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Norms of tensor product identities

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Abstract. For two symmetric Banach sequence spaces E and F, each either 2-convex or 2-concave, we derive asymptotically optimal estimates for the norms of identity maps $E^n \otimes_\varepsilon F^m \hookrightarrow \ell_2^{nm}$, where E^n and F^m denote the n-th and m-th sections of E and F, respectively, and $E^n \otimes_\varepsilon F^m$ their injective tensor product. This generalizes classical results of Hardy and Littlewood as well as of Schütt for ℓ_p-spaces. Based upon this, we give applications to Banach–Mazur distances, volume ratios and projection constants of tensor products, and approximation numbers of certain tensor product identities. As examples we consider powers of sequence spaces as well as Lorentz sequence spaces. Finally, we study the more general context of tensor products of spaces with enough symmetries. In particular, we consider tensor products involving finite-dimensional Schatten classes.

Keywords: Tensor products, symmetric Banach sequence spaces, Banach–Mazur distance, volume ratio, projection constant, approximation numbers, Schatten classes

MSC 2000 classification: primary 46M05, 46B07, 46B45, secondary 46B70, 47B06, 47B10

To the memory of our teacher Klaus Floret

Contents

1 Preliminaries

For $\mathcal{I} \subset \mathbb{N} \times \mathbb{N}$ and two sequences $(a_{n,m})$ and $(b_{n,m})$ of scalars, we write $a_{n,m} \overset{\mathcal{I}}{\prec} b_{n,m}$ (or $b_{n,m} \overset{\mathcal{I}}{\succ} a_{n,m}$) whenever there exists $c > 0$ such that $a_{n,m} \leq c\, b_{n,m}$ for all $(n,m) \in \mathcal{I}$, and $a_{n,m} \overset{\mathcal{I}}{\asymp} b_{n,m}$ whenever $a_{n,m} \overset{\mathcal{I}}{\prec} b_{n,m}$ and $b_{n,m} \overset{\mathcal{I}}{\prec} a_{n,m}$. We simply write "$\prec$" etc. when it is clear what $\mathcal{I}$ actually is, e.g., when $\mathcal{I} = \mathbb{N} \times \mathbb{N}$ or when we consider sequences depending on one index only. Whenever a third index k is involved, the formulas then are meant to hold "for all k", giving restrictions on k in advance. Very often the special set $\mathcal{D} = \{(n,n);\, n \in \mathbb{N}\}$ will be considered. For $1 \leq p \leq \infty$, its conjugate number p' is defined by $1/p + 1/p' = 1$.

We shall use standard notation and notions from Banach space theory, as presented e. g. in [5,13,19,33]. If E is a Banach space, then B_E is its (closed) unit ball and E' its dual. As usual $\mathcal{L}(E,F)$ denotes the Banach space of all (bounded and linear) operators from E into F endowed with the operator norm.

Throughout the paper by a Banach sequence space we mean a real Banach lattice E which is modeled on $\mathbb{N}$ and contains a sequence x with $\operatorname{supp} x = \mathbb{N}$. A Banach sequence space E is said to be symmetric, if $\|x\|_E = \|x^*\|_E$, where x^* as usual stands for the decreasing rearrangement of x, and E is called maximal provided its unit ball B_E is closed in the pointwise convergence topology on the space $\omega := \mathbb{R}^{\mathbb{N}}$ of all real sequences. The Köthe dual

$$E^{\times} := \{(x_n) \in \omega;\, \textstyle\sum_{n=1}^{\infty} |x_n y_n| < \infty \text{ for all } y \in E\}$$

equipped with the norm

$$\|x\|_{E^\times} := \sup_{y\in B_E} \textstyle\sum_{n=1}^{\infty}|x_n y_n|$$

is a maximal Banach sequence space which is symmetric provided E is. Recall that if E is separable, then E' is order isometrically isomorphic to $E^\times$ (in short $E' \cong E^\times$), and $E^{\times\times} = E$ if and only if E is maximal.

The fundamental function λ_E of a Banach sequence space E is defined to be

$$\lambda_E(n) := \|\textstyle\sum_{k=1}^{n} e_k\|_E$$

where (e_k) denotes as usual the standard unit vector basis in c_0. It is well-known that if E is symmetric, then we have $\lambda_E(n)\,\lambda_{E^\times}(n) = n$. By E^n we denote the span of the first n standard unit vectors in $\mathbb{R}^n$, endowed with the norm induced by E.

For two Banach sequence spaces F and E the space of multipliers $M(F,E)$ from F into E consists of all real sequences x such that the associated multiplication operator $(y_n) \mapsto (x_n\, y_n)$ is defined and bounded from F into E. Note that $M(F,E)$ equipped with the norm

$$\|x\|_{M(F,E)} := \sup\{\|xy\|_E\,;\, y \in B_F\},$$

is a Banach sequence space (symmetric provided F and E are). If E is maximal, then $M(F,E)$ is maximal.

As usual, define $E(X) := \{(x_i) \subset X;\, (\|x_i\|_X) \in E\}$, where E is a Banach sequence space and X a Banach space. Together with the norm $\|(x_i)\|_{E(X)} := \|(\|x_i\|_X)\|_E$ this space becomes a Banach space. The Banach space $E^n(X)$ is defined similarly.

Let $E = (\mathbb{R}^n, \|\cdot\|)$ be an n-dimensional Banach space. We say that E has enough symmetries in $\mathcal{O}(n)$ if there is a subgroup G of $GL(n)$ such that all $g \in G$ are isometries on ℓ_2^n as well as on E, and

$$\forall\, u \in \mathcal{L}(E) \text{ with } ug = gu \text{ for all } g \in G\ \exists\, c \in \mathbb{R} : u = c \cdot \mathrm{id}_E\,. \tag{1}$$

Basic examples of spaces with enough symmetries in the orthogonal group are the finite-dimensional spaces E^n, F^m associated to symmetric Banach sequence spaces E and F, and certain tensor products of spaces with enough symmetries, i.e., $X^n \otimes_\alpha Y^m$, where α is symmetrically invariant norm on $X^n \otimes Y^m$ and X^n, Y^m spaces with enough symmetries in $\mathcal{O}(n)$ and $\mathcal{O}(m)$, respectively. We will not recall this notion here, since we will only be dealing with the injective tensor product $X \otimes_\varepsilon Y$, and the projective tensor product $X \otimes_\pi Y$, where X and Y are Banach spaces. We will use very often the facts that $X \otimes_\varepsilon Y = \mathcal{L}(X', Y)$

and $X \otimes_\pi Y = \mathcal{N}(X', Y)$ (the space of all nuclear operators from X' into Y) whenever X and Y are finite-dimensional.

For the notion of type and cotype of Banach spaces, and convexity and concavity of Banach lattices, we refer to [19]. For a Banach space X we denote by $\mathbf{T_2}(X)$ and $\mathbf{C_2}(X)$ its type 2 and cotype 2 constant, respectively. For a Banach sequence space E we write $\mathbf{M^{(2)}}(E)$ and $\mathbf{M_{(2)}}(E)$ for its 2-convexity and 2-concavity constants, respectively. Clearly, this notation transfers to the sublattice E^n. It is known (see, e.g., [19]) that a Banach sequence space E is of cotype 2 if and only if it is 2-concave, and it is of type 2 if and only if it is 2-convex and of finite concavity.

2 Abstract tools

We will need the following basic properties.

1 Lemma. *Let E and F be symmetric Banach sequence spaces.*

(i) *If E is 2-convex or 2-concave, then*

$$\|E^n \hookrightarrow \ell_2^n\| \, \|\ell_2^n \hookrightarrow E^n\| \asymp \max\left(\frac{n^{1/2}}{\lambda_E(n)}, \frac{\lambda_E(n)}{n^{1/2}}\right). \tag{2}$$

(ii) *If E is 2-convex and of finite concavity, or 2-concave and of non-trivial convexity, then*

$$\mathbf{M^{(2)}}(E^n)\, \mathbf{M_{(2)}}(E^n) \asymp \max\left(\frac{n^{1/2}}{\lambda_E(n)}, \frac{\lambda_E(n)}{n^{1/2}}\right). \tag{3}$$

(iii) *If E is 2-concave and F is 2-convex, then*

$$\|F^n \hookrightarrow E^n\| = \lambda_{M(F,E)}(n) \asymp \frac{\lambda_E(n)}{\lambda_F(n)}. \tag{4}$$

(iv) *If E and F both are 2-convex or both are 2-concave, then*

$$\|M(F^n, E^n) \hookrightarrow \ell_2^n\| \asymp \frac{\sqrt{n}}{\|F^n \hookrightarrow E^n\|}. \tag{5}$$

Proof. For (i) see, e.g., [7, 3.5], and for (ii) [6, 6.2]. The upper estimate in (iii) follows from (i) by factorization through ℓ_2^n, the lower one is trivial. For (iv) observe that if E and F are both 2-convex, then the quasi-normed space $M(F,E)_{1/2} = M(F_{1/2}, E_{1/2})$ (for this notation see Section 7) admits a norm equivalent to the original quasi-norm, hence, it is 2-convex, and then (i) yields the formula. For E and F both 2-concave, the formula follows by duality. QED

Now we turn our attention to absolutely summing operators. Let E and F be symmetric Banach sequence spaces such that $F \hookrightarrow E$ with norm one. An operator $T : X \to Y$ is said to be (E,F)-summing if there is some constant $C > 0$ such that for each choice of $x_1, \dots, x_n \in X$ the following inequality holds:

$$\| \sum_{k=1}^{n} \|Tx_k\|_Y e_k \|_E \le C \sup_{\|x'\|_{X'} \le 1} \| \sum_{k=1}^{n} |x'(x_k)|\, e_k \|_F;$$

we denote the smallest constant C with this property by $\pi_{E,F}(T)$. The class $\Pi_{E,F}$ of all (E,F)-summing operators between Banach spaces together with the norm $\pi_{E,F}$ forms a Banach operator ideal. For $E = \ell_p$ and $F = \ell_q$, $1 \le p \le q < \infty$ we obtain the well-known ideal $(\Pi_{q,p}, \pi_{q,p})$ of all absolutely (q,p)-summing operators. In the case $p = q$, this ideal is usually denoted by (Π_p, π_p).

We will need the following formula for the absolutely 2-summing norm of identity maps due to [11]; for the purposes needed here, tensor products of symmetric spaces, this goes back to [3, p. 233].

2 Lemma. *Let E^N and F^N have enough symmetries in $\mathcal{O}(N)$. Then*

$$\pi_2(E^N \hookrightarrow F^N) = N^{1/2} \frac{\|\ell_2^N \hookrightarrow F^N\|}{\|\ell_2^N \hookrightarrow E^N\|}. \tag{6}$$

Our next tool needed is essentially due to [22] (see also [9]).

3 Lemma. *Let E and F be two symmetric Banach sequence spaces. Then $\Pi_{E,1} \subset \Pi_{M(F^\times,E),F^{\times\times}}$, and $\pi_{M(F^\times,E),F^{\times\times}}(T) \le \pi_{E,1}(T)$ for any $T \in \Pi_{E,1}$.*

The following result due to [8, 4.1] improves upon classical results of Hardy and Littlewood [16] as well as of Bennett [1], Carl [2], and Maligranda and Mastyło [20]. It allows applications to a wide range of topics, as e.g. strictly singular operators, approximation numbers, eigenvalues of compact operators and interpolation theory (see [7–10]). Here, it is crucial for most of our upper estimates for norms of tensor product identities.

4 Proposition. *Let E be a 2-concave symmetric Banach sequence space. Then the identity map $id : E \hookrightarrow \ell_2$ is $(E,1)$-summing.*

For some lower estimates and for volume ratio estimates, the ℓ-norm of an operator $T : \ell_2^n \to X$, defined by

$$\ell(T) := \left(\int_{\mathbb{R}^n} \| \sum_{k=1}^{n} g_i T e_i \|_X^2 \, d\lambda \right)^{1/2}$$

($g_1, \dots, g_n$ a collection of independent standard Gaussian variables), is crucial. We will frequently use the fact that $\ell(\mathrm{id}_{\ell_2^n}) = \sqrt{n}$.

5 Proposition. *Let E and F be symmetric Banach sequence spaces. Then*

$$\ell(\ell_2^{nm} \hookrightarrow E^n \otimes_\varepsilon F^m)$$

is asymptotically equivalent to

$\dfrac{\lambda_E(n)\lambda_F(m)}{\min(\sqrt{n},\sqrt{m})}$	*whenever E and F are 2-concave;*
$\max(\lambda_F(m),\lambda_E(n))$	*whenever E and F have type 2;*
$\lambda_F(m)\,\max(1,\lambda_E(n)/\sqrt{m})$	*whenever E has type 2 and F is 2-concave.*

Proof. This is a consequence of Chevet's inequality (see, e.g., [33, (43.2)]) which in our case says that

$$\ell(\mathrm{id}) \asymp \max(\|\ell_2^n \hookrightarrow E^n\|\,\ell(\ell_2^m \hookrightarrow F^m), \|\ell_2^m \hookrightarrow F^m\|\,\ell(\ell_2^n \hookrightarrow E^n)). \tag{7}$$

To conclude, use the well-known fact that $\ell(\ell_2^n \hookrightarrow E^n) \asymp \lambda_E(n)$ whenever E has finite concavity, together with (2). QED

To compute the ℓ-norm of embeddings into projective tensor products, we require the following formula for the type 2 constant thereof. It is a counterpart of [6, 9.1] for type 2 constants instead of cotype 2 constants.

