambda) := f(x) + \lambda$. It is easily seen that $\mathcal{B}(A_1)$ is locally equicontinuous, since $\mathcal{B}(A)$ has this property. So as we proved above, $\mathcal{B}(A_1) = \varinjlim \mathcal{B}((A_p)_1)$, up to a homeomorphism, where $(A_p)_1$ is the unitization of the Banach algebra A_p, that coincides (topologically) with the Banach algebra $(A_1)_{p_1}$. Applying now the same arguments as in the proof of [21, p. 169, (6.6)], we are led to the claim of the theorem. $\boxed{QED}$

4 Theorem. *Let $A[\tau_\Gamma]$ be a barrelled m^*-convex algebra with a bai, such that $\mathcal{B}(A)$ is locally equicontinuous in A_s^*. Then, the natural topology τ_δ on $\mathcal{R}(A)$ coincides with the inductive limit topology $\tau_{\varinjlim}$ on $\mathcal{R}(A)$ induced by $\varinjlim \mathcal{R}(A_p)$ according to* (5).

Proof. The commutativity of the diagram before (5), yields the existence of the unique continuous map

$$\varinjlim \delta_p : \varinjlim \mathcal{B}(A_p) \longrightarrow \varinjlim \mathcal{R}(A_p),$$

such that the diagram

$$\begin{tikzcd}
\mathcal{B}(A_p) \arrow[r, "\delta_p"] \arrow[d, "\beta_p"'] & \mathcal{R}(A_p) \arrow[d, "i_p"] \\
(\mathcal{B}(A), \tau^{\mathcal{B}}_{\varinjlim}) \arrow[r, "\varinjlim \delta_p"'] & (\mathcal{R}(A), \tau_{\varinjlim})
\end{tikzcd}$$

is commutative, where i_p, β_p are the continuous natural embeddings of $\mathcal{R}(A_p)$ into $(\mathcal{R}(A), \tau_{\varinjlim})$ and $\mathcal{B}(A_p)$ into $(\mathcal{B}(A), \tau^{\mathcal{B}}_{\varinjlim})$ respectively. We show that i_p is also continuous when $\mathcal{R}(A)$ carries the topology τ_δ. Note that from Theorem 3, $\tau^{\mathcal{B}}_{\varinjlim} = \tau_{\mathcal{B}}$. On the other hand, if in the preceding diagram we replace $\tau_{\varinjlim}$ with τ_δ and $\varinjlim \delta_p$ with δ (see (4)), the diagram remains commutative. Thus $i_p : \mathcal{R}(A_p) \to (\mathcal{R}(A), \tau_\delta)$ is continuous if and only if $i_p \circ \delta_p$ is continuous, which is true since $i_p \circ \delta_p = \delta \circ \beta_p$ and both of δ, β_p are continuous. Hence, from the definition of $\tau_{\varinjlim}$ we conclude that $\tau_\delta \preceq \tau_{\varinjlim}$. We show the inverse inequality. Let $G \subseteq (\mathcal{R}(A), \tau_{\varinjlim})$ be open, i.e. $i_p^{-1}(G)$ is open in $\mathcal{R}(A_p)$, $\forall p \in \Gamma$; then $\delta_p^{-1}(i_p^{-1}(G))$ is open in $\mathcal{B}(A_p)$, $\forall p \in \Gamma$. But, $\delta_p^{-1}(i_p^{-1}(G)) = \beta_p^{-1}(\delta^{-1}(G))$, $\forall p \in \Gamma$. Hence $\delta^{-1}(G) \cap \mathcal{B}(A_p)$ is open in $\mathcal{B}(A_p)$, $\forall p \in \Gamma$, which means that $\delta^{-1}(G)$ is open in $(\mathcal{B}(A), \tau^{\mathcal{B}}_{\varinjlim} = \tau_{\mathcal{B}})$. Thus G is τ_δ-open, therefore $\tau_{\varinjlim} \preceq \tau_\delta$. It follows that

$$\tau_\delta = \tau_{\varinjlim} \quad \text{and} \quad \delta = \varinjlim \delta_p,$$

since $\varinjlim \delta_p$ is the unique continuous map making the last diagram commutative.