6 Proposition. *Let E and F be symmetric Banach sequence spaces of non-trivial type, each either 2-concave or 2-convex. Then*

$$\mathbf{T_2}(E^n \otimes_\pi F^m) \asymp \min(\sqrt{n}\,\mathbf{M^{(2)}}(F^m), \sqrt{m}\,\mathbf{M^{(2)}}(E^n)). \tag{8}$$

Proof. Recall first that if X is a Banach lattice of non-trivial type, then X itself as well as its dual X' have finite concavity (see, e.g., [19, 1.f.3, 1.f.13, 1.e.17, 1.f.9]). We start with the lower estimate: from [6, 9.1] we know that

$$\mathbf{C_2}((E^n)' \otimes_\varepsilon (F^m)') \asymp \min(\sqrt{n}\,\mathbf{M_{(2)}}((E^n)'), \sqrt{m}\,\mathbf{M_{(2)}}((F^m)')),$$

hence, by duality of ε and π (see, e.g., [5]) as well as $\mathbf{T_2}$ and $\mathbf{C_2}$ (see, e.g., [33]),

$$\min(\sqrt{n}\,\mathbf{M^{(2)}}((E^n)'), \sqrt{m}\,\mathbf{M^{(2)}}((F^m)')) \prec \mathbf{C_2}((E^n)' \otimes_\pi (F^m)') \prec \mathbf{T_2}(E^n \otimes_\varepsilon F^m).$$

For the upper estimate, we will use the facts that

$$\|\,\mathrm{id} : E^n(F^m) \hookrightarrow E^n \otimes_\pi F^m\| \le \min(n/\lambda_E(n), K_G\,\mathbf{M^{(2)}}(F^m)\,m^{1/2}) \tag{9}$$

(use [6, 5.4, 6.3] and duality; here, K_G denotes Grothendieck's constant) and

$$\mathbf{T_2}(E^n(F^m)) \prec \mathbf{M^{(2)}}(E^n(F^m)) \prec \mathbf{M^{(2)}}(E^n)\,\mathbf{M^{(2)}}(F^m) \tag{10}$$

(for the first inequality, use the fact that $E(F)$ has finite concavity, hence, [19, 1.f.17] applies; for the second one use again duality plus the fact that $\mathbf{M}_{(\mathbf{2})}((E^n)'((F^m)')) \prec \mathbf{M}_{(\mathbf{2})}((E^n)')\,\mathbf{M}_{(\mathbf{2})}((F^m)'))$. Assume now that F is 2-convex. Then by the trivial factorization

$$\begin{array}{ccc} E^n \otimes_\pi F^m & \xrightarrow{\quad\text{id}\quad} & E^n \otimes_\pi F^m \\ & {\scriptstyle \text{id}}\searrow \quad \nearrow{\scriptstyle \text{id}} & \\ & E^n(F^m) & \end{array}$$

we obtain from (9) and (10)

$$\begin{aligned} \mathbf{T_2}(E^n \otimes_\pi F^m) &\le K_G\,\mathbf{M^{(2)}}(F^m)\,m^{1/2}\,\mathbf{T_2}(E^n(F^m)) \\ &\prec \mathbf{M^{(2)}}(F^m)\,m^{1/2}\,\mathbf{M^{(2)}}(E^n)\,\mathbf{M^{(2)}}(F^m) \\ &\prec m^{1/2}\,\mathbf{M^{(2)}}(E^n). \end{aligned}$$

If E is 2-concave, the same reasoning yields

$$\mathbf{T_2}(E^n \otimes_\pi F^m) \prec \frac{n}{\lambda_E(n)}\,\mathbf{M^{(2)}}(E^n)\,\mathbf{M^{(2)}}(F^m),$$

hence, we obtain from (3) that

$$\mathbf{T_2}(E^n \otimes_\pi F^m) \prec n^{1/2}\,\mathbf{M^{(2)}}(F^m).$$

This completes the proof. $\boxed{QED}$

The following estimates are a proper extension of [3, p. 247].

7 Proposition. *Let E and F be symmetric Banach sequence spaces of non-trivial type. Then*

$$\ell(\ell_2^{nm} \hookrightarrow E^n \otimes_\pi F^m)$$

is asymptotically equivalent to

$\min(\sqrt{n}, \sqrt{m})\,\lambda_E(n)\,\lambda_F(m)$	*whenever E and F are 2-convex;*
$\min(n\,\lambda_F(m), m\,\lambda_E(n))$	*whenever E and F are 2-concave;*
$\lambda_E(n)\,\min(m, \sqrt{n}\,\lambda_F(m))$	*whenever E is 2-convex and F is 2-concave.*

Proof. The upper estimate follows from the fact that for $S \in \mathcal{L}(\ell_2^N, Y)$ one has $\ell(S) \le \mathbf{T_2}(Y)\,\pi_2(S')$ (see, e.g., [33, p. 83]): by (6) we have

$$\pi_2(E^{\times n} \otimes_\varepsilon F^{\times m} \hookrightarrow \ell_2^{nm}) = \frac{\sqrt{nm}}{\|E^n \hookrightarrow \ell_2^n\|\,\|F^m \hookrightarrow \ell_2^m\|}.$$

Thus, (8) together with (2) gives the upper estimate. The lower one is a consequence of $\ell(\mathrm{id})\,\ell((\mathrm{id}')^{-1}) \geq nm$ (see, e.g., [3, Lemma 2]), and Proposition 5. QED

3 Norms of tensor product identities

Based on results of Hardy and Littlewood [16], Schütt in [27, Proposition 17] and [28, 3.4] proved the following norm estimates:

$$\|\ell_p^n \otimes_\varepsilon \ell_p^m \hookrightarrow \ell_2^{nm}\| \asymp \begin{cases} 1 & 1 \leq p \leq 4/3 \\ \min(n,m)^{3/2-2/p} & 4/3 \leq p \leq 2 \\ \sqrt{nm}\,\max(n,m)^{-1/p} & 2 \leq p \leq \infty, \end{cases}$$

and

$$\|\ell_p^n \otimes_\varepsilon \ell_q^n \hookrightarrow \ell_2^{n^2}\| \asymp \begin{cases} n^{1-1/\min(p,q)} & 2 \leq p,q \leq \infty \\ n^{1/2-\max(1/p+1/q-1,0)} & 1 \leq p \leq 2 \leq q \leq \infty \\ n^{3/2-1/p-1/q} & 1 \leq p,q \leq 2, 1/p+1/q \leq 3/2 \\ 1 & 1 \leq p,q \leq 2, 1/p+1/q \geq 3/2, \end{cases}$$

respectively.

We improve upon these results by considering injective tensor products $E^n \otimes_\varepsilon F^m$, where E and F are symmetric Banach sequence spaces which are each either 2-convex or 2-concave. The following is the key result for most of our asymptotic estimates in this paper, based on Proposition 4.

8 Proposition. *Let E and F be symmetric Banach sequence spaces and $m \leq n$. Then*

$$\|E^n \otimes_\varepsilon F^m \hookrightarrow \ell_2^{nm}\| \geq \|M(F^{\times m}, E^m) \hookrightarrow \ell_2^m\|. \tag{11}$$

If in addition E is 2-concave, then

$$\|E^n \otimes_\varepsilon F^m \hookrightarrow \ell_2^{nm}\| \asymp \|M(F^{\times m}, E^m) \hookrightarrow \ell_2^m\|. \tag{12}$$

PROOF. To obtain the lower estimate, simply use the factorization

$$M(F^{\times m}, E^m) \hookrightarrow \mathcal{L}(F^{\times m}, E^n) \hookrightarrow \mathcal{HS}(\ell_2^m, \ell_2^n)$$

(the latter denotes $L(\mathbb{R}^m, \mathbb{R}^n)$ equipped with the Hilbert–Schmidt norm). For the upper one recall Proposition 4: the inclusion map $E \hookrightarrow \ell_2$ is $(E,1)$-summing. Together with Lemma 3, this gives that $E \hookrightarrow \ell_2 \in \Pi_{M(F^\times,E),F^{\times\times}}$. Thus,

$$\|E^n \otimes_\varepsilon F^m \hookrightarrow M(F^{\times m}, E^m)(\ell_2^n)\| \asymp 1.$$

The conclusion now simply follows by factorization. QED

9 Theorem. *Let E and F be symmetric Banach sequence spaces and $m \leq n$. Then*

$$\|E^n \otimes_\varepsilon F^m \hookrightarrow \ell_2^{nm}\|$$

is asymptotically equivalent to

(i) $\dfrac{\sqrt{nm}}{\max(\lambda_E(n), \lambda_F(m))}$

(ii) $\dfrac{\sqrt{m}}{\|F^{\times m} \hookrightarrow E^m\|}$

(iii) $\dfrac{m^{3/2}}{\lambda_E(m)\,\lambda_F(m)}$

(iv) 1,

whenever

(i) *E and F have type 2;*

(ii) *E is 2-concave and F is 2-convex;*

(iii) *E and F are 2-concave and $M(F^\times, E)$ is 2-convex;*

(iv) *E and F are 2-concave and $M(F^\times, E)$ is contained in ℓ_2,*

respectively.

Proof. (i) The upper estimate easily follows from factorization through the spaces $\ell_\infty^n \otimes_\varepsilon F^m$ and $E^n \otimes_\varepsilon \ell_\infty^m$, respectively, together with (2); note that here we need 2-convexity only. For the lower estimate use the fact that $\|\operatorname{id}\| \geq \sqrt{nm}/\ell(\operatorname{id}^{-1})$, and Proposition 5.

(ii) This follows from (12) together with (5).

(iii) This is a consequence of (12) and (4) together with (2).

(iv) is clear by (12). QED

4 Application I: Banach–Mazur distances

For two Banach spaces X and Y isomorphic to each other, the Banach–Mazur distance of X and Y is defined by

$$d(X,Y) := \inf\{\|T\|\,\|T^{-1}\|;\ T : X \to Y \text{ invertible}\}.$$

For an N-dimensional normed space X with enough symmetries in $\mathcal{O}(N)$, let us denote by id the identity map $\ell_2^N \hookrightarrow X$ and by id^{-1} its inverse. It is known (see, e.g., [33, 16.4]) that for such a space X,

$$d(X, \ell_2^N) = \| \mathrm{id} \| \, \| \mathrm{id}^{-1} \|. \tag{13}$$

Since for two symmetric Banach sequence spaces E and F

$$\|\ell_2^{nm} \hookrightarrow E^n \otimes_\varepsilon F^m\| = \|\ell_2^n \hookrightarrow E^n\| \, \|\ell_2^m \hookrightarrow F^m\|,$$

we immediately obtain the following results from those of the previous section:

10 Theorem. *Let E and F be symmetric Banach sequence spaces and $m \le n$. Then*

$$d(E^n \otimes_\varepsilon F^m, \ell_2^{nm})$$

is asymptotically equivalent to

(i) $\dfrac{\sqrt{nm}}{\max(\lambda_E(n), \lambda_F(m))}$

(ii) $\left(\dfrac{m}{n}\right)^{1/2} \dfrac{\lambda_E(n)}{\|F^{\times m} \hookrightarrow E^m\|}$

(iii) $\dfrac{m}{\sqrt{n}} \dfrac{\lambda_E(n)}{\lambda_E(m)}$

(iv) $\dfrac{\lambda_E(n)\, \lambda_F(m)}{\sqrt{nm}}$

whenever the conditions (i)–(iv) *from Theorem 9 are satisfied, respectively.*

Note that in each of the above cases,

$$d(E^n \otimes_\varepsilon F^m, \ell_2^{nm}) \succ \min(\sqrt{n}, \sqrt{m}). \tag{14}$$

This is true for any choice of normed spaces E^n and F^m with enough symmetries in $\mathcal{O}(n)$ and $\mathcal{O}(m)$, respectively. Indeed, Chevet's formula (7) and $\ell(\ell_2^k \hookrightarrow G^k) \le \sqrt{k}\, \|\ell_2^k \hookrightarrow G^k\|$ for any k-dimensional normed space G^k give us

$$\ell(\mathrm{id}) \le K_C \max(\sqrt{n}, \sqrt{m})\, \|\ell_2^n \hookrightarrow E^n\| \, \|\ell_2^m \hookrightarrow F^m\|, \tag{15}$$

where $K_C > 0$ denotes the constant occurring in (7). Hence, since $\| \mathrm{id}^{-1} \| \ge \sqrt{nm}/\ell(\mathrm{id})$, by (13)

$$\begin{aligned} d(E^n \otimes_\varepsilon F^m, \ell_2^{nm}) &= \| \mathrm{id} \| \, \| \mathrm{id}^{-1} \| \\ &\ge \frac{\sqrt{nm}\, \|\ell_2^n \hookrightarrow E^n\| \, \|\ell_2^m \hookrightarrow F^m\|}{\ell(\mathrm{id})} \ge K_C^{-1} \min(\sqrt{n}, \sqrt{m}), \end{aligned}$$

the claim.

5 Application II: approximation numbers

In the following we will show how results on Banach–Mazur distances and norms of tensor product identities can be used to obtain asymptotically optimal estimates for the approximation numbers of certain tensor product identities. Recall that for any linear and bounded operator $T : X \to Y$ between Banach spaces X and Y its k-th approximation number, $k \in \mathbb{N}$, is defined by

$$a_k(T) := \inf\{\|T - S\|;\ S : X \to Y \text{ has rank} < k\}.$$

In [3, p. 250] it was shown, amongst other, that

$$a_k(\ell_2^{n^2} \hookrightarrow \ell_p^n \otimes_\varepsilon \ell_q^n) \asymp \max\left(\frac{1}{\|\operatorname{id}^{-1}\|}, \left(\frac{n^2-k+1}{n^2}\right)^{1/2} \|\operatorname{id}\|\right) \tag{16}$$

whenever $2 \le p, q < \infty$, or $1 \le p, q \le 2$ and $1/p + 1/q \le 3/2$. One ingredient of their proof is the general fact that if E and F are symmetric Banach sequence spaces and α a symmetrically invariant norm on the tensor product $E^n \otimes_\alpha F^m$ (for this notion we refer to the same paper; e.g., the injective and the projective norms have that property), then for all $1 \le k \le [nm/2]$

$$a_k(\ell_2^{nm} \hookrightarrow E^n \otimes_\alpha F^m) \asymp \|\operatorname{id}\|. \tag{17}$$

Analyzing their proof and using (6) instead of their proposition [3, p. 233], one can see that (17) also holds for E^n and F^m being spaces with enough symmetries in $\mathcal{O}(n)$ and $\mathcal{O}(m)$, respectively, with only absolute constants involved. For k larger than $[nm/2]$, they provided lower and upper estimates which then gave (16). For the injective norm, we will see that (17) can be extended in a certain way to $k \le nm - \max(n,m) + 1$, so the crucial point is to find appropriate estimates for k larger than $nm - \max(n,m)+1$. For this task, results on Banach–Mazur distances and norms of tensor product identities can be used.

11 Proposition. *Let $m, n \ge 2$, and E^n and F^m have enough symmetries in $\mathcal{O}(n)$ and $\mathcal{O}(m)$, respectively.*

(i) *For all $1 \le k \le nm - \max(n,m) + 1$,*

$$a_k(\ell_2^{nm} \hookrightarrow E^n \otimes_\varepsilon F^m) \asymp \left(\frac{nm-k+1}{nm}\right)^{1/2} \|\operatorname{id}\|.$$

(ii) *If $d(E^n \otimes_\varepsilon F^m, \ell_2^{nm}) \overset{\mathcal{I}}{\asymp} \min(\sqrt{n}, \sqrt{m})$ for $\mathcal{I} \subset \mathbb{N}_{>1} \times \mathbb{N}_{>1}$, then for all $(n,m) \in \mathcal{I}$ and $nm - \max(n,m) + 1 \le k \le nm$*

$$a_k(\ell_2^{nm} \hookrightarrow E^n \otimes_\varepsilon F^m) \overset{\mathcal{I}}{\asymp} \frac{1}{\|\operatorname{id}^{-1}\|} \overset{\mathcal{I}}{\asymp} \frac{\|\operatorname{id}\|}{\min(\sqrt{n}, \sqrt{m})}.$$

In particular, this holds whenever $\mathcal{I} = \mathcal{D} \setminus \{(1,1)\}$, *E and F are* 2*-concave symmetric Banach sequence spaces and* $M(F^\times, E)$ *is* 2*-convex.*

(iii) *If E and F are symmetric Banach sequence spaces with finite concavity such that* $d(E^n, \ell_2^n) \asymp \sqrt{n}/\lambda_E(n)$, $d(F^m, \ell_2^m) \asymp \sqrt{m}/\lambda_F(m)$, *then for all* $nm - \max(n,m) + 1 \le k \le nm$

$$a_k(\ell_2^{nm} \hookrightarrow E^n \otimes_\varepsilon F^m) \asymp \frac{\max(\lambda_E(n), \lambda_F(m), (nm-k+1)^{1/2})}{\sqrt{nm}}.$$

In particular, this holds whenever E and F have type 2.

Proof. (i) By what has been said above, the statement is clear for the case $1 \le k \le [nm/2]$. It was shown in [3, p. 249] that for for all $1 \le k \le nm$,

$$\max\left(\frac{1}{\|\,\mathrm{id}^{-1}\,\|}, \left(\frac{nm-k+1}{nm}\right)^{1/2} \|\,\mathrm{id}\,\|\right) \le a_k(\mathrm{id}), \tag{18}$$

which yields the lower estimate in (i). Also, in [3, p. 250] it was proved (based on [14, 2.2]) that for $[nm/2] \le k \le nm$ the following upper estimate holds (also valid within the extended framework of spaces with enough symmetries):

$$a_k(\mathrm{id}) \prec \max\left(\frac{\ell(\mathrm{id})}{\sqrt{nm}}, \left(\frac{nm-k+1}{nm}\right)^{1/2} \|\,\mathrm{id}\,\|\right) \tag{19}$$

Since for $1 \le k \le nm - \max(n,m) + 1$ by (15),

$$\left(\frac{nm-k+1}{nm}\right)^{1/2} \|\,\mathrm{id}\,\| \ge \frac{\max(\sqrt{n}, \sqrt{m})\, \|\ell_2^n \hookrightarrow E^n\|\, \|\ell_2^m \hookrightarrow F^m\|}{\sqrt{nm}} \ge \frac{\ell(\mathrm{id})}{\sqrt{nm}},$$

the upper estimate in (i) follows.

(ii) By (18) and (19), it suffices to show that

$$\max\left(\frac{\ell(\mathrm{id})}{\sqrt{nm}}, \left(\frac{nm-k+1}{nm}\right)^{1/2} \|\,\mathrm{id}\,\|\right) \overset{\mathcal{I}}{\prec} \frac{1}{\|\,\mathrm{id}^{-1}\,\|}.$$

Indeed, on one hand, by (15) and the assumption,

$$\frac{1}{\|\,\mathrm{id}^{-1}\,\|} = \frac{\|\,\mathrm{id}\,\|}{d(E^n \otimes_\varepsilon F^m, \ell_2^{nm})} \overset{\mathcal{I}}{\succ} \frac{\|\ell_2^n \hookrightarrow E^n\|\, \|\ell_2^m \hookrightarrow F^m\|}{\min(\sqrt{n}, \sqrt{m})} \succ \frac{\ell(\mathrm{id})}{\sqrt{nm}}.$$

On the other hand, since $k \ge nm - \max(n,m) + 1$,

$$\begin{aligned}\left(\frac{nm-k+1}{nm}\right)^{1/2} \|\,\mathrm{id}\,\| &\le \frac{\|\ell_2^n \hookrightarrow E^n\|\, \|\ell_2^m \hookrightarrow F^m\|}{\min(\sqrt{n}, \sqrt{m})} \\ &= \frac{d(E^n \otimes_\varepsilon F^m, \ell_2^{nm})}{\min(\sqrt{n}, \sqrt{m})\, \|\,\mathrm{id}^{-1}\,\|} \overset{\mathcal{I}}{\prec} \frac{1}{\|\,\mathrm{id}^{-1}\,\|},\end{aligned}$$

the claim.

(iii) Again by (18) and (19), it suffices to show that

$$\ell(\mathrm{id}) \prec \max(\lambda_E(n), \lambda_F(m)) \prec \sqrt{nm}/\|\,\mathrm{id}^{-1}\,\|.$$

The assumptions on the Banach–Mazur distances imply that

$$\|\ell_2^n \hookrightarrow E^n\| \asymp \|\ell_2^m \hookrightarrow F^m\| \asymp 1,\; \|E^n \hookrightarrow \ell_2^n\| \asymp \frac{\sqrt{n}}{\lambda_E(n)},\; \|F^m \hookrightarrow \ell_2^m\| \asymp \frac{\sqrt{m}}{\lambda_F(m)}.$$

Thus, Chevet's formula (7) together with the concavity assumptions give

$$\ell(\mathrm{id}) \prec \max(\lambda_E(n), \lambda_F(m)).$$

By factorization through $\ell_\infty^n \otimes_\varepsilon F^m$ and $E^n \otimes_\varepsilon \ell_\infty^m$ one gets that

$$\sqrt{nm}/\|\,\mathrm{id}^{-1}\,\| \succ \max(\lambda_E(n), \lambda_F(m)),$$

which finishes the proof. QED

6 Application III: volume ratio estimates

Next we consider volume ratios. Let X be $\mathbb{R}^N$ equipped with some norm. The volume ratio of X is defined by

$$\mathrm{vr}(X) := \inf_{\mathcal{E} \subset B_X} \left(\frac{\mathrm{vol}(B_X)}{\mathrm{vol}(\mathcal{E})} \right)^{1/N},$$

where $\mathcal{E}$ is an ellipsoid. Szarek and Tomczak-Jaegermann in [30] proved that $\mathrm{vr}(\ell_2^n \otimes_\pi \ell_p^n) \asymp 1$ whenever $1 \le p \le 2$. Schütt in [28, 3.1] then gave the whole asymptotically optimal order of $\mathrm{vr}(\ell_p^n \otimes_\pi \ell_q^n)$:

$$\mathrm{vr}(\ell_p^n \otimes_\pi \ell_q^n) \asymp \begin{cases} 1 & 1 \le p, q \le 2, \text{ or } 2 \le p, q \le \infty \\ & \text{and } 1/p + 1/q \ge 1/2 \\ n^{1/2 - 1/q + \min(1/q + 1/p - 1, 0)} & 1 \le p \le 2 \le q \\ n^{1/2 - 1/p - 1/q} & 1/p + 1/q \le 1/2. \end{cases}$$

We will give analogs of his formulas for the general setting of projective tensor products of n-th sections of symmetric Banach sequence spaces. The crucial tool for this is the following:

12 Proposition. *There exists an absolute constant $K_{vr} > 0$ such that for all dimensions N and any space $X = (\mathbb{R}^N, \|\cdot\|_X)$ with enough symmetries in $\mathcal{O}(N)$,*

$$\frac{1}{K_{vr}} \frac{\sqrt{N}\, \|\ell_2^N \hookrightarrow X\|}{\ell(\ell_2^N \hookrightarrow X)} \leq \mathrm{vr}(X) \leq K_{vr} \frac{\ell(\ell_2^N \hookrightarrow X')\, \|\ell_2^N \hookrightarrow X\|}{\sqrt{N}}. \tag{20}$$

Proof. By [11], the ellipsoid of maximal volume contained in the unit ball B_X is $\|\ell_2^N \hookrightarrow X\|^{-1} B_{\ell_2^N}$. Hence, by [25, 7.2],

$$\mathrm{vr}(X) = \|\ell_2^N \hookrightarrow X\| \left(\frac{\mathrm{vol}(B_X)}{\mathrm{vol}(B_{\ell_2^N})}\right)^{1/N} \asymp \|\ell_2^N \hookrightarrow X\| \left(\frac{\mathrm{vol}(B_{\ell_2^N})}{\mathrm{vol}(B_{X'})}\right)^{1/N}.$$

Now Urysohn's inequality (see, e.g., [25, 3.14]), in this particular case given as

$$\left(\frac{\mathrm{vol}(B_X)}{\mathrm{vol}(B_{\ell_2^N})}\right)^{1/N} \prec \frac{\ell(\ell_2^N \hookrightarrow X')}{\sqrt{N}} \quad \text{and} \quad \left(\frac{\mathrm{vol}(B_{\ell_2^N})}{\mathrm{vol}(B_{X'})}\right)^{1/N} \succ \frac{\sqrt{N}}{\ell(\ell_2^N \hookrightarrow X)},$$

yields the desired result. QED

Now we are prepared to state the following general estimates for the volume ratio of projective tensor products:

13 Theorem. *Let E and F be symmetric Banach sequence spaces and $m \leq n$. Then*

$$\mathrm{vr}(E^n \otimes_\pi F^m)$$

is asymptotically equivalent to

(i) 1

(ii) $\dfrac{\sqrt{n}}{\lambda_E(n)\, \|F^m \hookrightarrow E^{\times m}\|}$

(iii) $\left(\dfrac{n}{m}\right)^{1/2} \dfrac{\lambda_E(m)}{\lambda_E(n)}$

(iv) $\dfrac{\sqrt{n}}{\lambda_E(n)\, \lambda_F(m)}$

whenever

(i) *E and F are 2-concave and of non-trivial convexity;*

(ii) *E has type 2, and F is 2-concave and of non-trivial convexity;*

(iii) *E and F have type 2 and $M(F, E^\times)$ is 2-convex.*

(iv) *E and F have type 2 and $M(F, E^\times)$ is contained in ℓ_2,*

respectively.