$\boxed{QED}$

3 Structure space of topological tensor-product ∗-algebras

Let $A[\tau_\Gamma]$, $B[\tau_{\Gamma'}]$ be m^*-convex algebras and τ an admissible topology on $A \otimes B$ (cf. [14, Definition 3.1]). Suppose that $\{t_{pq}\}_{(p,q)\in\Gamma\times\Gamma'}$ is a family of m^*-seminorms defining the topology τ. Then τ is called *pq-admissible* [14, pp. 29, 30] on $A \otimes B$, if for any $(p,q) \in \Gamma \times \Gamma'$ an admissible norm $\|\cdot\|_{pq}$ (in the sense of the first of the above citations) is defined on $A_p \otimes B_q$, such that

$$\left\|\sum_{i=1}^{n} x_{i,p} \otimes y_{i,q}\right\|_{pq} = t_{p,q}\left(\sum_{i=1}^{n} x_i \otimes y_i\right),$$

for any $\sum\limits_{i=1}^{n} x_i \otimes y_i \in A \otimes B$, where $x_{i,p} \equiv x_i + N_p \in A/N_p$ and $y_{i,q} \equiv y_i + N_q \in B/N_q$, $i = 1, \ldots, n$.

The pq-admissible tensorial topologies have the nice property to preserve inverse limits; indeed, from (1)

$$A[\tau_\Gamma] \hookrightarrow \varprojlim_{p\in\Gamma} A_p \quad \text{and} \quad B[\tau_{\Gamma'}] \hookrightarrow \varprojlim_{q\in\Gamma'} B_q,$$

up to topological ∗-monomorphisms; if τ is a pq-admissible topology on $A \otimes B$, denoting by $A\widehat{\otimes}_\tau B$ the completion of $(A \otimes B, \tau)$ and by $A_p\widetilde{\otimes}B_q$ the completion of $(A_P \otimes B_q, \|\cdot\|_{pq})$, we have (see [14, 4.7 Theorem]) that

$$A\widehat{\otimes}_\tau B = \varprojlim A_p\widetilde{\otimes}B_q, \tag{8}$$

with respect to a topological ∗-isomorphism. Such topologies have been also studied in [7]. Because of (8), in the sequel we shall use the term *inverse-limit preserving tensorial topology* for τ, instead of pq-admissible topology. Examples and properties of inverse-limit preserving tensorial topologies are given in [14]. Clearly (8) leads to a corresponding to (5) set-theoretical identification; i.e.

$$\mathcal{R}(A\widehat{\otimes}_\tau B) = \varinjlim \mathcal{R}(A_p\widetilde{\otimes}B_q), \tag{9}$$

where the index set $\Gamma \times \Gamma'$ gets a preorder by $(p,q) \le (p',q')$ if both $p \le p'$ and $q \le q'$. So $\Gamma \times \Gamma'$ is a directed set and $(\mathcal{R}(A_p) \times \mathcal{R}(B_q))_{(p,q)\in\Gamma\times\Gamma'}$ (as well as $((\mathcal{R}(A_p\widetilde{\otimes}B_q))_{(p,q)\in\Gamma\times\Gamma'})$ is an inductive system of topological spaces, together with the continuous maps (see discussion before (5))

$$R_{p'p} \times R_{q'q} : \mathcal{R}(A_p) \times \mathcal{R}(B_q) \longrightarrow \mathcal{R}(A_{p'}) \times \mathcal{R}(B_{q'}),$$

with $(p,q) \leq (p',q')$ in $\Gamma \times \Gamma'$ (cf. [9, p. 425, 1.9]).

Suppose now that either of the algebras $A[\tau_\Gamma]$, $B[\tau_{\Gamma'}]$ is of type I [14, p. 25] and both of them have a bai. Let $A[\tau_\Gamma]$ be of type I. Then, each Banach $*$-algebra A_p is of type I (ibid., p. 43, Lemma 5.11). If τ is an inverse-limit preserving tensorial topology on $A \otimes B$, K.B. Laursen proved in [20] that

$$\mathcal{R}(A_p \widetilde{\otimes} B_q) = \mathcal{R}(A_p) \times \mathcal{R}(B_q),$$

under a homeomorphism, which we denote by G_{pq}. Note that $([\mu_p],[\mu_q]) \in \mathcal{R}(A_p) \times \mathcal{R}(B_q)$ goes through G_{pq}^{-1} to $[\mu_p \otimes \mu_q]$ (where $\mu_p \otimes \mu_q$ also denotes the unique extension of $\mu_p \otimes \mu_q$ to $A_p \widetilde{\otimes} B_q$ [20]). Now, from [9, p. 422, 1.5 Theorem], the map

$$G \equiv \varinjlim G_{pq} : \varinjlim \mathcal{R}(A_p \widetilde{\otimes} B_q) \longrightarrow \varinjlim (\mathcal{R}(A_p) \times \mathcal{R}(B_q)), \tag{10}$$

is a homeomorphism and since the map

$$\varinjlim (\mathcal{R}(A_p) \times \mathcal{R}(B_q)) \longrightarrow \varinjlim \mathcal{R}(A_p) \times \varinjlim \mathcal{R}(B_q),$$

is a continuous bijection [9, p. 425, 1.9, (3)], combined with (10) gives that

$$G : \varinjlim \mathcal{R}(A_p \widetilde{\otimes} B_q) \longrightarrow \varinjlim \mathcal{R}(A_p) \times \varinjlim \mathcal{R}(B_q), \tag{11}$$

is a continuous bijection too.

5 Theorem. *Let $A[\tau_\Gamma]$, $B[\tau_{\Gamma'}]$ be Fréchet $*$-algebras, such that each one of them has a bai and one of them is of type I. Let τ be an inverse-limit preserving tensorial topology on $A \otimes B$, such that $\mathcal{B}(\mathcal{E}(A \widehat{\otimes}_\tau B))$ is locally equicontinuous. Then, up to a homeomorphism, we have that:*

$$\mathcal{R}(A \widehat{\otimes}_\tau B) = \mathcal{R}(A) \times \mathcal{R}(B).$$

Proof. The local equicontinuity of $\mathcal{B}(\mathcal{E}(A \widehat{\otimes}_\tau B))$ implies that of $\mathcal{B}(A \widehat{\otimes}_\tau B)$ as well as of $\mathcal{B}(\mathcal{E}(A) \widehat{\otimes}_\alpha \mathcal{E}(B))$, where α is the injective tensorial locally C^*-topology on $\mathcal{E}(A) \otimes \mathcal{E}(B)$ (cf. [14, p. 27 and p. 44, Corollary 5.12]). In their turn the preceding locally equicontinuous sets imply local equicontinuity for the sets $\mathcal{B}(A)$, $\mathcal{B}(B)$ and $\mathcal{B}(\mathcal{E}(A))$, $\mathcal{B}(\mathcal{E}(B))$ respectively [11, Theorem 5.2]. Thus, from Theorems 3 and 4, we conclude that the natural topologies of $\mathcal{B}(A)$, $\mathcal{B}(B)$, $\mathcal{B}(A \widehat{\otimes}_\tau B)$, $\mathcal{R}(A)$, $\mathcal{R}(B)$, $\mathcal{R}(A \widehat{\otimes}_\tau B)$ coincide with their corresponding inductive limit topologies. Therefore, looking at (5), (9) and (11) we get that

$$G : \mathcal{R}(A \widehat{\otimes}_\tau B) \longrightarrow \mathcal{R}(A) \times \mathcal{R}(B)$$

is a continuous bijection. It remains to prove that G^{-1} is continuous. For this purpose, consider the following commutative diagram