Proof. Everything follows from the dualized version of Theorem 9 and (20) together with Proposition 5 and Proposition 7. $\boxed{QED}$

Note that in all cases from above, and in general for two spaces E^n and F^m with enough symmetries in $\mathcal{O}(n)$ and $\mathcal{O}(m)$, respectively,

$$\mathrm{vr}(E^n \otimes_\pi F^m) \leq K_C\, K_{vr}\, \max(\sqrt{n}, \sqrt{m}). \tag{21}$$

Indeed, (20), (15) and (13) yield

$$\begin{aligned}
&\mathrm{vr}(E^n \otimes_\pi F^m)\\
&\qquad \leq K_{vr}\, \frac{\ell(\ell_2^{nm} \hookrightarrow (E^n)' \otimes_\pi (F^m)')\, \|\ell_2^{nm} \hookrightarrow E^n \otimes_\pi F^m\|}{\sqrt{nm}}\\
&\qquad \leq K_C\, K_{vr}\, \frac{\max(\sqrt{n}, \sqrt{m})\, \|E^n \hookrightarrow \ell_2^n\|\, \|F^m \hookrightarrow \ell_2^m\|\, \|\ell_2^{nm} \hookrightarrow E^n \otimes_\pi F^m\|}{\sqrt{nm}}\\
&\qquad = K_C\, K_{vr}\, \frac{\max(\sqrt{n}, \sqrt{m})\, d(E^n \otimes_\pi F^m, \ell_2^{nm})}{\sqrt{nm}} \leq K_C\, K_{vr}\, \max(\sqrt{n}, \sqrt{m}).
\end{aligned}$$

Compared to this, the volume ratio of injective tensor products is rather large: for any choice of symmetric Banach sequence spaces E and F,

$$\mathrm{vr}(E^n \otimes_\varepsilon F^m) \geq (K_C\, K_{vr})^{-1} \min(\sqrt{n}, \sqrt{m}). \tag{22}$$

This also follows from (20), (15) and (13):

$$\begin{aligned}
\mathrm{vr}(E^n \otimes_\varepsilon F^m) &\geq \frac{\sqrt{nm}\, \|\ell_2^n \hookrightarrow E^n\|\, \|\ell_2^m \hookrightarrow F^m\|}{K_{vr}\, \ell(\ell_2^{nm} \hookrightarrow E^n \otimes_\varepsilon F^m)}\\
&\succ \frac{\sqrt{nm}\, \|\ell_2^n \hookrightarrow E^n\|\, \|\ell_2^m \hookrightarrow F^m\|}{K_{vr}\, K_C\, \max(\sqrt{n}, \sqrt{m})\, \|\ell_2^n \hookrightarrow E^n\|\, \|\ell_2^m \hookrightarrow F^m\|}\\
&= (K_{vr}\, K_C)^{-1} \min(\sqrt{n}, \sqrt{m}).
\end{aligned}$$

We illustrate this fact by giving the following counterpart to the preceding theorem, which immediately follows from (20) and the results on ℓ-norms.

14 Theorem. *Let E and F be symmetric Banach sequence spaces of non-trivial type. Then*

$$\mathrm{vr}(E^n \otimes_\varepsilon F^m)$$

is asymptotically equivalent to

(i) $\dfrac{\sqrt{nm}}{\max(\lambda_E(n), \lambda_F(m))}$

(ii) $\sqrt{n}\, \min\left(1, \dfrac{\sqrt{m}}{\lambda_E(n)}\right)$

(iii) $\min(\sqrt{n}, \sqrt{m})$

whenever

(i) *E and F are 2-convex;*

(ii) *E is 2-convex and F is 2-concave;*

(iii) *E and F are 2-concave,*

respectively.

We continue with a remark about a connection of the minimality of Banach–Mazur distances and volume ratios.

15 Proposition. *For $(n,m) \in \mathcal{I} \subset \mathbb{N} \times \mathbb{N}$ let E^n and F^m be spaces with enough symmetries in $\mathcal{O}(n)$ and $\mathcal{O}(m)$, respectively. Then the following are equivalent:*

(i) $d(E^n \otimes_\varepsilon F^m, \ell_2^{nm}) \overset{\mathcal{I}}{\asymp} \min(\sqrt{n}, \sqrt{m})$*;*

(ii) $\mathrm{vr}(E^n \otimes_\varepsilon F^m) \overset{\mathcal{I}}{\asymp} \min(\sqrt{n}, \sqrt{m})$ *and* $\mathrm{vr}((E^n)' \otimes_\pi (F^m)') \overset{\mathcal{I}}{\asymp} 1$.

Proof. The equations (20) and (13) show that

$$\mathrm{vr}(E^n \otimes_\varepsilon F^m)\ \mathrm{vr}((E^n)' \otimes_\pi (F^m)') \asymp d(E^n \otimes_\varepsilon F^m, \ell_2^{nm}). \tag{23}$$

Then it is clear that (ii) implies (i), and the converse follows by (20). QED

Let us also indicate a connection of the minimality of volume ratios of projective tensor products and the maximality of projection constants thereof. Recall that the projection constant $\lambda(X)$ of a finite-dimensional normed space X is defined as

$$\lambda(X) := \sup\{\lambda(i(X), Z);\ i : X \hookrightarrow Z \text{ is an isometric embedding into } Z\},$$

where for a subspace Y of a Banach space Z,

$$\lambda(Y, Z) := \inf\{\|P\|;\ P \in \mathcal{L}(Z, Z) \text{ a projection onto } Y\}.$$

For injective tensor products, the projection constant is the product of those of the underlying spaces: $\lambda(X \otimes_\varepsilon Y) = \lambda(X)\,\lambda(Y)$ whenever X and Y are finite-dimensional normed spaces (see, e.g., [5, 34.6]). This is not true for projective tensor products, which follows, e.g., from the lower bound provided below. However, the following holds:

16 Proposition. *For $(n,m) \in \mathcal{I} \subset \mathbb{N} \times \mathbb{N}$ let E^n and F^m be spaces with enough symmetries in $\mathcal{O}(n)$ and $\mathcal{O}(m)$, respectively. Then:*

(i) $\lambda(E^n \otimes_\pi F^m) \geq C \min(\sqrt{n}, \sqrt{m})$ *for some absolute constant $C > 0$.*

(ii) $\mathrm{vr}(E^n \otimes_\pi F^m) \overset{\mathcal{I}}{\asymp} 1$ *implies* $\lambda(E^n \otimes_\pi F^m) \overset{\mathcal{I}}{\asymp} \sqrt{nm}$.

Proof. Recall the definition of the cubic ratio of an N-dimensional normed space X:

$$\mathrm{evr}(X, Q_N) := \inf_{\mathcal{P} \supset B_X} \left(\frac{\mathrm{vol}(\mathcal{P})}{\mathrm{vol}(B_X)} \right)^{1/N},$$

where $\mathcal{P}$ is a parallelotope. It was shown in [15, Corollary 2] that $\mathrm{evr}(X, Q_N) \asymp \sqrt{N}/\mathrm{vr}(X)$. Hence, by [15, Corollary 7],

$$\lambda(X) \geq \mathrm{evr}(X, Q_N) \succ \sqrt{N}/\mathrm{vr}(X).$$

Now (21) gives (i). The general estimate $\lambda(X) \leq \sqrt{N}$ (see, e.g., [33, 9.12]) and the assumption on the volume ratio yield (ii). QED

17 Corollary. *Let E and F be symmetric Banach sequence spaces and $\mathcal{I} \subset \mathbb{N} \times \mathbb{N}$. Then*

$$\lambda(E^n \otimes_\pi F^m) \overset{\mathcal{I}}{\asymp} \sqrt{nm}$$

whenever one of the following holds:

(i) $\mathcal{I} = \mathbb{N} \times \mathbb{N}$, *and E and F are 2-concave and of non-trivial convexity;*

(ii) $\mathcal{I} = \mathcal{D}$, *and E, F and $M(F^\times, E)$ are 2-convex.*

Proof. Everything follows from the above proposition and Theorem 13; simply note that for $m = n$ the type 2 assumption occurring in Theorem 13 (iii) can be omitted. QED

7 Case study I: powers of a sequence space

We continue with a case study where we apply our abstract results to powers of sequence spaces E_p, which in particular leads to the known results for ℓ_p-spaces (cf. [28]) and new results for Lorentz sequence spaces $d(w,p)$.

Let E be a symmetric Banach sequence space. For $1 \leq p < \infty$, the space E_p consists of all $x \in \ell_\infty$ such that $|x|^p \in E$. Introducing the quasi-norm

$$\|x\|_{E_p} := \||x|^p\|_E^{1/p},$$

this again forms a symmetric Banach sequence space, which is always p-convex. When $E = \ell_1$, then $E_p = \ell_p$. Another prominent example is $E = d(w,1)$, which gives $E_p = d(w,p)$. Here, for $1 \leq p < \infty$, the symmetric Banach sequence space $d(w,p)$ is defined as follows: Let (w_n) be a non-increasing sequence of positive real numbers such that $w_1 = 1$, $\lim_{n\to\infty} w_n = 0$ and $\sum_{n=1}^\infty w_n = \infty$ (called a Lorentz function). The symmetric Banach sequence space of all sequences of scalars $x = (x_n)$ for which $\|x\| := \sup_\pi (\sum_{n=1}^\infty |x_{\pi(n)}|^p w_n)^{1/p} < \infty$, where π ranges over all the permutations of the integers, is denoted by $d(w,p)$ and called a Lorentz sequence space. It is easy to see that $\lambda_{d(w,p)}(n) = (\sum_{i=1}^n w_i)^{1/p}$. Reisner in [26] showed that $d(w,1)$ is r-concave, $1 < r < \infty$, whenever the weight (w_n) is r'-regular, that is, $\sum_{i=1}^n w_i^{r'} \asymp n\, w_n^{r'}$.

We list up some well-known facts needed for our purposes.

18 Lemma. *Let E be a symmetric Banach sequence space and $1 \leq p < \infty$.*

(i) $\lambda_{E_p}(n) = \lambda_E(n)^{1/p}$.

(ii) *E_p is 2-convex whenever $p \geq 2$, and for $1 \leq p \leq 2$ it is 2-concave whenever E is $2/p$-concave.*

(iii) *$M(E_q^n, E_p^n) = E_r^n$ whenever $1 \leq p < q < r < \infty$ such that $1/r = 1/p - 1/q$, and $M(E_q^n, E_p^n) = \ell_\infty$ whenever $1 \leq q \leq p < \infty$.*

Proof. (i) is easy, for (ii) and (iii) see, e.g., [4, Lemma 2] and [21, Theorem 5]. QED

Using the above facts and our abstract results from the previous sections, we are now able to formulate the following results. We omit detailed formulations for Lorentz sequence spaces, since these would be very similar to the more general case presented here. To translate, simply use the facts mentioned right after the definition of Lorentz sequence spaces. In particular, for $1 < r \leq 2$, the space $E = d(w,1)$ is $2/r$-concave whenever the weight (w_n) is $2/(2-r)$-regular, and it has finite concavity whenever (w_n) is at least 1-regular.

19 Example. Let E be a symmetric Banach sequence space. Then

$$\|\mathcal{L}(E_q^n, E_p^n) \hookrightarrow \ell_2^{n^2}\|$$

is asymptotically equivalent to

(i) 1

(ii) $\sqrt{n}$

(iii) $n^{1/2}\,\lambda_E(n)^{1/q-1/p}$

(iv) $\min(\lambda_E(n)^{1/q}, n/\lambda_E(n)^{1/p})$

whenever

(i) $1 \le p \le 2 \le q < \infty$, $1/r := 1/p - 1/q > 1/2$ and E is $2/r$-concave;

(ii) $2 \le q < p < \infty$;
$1 \le q \le p \le 2$ and E is $2/q$-concave;

(iii) $2 \le p < q < \infty$;
$1 \le p \le 2 \le q < \infty$ and $1/p - 1/q \le 1/2$;
$1 \le p < q \le 2$ and E is $2/p$-concave;

(iv) $1 < q \le 2 \le p < \infty$ and E is $2/q$-concave,

respectively.

20 Example. Let E be a symmetric Banach sequence space. Then

$$d(\mathcal{L}(E_q^n, E_p^n), \ell_2^{n^2})$$

is asymptotically equivalent to

(i) $\sqrt{n}$

(ii) $\lambda_E(n)^{1/p-1/q}$

(iii) $\max(\lambda_E(n)^{1/q}, n/\lambda_E(n)^{1/p})$

(iv) $\min(\lambda_E(n)^{1/q}, n/\lambda_E(n)^{1/p})$

whenever

(i) $2 \le q \le p < \infty$;
$1 \le q \le p \le 2$ and E is $2/p$-concave;
$1 \le p \le 2 \le q < \infty$, E is $2/p$-concave and $1/r := 1/p - 1/q \le 1/2$;

(ii) $1 \le p \le 2 \le q < \infty$, $1/r := 1/p - 1/q > 1/2$ and E is $2/r$-concave;

(iii) $2 \le p < q < \infty$;
$1 \le p < q \le 2$ and E is $2/q$-concave;

(iv) $1 < q \le 2 \le p < \infty$ and E is $2/q$-concave,

respectively.

21 Example. Let E be a symmetric Banach sequence space. Then

$$\mathrm{vr}(\mathcal{N}(E_p^n, E_q^n))$$

is asymptotically equivalent to

(i) 1

(ii) $\dfrac{n^{1/2}}{\lambda_E(n)^{1/\min(p,q)}}$

(iii) $\dfrac{\lambda_E(n)^{1/\max(p,q)}}{n^{1/2}}$

(iv) $\dfrac{\lambda_E(n)^{1/p-1/q}}{n^{1/2}}$

whenever

(i) $1 < q < 2 \le p < \infty$ and E is $2/q$-concave;
$1 \le p < 2 \le q < \infty$, $1/p - 1/q \le 1/2$ and E is $2/p$-concave;

(ii) $2 \le p, q < \infty$ and E has finite concavity;

(iii) $1 < p, q \le 2$ and E is $2/\max(p,q)$-concave;

(iv) $1 < p < 2 \le q < \infty$, $1/r := 1/p - 1/q > 1/2$ and E is $2/r$-concave,

respectively.