$$\begin{tikzcd}
\mathcal{B}(A) \times \mathcal{B}(B) \arrow[r, "H"] \arrow[d, "\delta_A \times \delta_B"'] & \mathcal{B}(A \widehat{\underset{\tau}{\otimes}} B) \arrow[d, "\delta_\otimes"] \\
\mathcal{R}(A) \times \mathcal{R}(B) \arrow[r, "G^{-1}"'] & \mathcal{R}(A \widehat{\underset{\tau}{\otimes}} B)
\end{tikzcd}$$

where δ_A, δ_B, $\delta_\otimes$ are the corresponding to (4) GNS-maps. H and G^{-1} are defined as follows: $H(f,g) := f \otimes g$ (cf. [11, Theorem 5.2]) and $G^{-1}([\mu],[\mu']) = [\mu \otimes \mu']$ (ibid., p. 23). For simplicity we retain the symbols $f \otimes g$, $\mu \otimes \mu'$ for the unique extensions of $f \otimes g$, $\mu \otimes \mu'$ on $A \widehat{\underset{\tau}{\otimes}} B$. Commutativity of the above diagram gives $G^{-1} \circ (\delta_A \times \delta_B) = \delta_\otimes \circ H$, where δ_i, $i = A, B, \otimes$ are continuous by the definition of the topologies τ_{δ_i}, and H from [11, Theorem 5.2]. Thus, $G^{-1} \circ (\delta_A \times \delta_B)$ is continuous with $\delta_A \times \delta_B$ continuous and open; for the openness of δ_A, δ_B see [12, Theorem 6.4]); the local equicontinuity of the sets $\mathcal{B}(\mathcal{E}(A))$ and $\mathcal{B}(\mathcal{E}(B))$ (mentioned above) is essential for the result of the last citation. The continuity of G^{-1} follows now from [19, p. 95, Theorems 8, 9]. $\boxed{QED}$

There is a plethora of non-normed m^*-convex algebras, that attain a (normed) C^*-enveloping algebra (cf. [3]). In this respect, we have the following

6 Corollary. *Let $A[\tau_\Gamma]$, $B[\tau_{\Gamma'}]$ be Fréchet $*$-algebras such that both of them have a bai and a C^*-enveloping algebra and one of them is of type I. Then, for every inverse-limit preserving tensorial topology τ on $A \otimes B$, the next equality holds up to a homeomorphism:*

$$\mathcal{R}(A \widehat{\underset{\tau}{\otimes}} B) = \mathcal{R}(A) \times \mathcal{R}(B).$$

Proof. Since $\mathcal{E}(A)$, $\mathcal{E}(B)$ are C^*-algebras, the same is true for $\mathcal{E}(A \widehat{\underset{\tau}{\otimes}} B)$ (see [14, Corollary 5.12]). Thus $\mathcal{B}(\mathcal{E}(A \widehat{\underset{\tau}{\otimes}} B))$ is equicontinuous, hence locally equicontinuous. The assertion now follows from Theorem 5. $\boxed{QED}$

4 Applications

In this Section we apply the results of Section 4 to some concrete m^*-convex algebras.

7 Proposition. *Let G be a (non-abelian) locally compact group, $C^\infty(X)$ the Fréchet $*$-algebra of all smooth functions on a 2nd countable compact finite-dimensional smooth manifold X and $L^1_{C^\infty(X)}(G)$ the generalized group algebra of G. Then, up to a homeomorphism, one has that*

$$\mathcal{R}(L^1_{C^\infty(X)}(G)) = \mathcal{R}(C^*(G)) \times X, \tag{12}$$

where $C^*(G)$ is the group C^*-algebra of G.

Proof. Note that $L^1_{C^\infty(X)}(G) = L^1(G)\widehat{\otimes}_\pi C^\infty(X)$, up to a topological $*$-isomorphism, with π the projective tensorial topology [21, p. 406]. The topology π is an inverse-limit preserving tensorial topology (see [14, (4.15)]) and both of $L^1(G)$, $C^\infty(X)$ have a C^*-enveloping algebra (for the second one, cf. [21, p. 498, (6.4)]). In particular, $\mathcal{E}(C^\infty(X)) = \mathcal{C}(X)$; hence $\mathcal{E}(C^\infty(X))$ as a commutative C^*-algebra is of type I [8, 4.2.2, 4.3.1, 5.5.2], therefore by [14, Lemma 5.11] $C^\infty(X)$ is of type I too. Furthermore, since $C^\infty(X)$ is commutative, its structure space $\mathcal{R}(C^\infty(X))$ coincides with its topological spectrum (Gel'fand space), which is homeomorphic to X [21, p. 227]. The conclusion now follows from Corollary 6, provided that $\mathcal{R}(L^1(G)) = \mathcal{R}(C^*(G))$ up to a homeomorphism [20, Proposition 2.10]. $\boxed{QED}$

A unital locally convex algebra $A[\tau_\Gamma]$ is called *Q-algebra*, if the group of its invertible elements is open ($C^\infty(X)$, X as in Proposition 7, is such an algebra). Every m^*-convex Q-algebra with a bai, has a C^*-enveloping algebra (cf. [3]). Thus, in Proposition 7, $C^\infty(X)$ can be replaced, for instance, by any unital Fréchet Q-$*$-algebra $A[\tau_\Gamma]$, so that the place of X in (12) will be taken by $\mathcal{R}(A)$.

8 Proposition. *Let X be as in Proposition 7 and A a* (non-commutative) *Banach $*$-algebra with a bai. Let $C^\infty(X, A)$ be the Fréchet $*$-algebra of all A-valued smooth functions on X. Then, up to a homeomorphism, one has that:*

$$\mathcal{R}(C^\infty(X, A)) = X \times \mathcal{R}(A).$$

Proof. From [21, p. 394, (2.8)] one has $C^\infty(X, A) = C^\infty(X) \widehat{\otimes}_{\pi=\varepsilon} A$ (topologically $*$-isomorphically), where ε is the injective tensorial topology. According to the proof of Proposition 7, we have that $C^\infty(X)$ is of type I, has a C^*-enveloping algebra, $\mathcal{R}(C^\infty(X))$ is homeomorphic to X and π is an inverse-limit preserving tensorial topology. So the result is again taken by Corollary 6. $\boxed{QED}$

A $*$-algebra A is called *symmetric* if every element of the form x^*x, $x \in A$, has its spectrum in $\mathbb{R}_+$.

9 Proposition. *Let X be a compact space and $A[\tau_\Gamma]$ a unital symmetric Fréchet $*$-algebra, the spectral radius of which is finite on the self-adjoint elements. Let $\mathcal{C}(X,A)$ be the unital Fréchet $*$-algebra of all A-valued continuous functions on X. Then, the following equality holds up to a homeomorphism:*

$$\mathcal{R}(\mathcal{C}(X,A)) = X \times \mathcal{R}(A).$$

Proof. If $\Gamma = \{p_n\}$, $n \in \mathbb{N}$, the topology of $C(X,A)$ is defined by the family $q_n(f) := \sup\{p_n(f(x)) : x \in X\}$, $n \in \mathbb{N}$, $f \in \mathcal{C}(X,A)$ of m^*-seminorms. From [21, p. 391, Theorem 1.1] we have that $\mathcal{C}(X,A) = \mathcal{C}(X)\widehat{\otimes}_{\varepsilon} A$, up to a topological $*$-isomorphism, with "ε" the injective tensorial topology. $\mathcal{C}(X)$ is of type I as a commutative C^*-algebra. A is a Q-algebra according to [5, Corollary 4.11], therefore it has a C^*-enveloping algebra (cf. comments after Proposition 7). Clearly, $\mathcal{R}(\mathcal{C}(X))$ coincides with the Gel'fand space of $\mathcal{C}(X)$, which is homeomorphic to X. Finally, ε is an inverse-limit preserving tensorial topology on $\mathcal{C}(X) \otimes A$ (cf. [14, Corollary 4.8]), so that the claimed result follows from Corollary 6. $\boxed{QED}$

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