8 Case study II: Lorentz sequence spaces

The following lemma gathers several facts for Lorentz sequence spaces, some of them already folklore and others easy to prove. However, for the convenience of the reader, we give some hints for the proofs.

22 Lemma. *Let $1 < p_0, p_1 < \infty$ and $1 \le q_0, q_1 \le \infty$.*

(i) *If $p_0 \ne p_1$, or $p_0 = p_1$ and $q_0 \le q_1$, then*

$$\|\ell_{p_0,q_0}^n \hookrightarrow \ell_{p_1,q_1}^n\| \asymp n^{\max(1/p_1 - 1/p_0, 0)}. \tag{24}$$

If $p_0 = p_1$ and $q_0 > q_1$, then

$$\|\ell_{p_0,q_0}^n \hookrightarrow \ell_{p_1,q_1}^n\| \asymp (1 + \log n)^{1/q_1 - 1/q_0}. \tag{25}$$

(ii) *If* $p_0 > p_1$*, then for* $1/p = 1/p_1 - 1/p_0$ *and* $1/q = \max(1/q_1 - 1/q_0, 0)$*,*

$$M(\ell_{p_0,q_0}, \ell_{p_1,q_1}) = \ell_{p,q}. \tag{26}$$

If $p_0 < p_1$*, or* $p_0 = p_1$ *and* $q_0 \le q_1$*, then*

$$M(\ell_{p_0,q_0}, \ell_{p_1,q_1}) = \ell_\infty. \tag{27}$$

(iii) *For* $1 < p < \infty$ *and* $1 \le q \le \infty$*,*

$$\ell(\ell_2^n \hookrightarrow \ell_{p,q}^n) \asymp n^{1/p}. \tag{28}$$

PROOF. (i) The lower estimates in (24) are trivial. For $p_0 < p_1$ or $p_0 = p_1$ and $q_0 \le q_1$, the space ℓ_{p_0,q_0}^n is contained in ℓ_{p_1,q_1}. The remaining cases follow by Hölder's Inequality (when $q_0 > q_1$) and interpolation. (25) follows by considering appropriate vectors with logarithmic weights for the lower estimates (see, e.g., [23, 2.1.12]) and again Hölder's inequality for the upper ones.

(ii) Obviously, (27) holds since in those cases, ℓ_{p_0,q_0} is contained in ℓ_{p_1,q_1}. In the case $p_0 > p_1$ and $q_0 \ge q_1$, (26) is essentially known (see, e.g., [23, 2.1.13]), and for $q_0 < q_1$ it follows by factorization that

$$\ell_{p,\infty} = M(\ell_{p_0,q_1}, \ell_{p_1,q_1}) \hookrightarrow M(\ell_{p_0,q_0}, \ell_{p_1,q_1}).$$

On the other hand, (24) implies that $\lambda_{M(\ell_{p_0,q_0},\ell_{p_1,q_1})}(n) \asymp n^{1/p}$, hence we get that $M(\ell_{p_0,q_0}, \ell_{p_1,q_1}) \hookrightarrow \ell_{p,\infty}$.

(iii) For $q < \infty$, this is clear since then $\ell_{p,q}$ has finite concavity. If $q = \infty$, then the upper estimate follows by factorization from the case $q < \infty$, and the lower one by comparison of Gauss and Rademacher averages. QED

In the case of tensor products of Lorentz sequence spaces, the restrictions in Proposition 5 and Proposition 7 can be essentially weakened. This allows us to cover the whole range of possible indices for the injective tensor product.

23 Lemma. *Let* $1 < p_0 \le p_1 < \infty$ *and* $1 \le q_0, q_1 \le \infty$*. Then*

$$\ell(\ell_2^{n^2} \hookrightarrow \ell_{p_0,q_0}^n \otimes_\varepsilon \ell_{p_1,q_1}^n)$$

is asymptotically equivalent to

(i) n^{1/p_0}

(ii) $n^{1/p_0 + 1/p_1 - 1/2}$

(iii) $(1 + \log n)^{1/\min(q_0,q_1) - 1/2}\, n^{1/2}$

whenever

(i) $p_1 > 2$;

(ii) $p_0 < p_1 \leq 2$, *or* $p_1 < 2$, *or* $p_0 = p_1 = 2$ *and* $q_0, q_1 \geq 2$;

(iii) $p_0 = p_1 = 2$ *and* $\min(q_0, q_1) < 2$,

respectively.

PROOF. Everything here follows from Chevet's formula (7) together with the above lemma. QED

For the projective tensor product, the only case missing is when $p_0 = p_1 = 2$ and $\max(q_0, q_1) > 2$.

24 Lemma. *Let* $1 < p_0, p_1 < \infty$ *and* $1 \leq q_0, q_1 \leq \infty$. *Then*

$$\ell(\ell_2^{n^2} \hookrightarrow \ell_{p_0,q_0}^n \otimes_\pi \ell_{p_1,q_1}^n)$$

is asymptotically equivalent to

(i) n^{1+1/p_0}

(ii) $n^{1/2+1/p_0+1/p_1}$

whenever

(i) $p_1 \leq p_0 < 2$, *or* $p_1 < 2 \leq p_0$;

(ii) $p_0 \geq p_1 > 2$, *or* $p_0 > p_1 \geq 2$, *or* $p_0 = p_1 = 2$ *and* $q_0, q_1 \leq 2$,

respectively.

PROOF. All the lower estimates follow from the relation $\ell(\mathrm{id})\,\ell((\mathrm{id}')^{-1}) \geq n^2$, and the upper estimates as follows:

(ii) If $p_0 \geq p_1 > 2$ and $2 \leq q_0, q_1 < \infty$, the upper estimate follows from Proposition 7, and then also for all remaining cases in (ii) by factorization through $\ell_p^n \otimes_\pi \ell_p^n$ with $p > \max(p_0, p_1)$.

(i) If $p_1 \leq p_0 < 2$ and $q_0, q_1 \leq 2$, or $p_1 < 2 < p_0$ and $q_1 \leq 2 \leq q_0 < \infty$, then again Proposition 7 applies. Thus, by factorization through the space $\ell_{p_0,\min(q_0,2)}^n \otimes_\pi \ell_{p_1,\min(q_1,2)}^n$ we can eliminate the restrictions on q_0, q_1 in the first case. Similarly, the restriction on q_1 in the second case can be omitted, and then the one on q_0 by factorization through $\ell_p^n \otimes_\pi \ell_{p_1,q_1}^n$ with $p > p_0$. The same factorization gives the upper estimate when $p_1 < 2 = p_0$. QED

After these preparations, we are able to formulate our results for norms of tensor product identities, Banach–Mazur distances and volume ratios for tensor products of n-th sections of Lorentz sequence spaces. The four indices involved each time make formulas as well as proofs look complicated, and some cases (few though) are left unsolved.

25 Example. Let $1 < p_0, p_1 < \infty$ and $1 \le q_0, q_1 \le \infty$. Then

$$\|\ell^n_{p_0,q_0} \otimes_\varepsilon \ell^n_{p_1,q_1} \hookrightarrow \ell^{n^2}_2\|$$

is asymptotically equivalent to

(i) $n^{1/p_0'}$

(ii) $\sqrt{n}$

(iii) $\dfrac{\sqrt{n}}{(1+\log n)^{1/\min(q_0,q_1)+1/\max(q_0,q_1,2)-1}}$

(iv) $n^{3/2-1/p_0-1/p_1}$

(v) 1

(vi) $(1+\log n)^{\min(3/2-1/q_0-1/q_1,1/2)}$

whenever

(i) $2 < p_0 \le p_1$;
$p_0 = 2 < p_1$ and $q_0 \le 2$;

(ii) $p_0 < 2 < p_1$ and $p_1' < p_0$;
$p_1' = p_0 \le 2$ and $q_1' \le q_0$;

(iii) $p_1' = p_0 < 2 < p_1$ and $q_0 < q_1' \le 2$;
$p_0 = p_1 = 2$ and $q_0 < q_1'$;

(iv) $p_0 < 2$, $p_0 < p_1'$ and $1/p_0 + 1/p_1 < 3/2$;

(v) $p_0, p_1 < 2$ and $1/p_0 + 1/p_1 > 3/2$;
$p_0, p_1 < 2$ and $1/p_0 + 1/p_1 = 3/2$ and $1/q_0 + 1/q_1 \ge 3/2$;

(vi) $1/p_0 + 1/p_1 = 3/2$, $1/q_0 + 1/q_1 < 3/2$ and $\min(q_0, q_1) \le 2$,

respectively.

Proof. (i) The upper estimate follows from factorization through $\ell^n_{p_0,q_0} \otimes_\varepsilon \ell^n_\infty$. For the lower estimate, $\|\,\mathrm{id}\,\| \geq n/\ell(\mathrm{id}^{-1}) \asymp n^{1/p_0'}$ gives the claim.

(ii) In both cases, $M(\ell_{p_1',q_1'}, \ell_{p_0,q_0}) = \ell_\infty$, which immediately gives us the lower estimate. The upper estimate in the case when $p_0 < 2 < p_1$ follows by factorization through $\ell^n_2 \otimes_\varepsilon \ell^n_{p_1,q_1}$, the second case in (i). When $p_0 = p_1 = 2$, the upper estimate follows by factorization through $\ell^n_{4/3} \otimes_\varepsilon \ell^n_{4/3}$.

(iii) The first case follows directly from Theorem 9 (ii). In the second case, for $q_0, q_1 \leq 2$ the lower estimate follows as in (i) and for $q_0 < 2 < q_1$ by factorizing the identity map $\mathrm{id} : \ell^n_{2,q_0} \otimes_\varepsilon \ell^n_{2,q_0'} \hookrightarrow \ell^{n^2}_2$ through $\ell^n_{2,q_0} \otimes_\varepsilon \ell^n_{2,q_1}$. For the upper estimate, consider first the special case $q_0 < q_1 = 2$. Then

$$\|\,\mathrm{id}\,\| \leq \pi_2(\ell^n_{2,q_0} \hookrightarrow \ell^n_2) = \frac{\sqrt{n}}{\|\ell^n_2 \hookrightarrow \ell^n_{2,q_0}\|} \asymp \frac{\sqrt{n}}{(1+\log n)^{1/q_0 - 1/2}}.$$

Now factorize through $\ell^n_{2,q_0} \otimes_\varepsilon \ell^n_2$ to obtain the upper estimate for q_1 arbitrary.

(iv) We immediately see that for some $s \geq 1$ we have that $M(\ell_{p_1',q_1'}, \ell_{p_0,q_0}) = \ell_{r,s}$, where $0 < 1/r = 1/p_0 + 1/p_1 - 1 < 1/2$. This gives the lower estimate, and by (12), in the case $q_0 \leq 2$ the upper one. For $q_0 > 2$, let $p < p_0$ such that still $1/p + 1/p_1 < 3/2$. Then the upper estimate follows by factorization through $\ell^n_{p,2} \otimes_\varepsilon \ell^n_{p_1,q_1}$.

(v) The assumptions yield that in both cases $M(\ell_{p_1',q_1'}, \ell_{p_0,q_0}) \hookrightarrow \ell_2$. By (12), this gives the claim in the second case, and in the first case when additionally $q_0 \leq 2$ is assumed. For $q_0 > 2$, take $p > p_0$ such that still $1/p + 1/p_1 > 3/2$, and factorize through $\ell^n_{p,2} \otimes_\varepsilon \ell^n_{p_1,q_1}$.

(vi) Here, $M(\ell_{p_1',q_1'}, \ell_{p_0,q_0}) = \ell_{2,s}$ with $1/s = \max(1/q_0 + 1/q_1 - 1, 0)$, and we can apply (12). QED

For the sake of completeness, let us list up the combinations of indices missing in the above:

- $p_0 = 2 < p_1$ and $q_0 > 2$;
- $p_1' = p_0 < 2$ and $q_1' > \max(q_0, 2)$;
- $1/p_0 + 1/p_1 = 3/2$, $1/q_0 + 1/q_1 < 3/2$ and $q_0, q_1 > 2$.

For Banach–Mazur distances, we arrive at the following (rather extensive) formulas (same combinations of indices missing as above):

26 Example. Let $1 < p_0, p_1 < \infty$ and $1 \leq q_0, q_1 \leq \infty$. Then

$$d(\ell^n_{p_0,q_0} \otimes_\varepsilon \ell^n_{p_1,q_1}, \ell^{n^2}_2)$$

is asymptotically equivalent to

(i) n^{1-1/p_0}

(ii) $\sqrt{n}\,(1+\log n)^{1/q_0-1/2}$

(iii) $\sqrt{n}\,(1+\log n)^{1/\min(q_0,q_1,2)-1/2}$

(iv) $\sqrt{n}\,(1+\log n)^{|1/\max(q_0,q_1)-1/2|}$

(v) n^{1/p_0}

(vi) $\dfrac{n^{1/p_0}}{(1+\log n)^{1/q_1+1/q_0-1}}$

(vii) n^{1-1/p_1}

(viii) $\sqrt{n}\,(1+\log n)^{1/\min(q_1,2)-1/2}$

(ix) $\sqrt{n}$

(x) $n^{1/p_0+1/p_1-1}$

(xi) $\sqrt{n}\,(1+\log n)^{\min(3/2-1/q_0-1/q_1,1/2)}$

whenever

(i) $2 < p_0 \le p_1$;

(ii) $p_0 = 2 < p_1$ and $q_0 \le 2$;

(iii) $p_0 = p_1 = 2$ and $q_1' \le q_0$;

(iv) $p_0 = p_1 = 2$ and $q_0 < q_1'$;

(v) $p_1' < p_0 < 2 < p_1$;
$p_1' = p_0 < 2 < p_1$ and $q_1' \le q_0$;

(vi) $p_1' = p_0 < 2 < p_1$ and $q_0 < q_1' \le 2$;

(vii) $p_0 < 2 < p_1$ and $p_0 < p_1'$;

(viii) $p_0 < 2 = p_1$;

(ix) $p_0, p_1 < 2$ and $1/p_0 + 1/p_1 < 3/2$;
$p_0, p_1 < 2$, $1/p_0 + 1/p_1 = 3/2$ and $1/q_0 + 1/q_1 \ge 3/2$;

(x) $p_0, p_1 < 2$ and $1/p_0 + 1/p_1 > 3/2$;

(xi) $p_0, p_1 < 2$, $1/p_0 + 1/p_1 = 3/2$, $1/q_0 + 1/q_1 < 3/2$ and $\min(q_0, q_1) \le 2$,

respectively.

PROOF. Everything follows by using the fact that for any symmetric Banach sequence spaces E and F,

$$d(E^n \otimes_\varepsilon F^n, \ell_2^{n^2}) = \|E^n \otimes_\varepsilon F^n \hookrightarrow \ell_2^{n^2}\| \, \|\ell_2^n \hookrightarrow E^n\| \, \|\ell_2^n \hookrightarrow F^n\|;$$

we leave the detailed calculations to the interested reader. QED

Finally, we consider volume ratios. Here, the combinations of indices missing are the dual ones to those missing in the above, plus $p_0 = p_1 = 2$ and $\min(q_0, q_1) < 2 < \max(q_0, q_1)$. Note that by Proposition 16 it follows that $\lambda(\ell_{p_0,q_0}^n \otimes_\pi \ell_{p_1,q_1}^n) \asymp n$ for all combinations of indices occurring in (i).

27 Example. Let $1 < p_0, p_1 < \infty$ and $1 \leq q_0, q_1 \leq \infty$. Then

$$\operatorname{vr}(\ell_{p_0,q_0}^n \otimes_\pi \ell_{p_1,q_1}^n)$$

is asymptotically equivalent to

(i) 1

(ii) $n^{1/2-1/p_0}$

(iii) $\dfrac{n^{1/2-1/p_0}}{(1+\log n)^{1-1/q_0-1/q_1}}$

(iv) $n^{1/p_1-1/2}$

(v) $n^{1/2-1/p_0-1/p_1}$

(vi) $(1+\log n)^{\min(1/q_0+1/q_1-1/2,1/2)}$

whenever

(i) $p_0, p_1 < 2$;
$p_1 < p_0 = 2$ and $q_0 \geq 2$;
$p_0 = p_1 = 2$ and $q_0, q_1 \leq 2$ or $q_0, q_1 \geq 2$;
$2 = p_1 < p_0$;
$2 < p_0, p_1$ and $1/p_0 + 1/p_1 > 1/2$;
$2 < p_0, p_1$, $1/p_0 + 1/p_1 = 1/2$ and $1/q_0 + 1/q_1 \leq 1/2$;

(ii) $p_1 < 2 < p_0 < p_1'$;
$p_1 < 2 < p_0 = p_1'$ and $q_0 \leq q_1'$;

(iii) $p_1 < 2 < p_0 = p_1'$ and $q_0' < q_1 \leq 2$;

(iv) $p_1 < 2 < p_1' < p_0$;

(v) $2 < p_0, p_1$ and $1/p_0 + 1/p_1 < 1/2$;

(vi) $2 < p_0, p_1$ and $1/p_0 + 1/p_1 = 1/2$, $1/q_0 + 1/q_1 > 1/2$ and $\max(q_0, q_1) \geq 2$,

respectively.

Proof. Everything follows by using (20). Note that for those cases where the volume ratio is asymptotically equal 1, only the upper estimate is needed. QED

We conclude with the volume ratio of injective tensor products. The only case missing is when $p_0 = p_1 = 2$ and $\min(q_0, q_1) < 2 < \max(q_0, q_1)$.

28 Example. Let $1 < p_0, p_1 < \infty$ and $1 \leq q_0, q_1 \leq \infty$. Then

$$\mathrm{vr}(\ell^n_{p_0,q_0} \otimes_\varepsilon \ell^n_{p_1,q_1})$$

is asymptotically equivalent to

(i) $\sqrt{n}$

(ii) $n^{1/p_0'}$

(iii) $\sqrt{n}\,(1+\log n)^{1/\min(q_1,2)-1/2}$

(iv) $\sqrt{n}\,(1+\log n)^{1/\min(q_0,2)-1/2}$

(v) $\sqrt{n}\,(1+\log n)^{1/\max(q_0,q_1)-1/2}$

whenever

(i) $p_0, p_1 < 2$;
$p_0 < 2 < p_1$;
$p_0 = p_1 = 2$ and $q_0, q_1 \geq 2$;

(ii) $2 < p_0 \leq p_1$;

(iii) $p_0 < p_1 = 2$;

(iv) $2 = p_0 < p_1$;

(v) $p_0 = p_1 = 2$ and $q_0, q_1 \leq 2$,

respectively.

Proof. Everything follows again from (20), except of the last case. Here, we apply (23) and the corresponding dual result for the volume ratio of the projective tensor product from above. QED

9 The Hilbert space case

In the case that one of the spaces is ℓ_2^n with dominating dimension, our techniques turn out to be quite efficient: norms of tensor product identities, Banach–Mazur distances, approximation numbers, volume ratios and projection constants, all can be estimated in an asymptotically optimal way regardless of any geometric assumptions on the second space involved. Note that the volume ratio estimate for the projective tensor product is a generalization of [30, 3.4] where it was shown that $\mathrm{vr}(\ell_2^n \otimes_\varepsilon \ell_p^n) \asymp 1$ whenever $1 \le p \le 2$.

29 Proposition. *Let F^m be a normed space with enough symmetries in $\mathcal{O}(m)$. Then for $n \ge m$ and $1 \le k \le nm$ the following asymptotically optimal formulas hold, with absolute constants involved only:*

(i) $\|\ell_2^n \otimes_\varepsilon F^m \hookrightarrow \ell_2^{nm}\| \asymp \dfrac{\sqrt{m}}{\|\ell_2^m \hookrightarrow F^m\|}$;

(ii) $d(\ell_2^n \otimes_\varepsilon F^m, \ell_2^{nm}) \asymp \sqrt{m}$;

(iii) $\mathrm{vr}(\ell_2^n \otimes_\varepsilon F^m) \asymp \sqrt{m}$ *and* $\mathrm{vr}(\ell_2^n \otimes_\pi F^m) \asymp 1$;

(iv) $\lambda(\ell_2^n \otimes_\pi F^m) \asymp \sqrt{nm}$.

(v) $a_k(\ell_2^{nm} \hookrightarrow \ell_2^n \otimes_\varepsilon F^m) \asymp \max\left(\left(\dfrac{nm-k+1}{n}\right)^{1/2}, 1\right) \dfrac{\|\ell_2^m \hookrightarrow F^m\|}{\sqrt{m}}$.

Proof. (i) follows from (6) and [33, 18.4] by the assumption $m \le n$:

$$\|\,\mathrm{id}\,\| \le \pi_2(F^m \hookrightarrow \ell_2^m) = \frac{\sqrt{m}}{\|\ell_2^m \hookrightarrow F^m\|} \le \sqrt{2}\,\|\,\mathrm{id}\,\|.$$

Then (ii) is clear by (13), since $\|\ell_2^{nm} \hookrightarrow \ell_2^n \otimes_\varepsilon F^m\|\|\ell_2^m \hookrightarrow F^m\|$. (iii)–(v) then follow from Proposition 15, Proposition 16 and Proposition 11, respectively, by taking $\mathcal{I} = \{(n,m);\, n \ge m\}$. QED

The Hilbert space ℓ_2 is unique (up to isomorphisms) with property (ii) (and therefore with property (iii)). Indeed, let E^n have enough symmetries in $\mathcal{O}(n)$. Then

$$\|E^n \otimes_\varepsilon \ell_\infty^m \hookrightarrow \ell_2^{nm}\|\|\ell_\infty^m(E^n) \hookrightarrow \ell_2^m(\ell_2^n)\| = \sqrt{m}\,\|E^n \hookrightarrow \ell_2^n\|.$$

Thus, $d(E^n \otimes_\varepsilon \ell_\infty^m, \ell_2^{nm}) = \sqrt{m}\, d(E^n, \ell_2^n)$.

10 Tensor products involving Schatten classes

In this section we consider tensor products where at least one part is the n-th section of a unitary ideal. Naturally, we now have to consider complex spaces; we leave the easy task to make the right adjustments to the definitions and results from the previous sections to the reader.

The unitary ideal $\mathcal{S}_E$ associated to a symmetric Banach sequence space E is the Banach space of all compact operators $T \in \mathcal{L}(\ell_2, \ell_2)$ with singular numbers $(s_i(T))$ in E endowed with the norm $\|T\|_{\mathcal{S}_E} := \|(s_i(T))\|_E$; with $\mathcal{S}_E^n$ we denote $\mathcal{L}(\ell_2^n, \ell_2^n)$ together with the norm $\|T\|_{\mathcal{S}_E^n} := \|(s_i(T))\|_E$, which is a space with enough symmetries in $\mathcal{O}(n^2)$. For $E = \ell_p$ ($1 \le p < \infty$) one gets the well-known Schatten-p-class $\mathcal{S}_p$; for simplicity put $\mathcal{S}_\infty := \mathcal{L}(\ell_2, \ell_2)$. It is well-known that if E is separable, then $\mathcal{S}_E' = \mathcal{S}_{E^\times}$ (see e. g. [29, 3.2]).

The first tool needed is an analogue of Proposition 5.

30 Lemma. *Let E and F be symmetric Banach sequence spaces. Then*

$$\ell(\ell_2^{mn^2} \hookrightarrow \mathcal{S}_E^n \otimes_\varepsilon F^m)$$

is asymptotically equivalent to

$\lambda_E(n)\,\lambda_F(m)\,\max\left(\frac{1}{\sqrt{n}}, \sqrt{\frac{n}{m}}\right)$	*whenever E and F are 2-concave;*
$\max(\sqrt{n}\,\lambda_E(n), \lambda_F(m))$	*whenever E is 2-convex and F has type 2;*
$\frac{\lambda_E(n)}{\sqrt{n}}\max(n, \lambda_F(m))$	*whenever E is 2-concave and F has type 2;*
$\lambda_F(m)\,\max\left(1, \sqrt{\frac{n}{m}}\,\lambda_E(n)\right)$	*whenever E is 2-convex, F is 2-concave.*

Proof. Everything follows in the usual way from (7) and the well-known facts that $\|\ell_2^{n^2} \hookrightarrow \mathcal{S}_E^n\| = \|\ell_2^n \hookrightarrow E^n\|$ and $\ell(\ell_2^{n^2} \hookrightarrow \mathcal{S}_E^n) \asymp \sqrt{n}\,\lambda_E(n)$ (see, e.g., [33, 45.1]). QED

As we could see in the proof of Proposition 7, ℓ-norms of identities into projective tensor products are quite difficult to estimate from above. We only prove the following formula needed for our purposes.

31 Lemma. *Let $2 \le p, q < \infty$. Then*

$$\ell(\ell_2^{n^4} \hookrightarrow \mathcal{S}_p^n \otimes_\pi \ell_q^{n^2}) \asymp n^{3/2+1/p+2/q}. \tag{29}$$

Proof. We first estimate $\mathbf{T_2}(\mathcal{S}_p^n \otimes_\pi \ell_q^{n^2})$. First observe that by [33, 32.7] one has that $\gamma_\infty(\mathrm{id}_{\mathcal{S}_{p'}^n}) = \lambda(\mathcal{S}_{p'}^n) \asymp n$ since by [32] the space $\mathcal{S}_{p'}$ has cotype 2.

Thus, by [33, 16.1] we obtain

$$\pi_1(\mathrm{id}_{\mathcal{S}^n_{p'}}) = \frac{n^2}{\gamma_\infty(\mathrm{id}_{\mathcal{S}^n_{p'}})} \asymp n.$$

Now by [13, 2.8] and the above, we have

$$\|\ell_q^{n^2}(\mathcal{S}_p^n) \hookrightarrow \ell_q^{n^2} \otimes_\pi \mathcal{S}_p^n\| = \|\ell_{q'}^{n^2} \otimes_\varepsilon \mathcal{S}_{p'}^n \hookrightarrow \ell_{q'}^{n^2}(\mathcal{S}_{p'}^n)\| \le \pi_{q'}(\mathrm{id}_{\mathcal{S}^n_{p'}}) \le \pi_1(\mathrm{id}_{\mathcal{S}^n_{p'}}) \prec n.$$

Thus, by factorization, [13, 11.12] and the fact that $\mathcal{S}_p$ has type 2 (see [32]),

$$\mathbf{T_2}(\mathcal{S}_p^n \otimes_\pi \ell_q^{n^2}) \le \mathbf{T_2}(\ell_q^{n^2}(\mathcal{S}_p^n))\, \|\ell_q^{n^2}(\mathcal{S}_p^n) \hookrightarrow \ell_q^{n^2} \otimes_\pi \mathcal{S}_p^n\| \prec n.$$

Now we obtain, as in the proof of Proposition 7, by (6)

$$\begin{aligned}\ell(\ell_2^{n^4} \hookrightarrow \mathcal{S}_p^n \otimes_\pi \ell_q^{n^2}) &\le \mathbf{T_2}(\mathcal{S}_p^n \otimes_\pi \ell_q^{n^2})\, \pi_2(\mathcal{S}_{p'}^n \otimes_\varepsilon \ell_{q'}^{n^2} \hookrightarrow \ell_2^{n^4}) \\ &\prec \frac{n^3}{\|\mathcal{S}_p^n \hookrightarrow \mathcal{S}_2^n\|\, \|\ell_q^{n^2} \hookrightarrow \ell_2^{n^2}\|} = n^{3/2+1/p+2/q}.\end{aligned}$$

The lower estimate follows from $\ell(\mathrm{id})\,\ell((\mathrm{id}')^{-1}) \ge n^4$ and the preceding lemma. QED

Now we are able to prove the following:

32 Proposition. *Let $1 \le p,q \le 2$. Then the following hold for $m \ge n^2$:*

(i) $\|\mathcal{S}_p^n \otimes_\varepsilon \ell_q^m \hookrightarrow \ell_2^{mn^2}\| \asymp \begin{cases} n^{5/2-1/p-2/q} & 1/p+1/q \le 3/2 \\ n^{1-1/q} & 1/p+1/q > 3/2; \end{cases}$

(ii) $d(\mathcal{S}_p^n \otimes_\varepsilon \ell_q^m, \ell_2^{mn^2}) \asymp \begin{cases} n^{2-2/q}\, m^{1/q-1/2} & 1/p+1/q \le 3/2 \\ n^{1/2+1/p-1/q}\, m^{1/q-1/2} & 1/p+1/q > 3/2; \end{cases}$

(iii) $a_k(\ell_2^{n^4} \hookrightarrow \mathcal{S}_p^n \otimes_\varepsilon \ell_q^{n^2}) \asymp \max\left(\frac{1}{\|\mathrm{id}^{-1}\|}, \left(\frac{n^4-k+1}{n^4}\right)^{1/2} \|\mathrm{id}\|\right)$ *for all* $1 \le k \le n^4$ *whenever* $1/p+1/q \le 3/2$.

Proof. Let $1 \le p,q \le 2$, and set $1/s = 3/2 - 1/q$, $1/r = 1 - 1/q$. Then by [18] (see also Lemma 3) one has $\Pi_{s,1} \subset \Pi_{2,q} \subset \Pi_{r,2}$, thus, by [31], $\Pi_{2,q}(X,Y) = \Pi_{r,2}(X,Y)$ whenever X is a Banach space with cotype 2. Now [11, 7.2] and the fact that $\mathcal{S}_p$ has cotype 2 (see [32]) imply that

$$\|\mathcal{S}_p^n \otimes_\varepsilon \ell_q^m \hookrightarrow \ell_2^{mn^2}\| \le \pi_{2,q}(\mathcal{S}_p^n \hookrightarrow \mathcal{S}_2^n) \asymp \begin{cases} n^{5/2-1/p-2/q} & 1/p+1/q \le 3/2 \\ n^{1-1/q} & 1/p+1/q > 3/2, \end{cases}$$

which gives the upper estimates in (i).

For $1/p+1/q \leq 3/2$, the lower estimate for all $m \geq n^2$ follows from the case $m = n^2$, and this itself from $\|\,\mathrm{id}\,\| \geq n^2/\ell(\mathrm{id}^{-1})$ and Lemma 30. In the case $1/p+1/q > 3/2$ the lower estimate is easy, since $\mathcal{S}_p^n \otimes_\varepsilon \ell_q^{n^2}$ contains $\ell_2^n \otimes_\varepsilon \ell_q^n$, and use Proposition 29. (ii) now follows from (13) together with

$$\|\ell_2^{mn^2} \hookrightarrow \mathcal{S}_p^n \otimes_\varepsilon \ell_q^m\| = \|\mathcal{S}_2^n \hookrightarrow \mathcal{S}_p^n\|\,\|\ell_2^m \hookrightarrow \ell_q^m\| = n^{1/p-1/2} m^{1/q-1/2}.$$

For $m = n^2$, statement (iii) follows from (ii) by Proposition 11 (ii). QED

33 Corollary. *Let $2 \leq p, q < \infty$. Then the following hold:*

(i) $\mathrm{vr}(\mathcal{S}_p^n \otimes_\pi \ell_q^{n^2}) \asymp n^{\max(1/2-1/p-1/q,0)}$;

(ii) $\lambda(\mathcal{S}_p^n \otimes_\pi \ell_q^{n^2}) \asymp n^2$ *whenever* $1/p+1/q \geq 1/2$.

Proof. (i) follows from (20) by the lemmas and the proposition above, and then (ii) by Proposition 16. QED

For $p, q \geq 2$, the above proposition has the following, more satisfying counterpart:

34 Proposition. *Let E and F be symmetric Banach sequence spaces such that E is 2-convex and F has type 2. Then the following hold:*

(i) $\|\mathcal{S}_E^n \otimes_\varepsilon F^m \hookrightarrow \ell_2^{mn^2}\| \asymp \dfrac{n\sqrt{m}}{\max(\sqrt{n}\,\lambda_E(n), \lambda_F(m))}$;

(ii) $a_k(\ell_2^{mn^2} \hookrightarrow \mathcal{S}_E^n \otimes_\varepsilon F^m) \asymp \max\left(\dfrac{1}{\|\,\mathrm{id}^{-1}\,\|}, \left(\dfrac{mn^2-k+1}{mn^2}\right)^{1/2} \|\,\mathrm{id}\,\|\right)$

for all $1 \leq k \leq mn^2$.

Proof. (i) The lower estimates all follow from Lemma 30 and the estimate $\|\,\mathrm{id}\,\| \geq n\sqrt{m}/\ell(\mathrm{id}^{-1})$. The upper ones can be obtained by factorization through the spaces $\ell_\infty^{n^2} \otimes_\varepsilon F^m$ and $\mathcal{S}_E^n \otimes_\varepsilon \ell_\infty^m$, respectively; note that $\|\mathcal{S}_E^n \hookrightarrow \ell_\infty^{n^2}\| \leq 1$ (see, e.g., [33, p. 336]). (ii) follows as in the proof of Proposition 11 (iii); to see this, just observe that here, $\ell(\mathrm{id}) \asymp n\sqrt{m}/\|\,\mathrm{id}^{-1}\,\|$. QED

35 Corollary. *Let E and F be symmetric Banach sequence spaces such that E and F are 2-concave and F is of non-trivial convexity. Then it is* $\mathrm{vr}(\mathcal{S}_E^n \otimes_\pi F^m) \asymp 1$ *and* $\lambda(\mathcal{S}_E^n \otimes_\pi F^m) \asymp n\sqrt{m}$.

Proof. This follows from the above together with (20), Lemma 30 and Proposition 16. QED

For spaces E and F lying on different sides of ℓ_2, things seem to be complicated. However, some partial answers for m considerably smaller than n^2 can be found in the upcoming more general result, which somewhat generalises Proposition 29. We conjecture that the assumption on E in the statement is superfluous when considering $m \leq n$ only. Note that a similar result holds in the case $\|E^n \hookrightarrow \ell_2^n\| \asymp \sqrt{n}/\lambda_E(n)$ when substituting $\mathcal{S}_E^n$ by E^n and considering $m \leq \lambda_E(n)^2$; we leave this task to the interested reader.

36 Proposition. *Let E be a symmetric Banach sequence space and m such that either $m \leq n$ and $E \hookrightarrow \ell_2$, or $m \leq n\,\lambda_E(n)^2$ and $\|E^n \hookrightarrow \ell_2^n\| \asymp \sqrt{n}/\lambda_E(n)$. Then for any normed space F^m with enough symmetries in $\mathcal{O}(m)$ the following hold:*

(i) $\|\mathcal{S}_E^n \otimes_\varepsilon F^m \hookrightarrow \ell_2^{mn^2}\| \asymp \sqrt{m}\,\dfrac{\|E^n \hookrightarrow \ell_2^n\|}{\|\ell_2^m \hookrightarrow F^m\|}$;

(ii) $d(\mathcal{S}_E^n \otimes_\varepsilon F^m, \ell_2^{mn^2}) \asymp \sqrt{m}\, d(E^n, \ell_2^n)$;

(iii) $a_k(\ell_2^{mn^2} \hookrightarrow \mathcal{S}_E^n \otimes_\varepsilon F^m) \asymp \max\left(\dfrac{1}{\|\operatorname{id}^{-1}\|}, \left(\dfrac{mn^2-k+1}{mn^2}\right)^{1/2} \|\operatorname{id}\|\right)$

for all $1 \leq k \leq mn^2$ whenever $\|E^n \hookrightarrow \ell_2^n\| \asymp \sqrt{n}/\lambda_E(n)$.

In particular, $\|\mathcal{S}_p^n \otimes_\varepsilon \ell_q^n \hookrightarrow \ell_2^{n^3}\| \asymp n^{3/2-\min(1/p,1/2)-\max(1/q,1/2)}$ for $1 \leq p, q \leq \infty$.

Proof. In (i), the upper estimate follows by factorization through $\mathcal{S}_2^n \otimes_\varepsilon F^m$ and Proposition 29. The lower one in the case $E \hookrightarrow \ell_2$ is clear by Proposition 29 and the assumption $m \leq n$, since $\mathcal{S}_E^n \otimes_\varepsilon F^m$ contains $\ell_2^n \otimes_\varepsilon F^m$. Furthermore, by (7) and the assumption $m \leq n\,\lambda_E(n)^2$,

$$\ell(\ell_2^{mn^2} \hookrightarrow \mathcal{S}_E^n \otimes_\varepsilon F^m) \asymp \sqrt{n}\,\lambda_E(n)\,\|\ell_2^m \hookrightarrow F^m\|, \tag{30}$$

which by $\|\operatorname{id}\| \geq n\sqrt{m}/\ell(\operatorname{id}^{-1})$ gives the lower estimate in the case when $\|E^n \hookrightarrow \ell_2^n\| \asymp \sqrt{n}/\lambda_E(n)$. (ii) then follows as usual by (13), and (iii) by (30) as in the proof of Proposition 11. QED

37 Corollary. *Let E be a symmetric Banach sequence space such that it holds $\|\ell_2^n \hookrightarrow E^n\| \asymp \lambda_E(n)/\sqrt{n}$. Then for all $m \leq n\,\lambda_E(n)^2$ and any normed space F^m with enough symmetries in $\mathcal{O}(m)$, it is $\operatorname{vr}(\mathcal{S}_E^n \otimes_\pi F^m) \asymp 1$ and $\lambda(\mathcal{S}_E^n \otimes_\pi F^m) \asymp n\sqrt{m}$.*

Proof. In the proof of the proposition above, we have seen that the lower asymptotic estimate $\ell(\operatorname{id}^{-1}) \succ n\sqrt{m}/\|\operatorname{id}\|$ holds. This together with Proposition 15 and Proposition 16 gives the claim. QED

Note that for $E = \ell_p$, $1 \le p < 2$, the upper bound n for m in the proposition above is sharp, since for q such that $1/p + 1/q \ge 3/2$, we have proved altogether that

$$\|\mathcal{S}_p^n \otimes_\varepsilon \ell_q^m \hookrightarrow \ell_2^{mn^2}\| \asymp n^{1-1/q} \text{ for all } m \ge n.$$

Furthermore, the proof of the following corollary fills a gap in the proof of the same result stated in [11, 7.3] and shows that for $E = \ell_p$, $2 \le p \le \infty$, the upper bound $n^{1+2/p}$ for m in the proposition above is also sharp.

38 Corollary. *Let $1 \le p, q \le \infty$ and $2 \le r \le \infty$. Then*

$$\pi_{r,2}(\mathcal{S}_p^n \hookrightarrow \mathcal{S}_q^n) \asymp n^{1/r}\, \pi_{r,2}(\ell_p^n \hookrightarrow \ell_q^n).$$

Proof. The proof of [11, 7.3] only lacks the appropriate lower estimate in the case $2 \le p \le \infty$, $2 \le r < \infty$ and $1/q \ge 1/r + 1/p - 2/pr$. We have to show that in this case

$$\pi_{r,2}(\mathcal{S}_p^n \hookrightarrow \mathcal{S}_q^n) \succ n^{1/r-1/p+1/q+2/pr}.$$

Using the proposition above and factorization, we obtain for $m \le n^{1+2/p}$

$$\begin{aligned}\pi_{r,2}(\mathcal{S}_p^n \hookrightarrow \mathcal{S}_q^n) &\ge \|\mathcal{S}_p^n \otimes_\varepsilon \ell_2^m \hookrightarrow \ell_r^m(\mathcal{S}_q^n)\| \\ &\ge \frac{\|\mathcal{S}_p^n \otimes_\varepsilon \ell_2^m \hookrightarrow \ell_2^m(\mathcal{S}_2^n)\|}{n^{1/2-1/q}\, m^{1/2-1/r}} \asymp n^{1/q-1/p}\, m^{1/r}.\end{aligned}$$

Taking $m = [n^{1+2/p}]$, this gives the claim. If $q \le 2$, we have by [13, 12.6], the fact that $\ell_2(\mathcal{S}_q)$ has cotype 2 (this follows from [32] together with [13, 11.12]) and (6)

$$\begin{aligned}\|\mathcal{S}_p^n \otimes_\varepsilon \ell_2^m \hookrightarrow \ell_2^m(\mathcal{S}_q^n)\| &\ge \frac{\ell(\ell_2^m(\mathcal{S}_2^n) \hookrightarrow \ell_2^m(\mathcal{S}_q^n))}{\ell(\ell_2^m(\mathcal{S}_2^n) \hookrightarrow \mathcal{S}_p^n \otimes_\varepsilon \ell_2^m)} \asymp \frac{\pi_2(\ell_2^m(\mathcal{S}_2^n) \hookrightarrow \ell_2^m(\mathcal{S}_q^n))}{\ell(\ell_2^m(\mathcal{S}_2^n) \hookrightarrow \mathcal{S}_p^n \otimes_\varepsilon \ell_2^m)} \\ &= \frac{n^{1/2+1/q}\sqrt{m}}{n^{1/2+1/p}} = n^{1/q-1/p}\sqrt{m}.\end{aligned}$$

Hence,

$$\pi_{r,2}(\mathcal{S}_p^n \hookrightarrow \mathcal{S}_q^n) \succ \frac{n^{1/q-1/p}\sqrt{m}}{m^{1/2-1/r}} = n^{1/q-1/p}\, m^{1/r}.$$

Now the claim follows as above by taking $m = [n^{1+2/p}]$. QED

11 Tensor products of Schatten classes

In this concluding section, we deal with tensor products of finite-dimensional unitary ideals. We first consider the situation when the underlying sequence spaces both are 2-convex.

39 Proposition. *Let E and F be 2-convex symmetric Banach sequence spaces. Then the following hold:*

(i) $\|\mathcal{S}_E^n \otimes_\varepsilon \mathcal{S}_F^m \hookrightarrow \mathcal{S}_2^{nm}\| \asymp \dfrac{nm}{\max(\sqrt{n}\,\lambda_E(n), \sqrt{m}\,\lambda_F(m))}$;

(ii) $a_k(\mathcal{S}_2^{nm} \hookrightarrow \mathcal{S}_E^n \otimes_\varepsilon \mathcal{S}_F^m) \asymp \max\left(\dfrac{1}{\|\operatorname{id}^{-1}\|}, \left(\dfrac{n^2m^2-k+1}{n^2m^2}\right)^{1/2} \|\operatorname{id}\|\right)$ *for all* $1 \le k \le n^2m^2$.

Proof. Using once more (7), it is easy to see that for E and F as above,

$$\ell(\mathcal{S}_2^{nm} \hookrightarrow \mathcal{S}_E^n \otimes_\varepsilon \mathcal{S}_F^m) \asymp \max(\sqrt{n}\,\lambda_E(n), \sqrt{m}\,\lambda_F(m)). \tag{31}$$

Thus, $\|\operatorname{id}\| \ge nm/\ell(\operatorname{id}^{-1})$ yields the lower estimates in (i), The upper ones follow by factorization through $\ell_\infty^{n^2} \otimes_\varepsilon \mathcal{S}_F^m$ and $\mathcal{S}_E^n \otimes_\varepsilon \ell_\infty^{m^2}$, respectively. (ii) then follows as in the proof of Proposition 11 (iii), since $\ell(\operatorname{id}) \asymp nm/\|\operatorname{id}^{-1}\|$. QED

40 Corollary. *Let E, F be 2-concave symmetric Banach sequence spaces. Then* $\operatorname{vr}(\mathcal{S}_E^n \otimes_\pi \mathcal{S}_F^m) \asymp 1$ *and* $\lambda(\mathcal{S}_E^n \otimes_\pi \mathcal{S}_F^m) \asymp nm$.

Proof. This follows from the above proposition, and (20), (31) and Proposition 16. QED

The remaining results all deal with $\mathcal{S}_p^n \otimes_\varepsilon \mathcal{S}_q^n$ only. Note that with regard to the proof of the forthcoming lemma and the remarks in [11], it is far from obvious what may happen for arbitrary finite-dimensional unitary ideals (in contrast to the sequence space case). The following is a non-commutative analogue of the fact (see, e.g., [17, p. 103], or [10, 4.2] for a more general result) that for the same combination of indices as stated below, the map $\ell_p \otimes_\varepsilon \ell_q \hookrightarrow \mathcal{S}_r$ is bounded.

41 Lemma. *Let $1 \le p, q \le 2$ and $1 \le r \le \infty$ such that $1/r = 1/p + 1/q - 1$. Then*

$$\|\mathcal{S}_p^n \otimes_\varepsilon \mathcal{S}_q^n \hookrightarrow \mathcal{S}_r^{n^2}\| \asymp n^{1/r}.$$

Proof. Consider first the case $p = 2$. Let $T : \mathcal{S}_2^n \to \mathcal{S}_2^n$, and denote by i_{q2}^n the inclusion map $\mathcal{S}_q^n \hookrightarrow \mathcal{S}_2^n$. Set $T_0 : \mathcal{S}_2^n \hookrightarrow \mathcal{S}_q^n$, $T_0 x := Tx$ for $x \in \mathcal{S}_2^n$. Then by [13, 10.3] and [11, 7.2]

$$\|T\|_{\mathcal{S}_r^{n^2}} = \pi_{r,2}(T) = \pi_{r,2}(i_{q2}^n\, T_0) \le C_q\, n^{1/r}\, \|T_0\|,$$

where $C_q > 0$ does not depend on n and T. Now let p be arbitrary. By Pisier's Factorization Theorem ([24, 4.1] or [5, 31.4]) and the fact that $\mathcal{S}_p$ and $\mathcal{S}_q$ have cotype 2 (see again [32]), there exists $C_{p,q}$ not depending on n such that for any operator $T_0 : \mathcal{S}_{p'}^n \to \mathcal{S}_q^n$ there exist $R_0 : \mathcal{S}_{p'}^n \to \mathcal{S}_2^n$ and $S_0 : \mathcal{S}_2^n \to \mathcal{S}_q^n$ such that $T_0 = S_0 R_0$ and $\|S_0\|\,\|R_0\| \le C_{p,q}\,\|T_0\|$. By the above together with duality, $S := i^n_{q2} S_0$ and $R := R_0 i^n_{2p'}$ satisfy $\|S\|_{\mathcal{S}^{n^2}_{s_0}} \le C_q\, n^{1/q-1/2}\, \|S_0\|$ and $\|R\|_{\mathcal{S}^{n^2}_{r_0}} \le C_p\, n^{1/p-1/2}\, \|R_0\|$, where $1/s_0 = 1/q - 1/2$ and $1/r_0 = 1/p - 1/2$. Hence, by [13, 6.3] for $T := i^n_{q2}\, T_0\, i^n_{2p'}$

$$\|T\|_{\mathcal{S}^{n^2}_r} = \|SR\|_{\mathcal{S}^{n^2}_r} \le \|S\|_{\mathcal{S}^{n^2}_{s_0}} \|R\|_{\mathcal{S}^{n^2}_{r_0}} \le C_p\, C_q\, C_{pq}\, n^{1/p+1/q-1}\, \|T_0\|,$$

which gives the upper estimate. The lower one is clear by considering the identity map $\mathrm{id} : \mathcal{S}^n_2 \to \mathcal{S}^n_2$. QED

Now we are able to formulate a non-commutative analogue of Schütt's result mentioned at the beginning of Section 3:

42 Proposition. *Let $1 \le p, q \le \infty$. Then the following hold:*

(i) $\|\mathcal{S}^n_p \otimes_\varepsilon \mathcal{S}^n_q \hookrightarrow \mathcal{S}^{n^2}_2\| \asymp \sqrt{n}\, \|\ell^n_p \otimes_\varepsilon \ell^n_q \hookrightarrow \ell^{n^2}_2\|$;

(ii) $d(\mathcal{S}^n_p \otimes_\varepsilon \mathcal{S}^n_q, \mathcal{S}^{n^2}_2) \asymp \sqrt{n}\, d(\ell^n_p \otimes_\varepsilon \ell^n_q, \ell^{n^2}_2)$;

(iii) $a_k(\mathcal{S}^{n^2}_2 \hookrightarrow \mathcal{S}^n_p \otimes_\varepsilon \mathcal{S}^n_q) \asymp \max\left(\frac{1}{\|\mathrm{id}^{-1}\|}, \left(\frac{n^4 - k + 1}{n^4}\right)^{1/2} \|\mathrm{id}\| \right)$ *for all* $1 \le k \le n^4$ *whenever* $1 \le p, q \le 2$ *and* $1/p + 1/q \le 3/2$, *or* $2 \le p, q \le \infty$.

PROOF. (i) We have to show the following:

$$\|\mathcal{S}^n_p \otimes_\varepsilon \mathcal{S}^n_q \hookrightarrow \mathcal{S}^{n^2}_2\| \asymp \begin{cases} n^{3/2-1/\min(p,q)} & 2 \le p, q \le \infty \\ n^{1-\max(1/p+1/q-1,0)} & 1 \le p \le 2 \le q \le \infty \\ n^{2-1/p-1/q} & 1 \le p, q \le 2, 1/p + 1/q \le 3/2 \\ \sqrt{n} & 1 \le p, q \le 2, 1/p + 1/q \ge 3/2. \end{cases}$$

The first case is contained within a more general setting in the previous proposition. For the remaining cases, observe that by the above lemma, for $1 \le p, q \le 2$ such that $1/p + 1/q = 3/2$ it is $\|\mathcal{S}^n_p \otimes_\varepsilon \mathcal{S}^n_q \hookrightarrow \mathcal{S}^{n^2}_2\| \asymp \sqrt{n}$. The rest of the upper estimates in (i) now follow by factorization from this special case and from factorization through $\mathcal{S}^n_p \otimes_\varepsilon \ell^{n^2}_\infty$, respectively. The lower estimates are true by $\|\mathrm{id}\| \ge n / \|\mathcal{S}^n_{p'} \hookrightarrow \mathcal{S}^n_q\|$ and considering the general lower bound $\sqrt{n}$, since $\mathcal{S}^n_p \otimes_\varepsilon \mathcal{S}^n_q$ contains $\ell^n_2 \otimes_\varepsilon \ell^n_2$. (ii) follows as usual, and for (iii) observe that in those cases, $\|\mathrm{id}\| \ge n^2 / \ell(\mathrm{id}^{-1})$, and proceed as usual. QED

43 Corollary. *Let $2 \le p, q \le \infty$ such that $1/p + 1/q \ge 1/2$. Then it is* $\mathrm{vr}(\mathcal{S}^n_p \otimes_\pi \mathcal{S}^n_q) \asymp 1$ *and* $\lambda(\mathcal{S}^n_p \otimes_\pi \mathcal{S}^n_q) \asymp n^2$.

For p, q arbitrary, the computation of the volume ratios of projective tensor products of Schatten classes turns out to be more complicated due to a lack of knowledge on ℓ-norms. We believe that the logarithmic factor in the second part of the upcoming statement is superfluous.

44 Corollary. *Let $1 \le p \le q \le \infty$. Then*

$$\mathrm{vr}(\mathcal{S}_p^n \otimes_\pi \mathcal{S}_q^n) \asymp \mathrm{vr}(\ell_p^n \otimes_\pi \ell_q^n)$$

whenever $1/p + 1/q \ge 1/2$ and $q \ne \infty$. If $1/p + 1/q < 1/2$ or $q = \infty$, then

$$\mathrm{vr}(\ell_p^n \otimes_\pi \ell_q^n)/(1 + \log n) \prec \mathrm{vr}(\mathcal{S}_p^n \otimes_\pi \mathcal{S}_q^n) \prec \mathrm{vr}(\ell_p^n \otimes_\pi \ell_q^n).$$

Proof. The upper estimates all follow from Proposition 42 by (7) and (20), and the lower ones with the additional logarithmic factor by taking into account that $\ell(\mathrm{id})\,\ell((\mathrm{id}^{-1})') \le C\,(1 + \log n)\,n^4$ for some universal constant $C > 0$ (see, e.g., [3, Lemma 2]). Taking into account the corollaries so far from this section, we are left with giving the appropriate lower estimate in the case $1 \le p \le 2 \le q < \infty$. First, by factorization, [13, 11.12] and [32],

$$\mathbf{T_2}(\mathcal{S}_p^n \otimes_\pi \mathcal{S}_q^n) \le \mathbf{T_2}(\ell_2^{n^2}(\mathcal{S}_q^n))\,\|\ell_2^{n^2}(\mathcal{S}_q^n) \hookrightarrow \ell_1^{n^2}(\mathcal{S}_q^n)\| \prec n.$$

Hence, as in the proof of Proposition 7, by (6)

$$\ell(\ell_2^{n^4} \hookrightarrow \mathcal{S}_p^n \otimes_\pi \mathcal{S}_q^n) \le \mathbf{T_2}(\mathcal{S}_p^n \otimes_\pi \mathcal{S}_q^n)\,\pi_2(\mathcal{S}_{p'}^n \otimes_\varepsilon \mathcal{S}_{q'}^n \hookrightarrow \ell_2^{n^4}) \prec n^{5/2+1/q}.$$

The claim now follows again by applying (20) and the proposition above. $\boxed{QED}$

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