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WHAT IS THE SHAPE OF A TRIANGLE?

DANA N. MACKENZIE

Abstract. *We consider the problem of describing the shape of a triangle $\triangle ABC$ by a single complex number $\sigma(\triangle ABC)$, which we call the shape invariant of the triangle. After giving a simple algebraic definition of σ, we prove a surprising geometric description: modulo a symmetry group of order 6, σ is the location of the orthocenter of the triangle, after it is rescaled so that the vertices lie on the unit circle and rotated so that an altitude of its Morley triangle points in the direction of the positive x-axis. We find the set of all possible values of σ, and discuss how the value of σ determines the «scaleneness» and «acuteness» of $\triangle ABC$. Finally, we give formulas for the «scaleneness» and «acuteness» in terms of the side lengths or angles of $\triangle ABC$, and compute some numerical examples where the angles are «unusual» rational multiples of π.*

In high-school geometry, we learn that two triangles have the same shape if and only if they are *similar* to each other. Hence the question in the title of this article could be answered very succinctly: the «shape» of a triangle may be defined as the equivalence class of all similar triangles. The more precise question I would like to address in this article is: how can we give a *numerical measure* of the shape of a triangle?

There are three requirements I would like to make of this numerical measure. First, it should treat the data describing the triangle (which may be the three side lenghts, the three angles, or the coordinates of the three vertices) *symmetrically.* Second, it should not contain redundant information. Finally, it should be «geometrically meaningful»: more than merely identifying the shape, it should actually tell us something about the triangle. This requirement is somewhat vague, but it is the real reason for being interested in the problem.

Perhaps the simplest solution to the problem of giving a numerical measure of shape is to list the three angles in increasing order. However, this fails the second test: it uses three parameters to describe the shape, when two would suffice. We could remedy this by listing the two smallest angles, $\angle A$ and $\angle B$, since $\angle C$ can then be found from the equation $\angle C = \pi - \angle A - \angle B$. But this solution does not treat the three angles symmetrically.

The «shape invariant» I will propose below is certainly not the only one that meets the above requirements, but I hope the reader will agree that it passes the test of «geometric meaningfulness» especially well.

DERIVATION OF THE SHAPE INVARIANT

We begin by observing that any triangle $\triangle ABC$ can be mapped by a dilation to a triangle $\triangle z_1' z_2' z_3'$ whose three vertices lie on the unit circle $|z| = 1$. If we think of z_1', z_2', z_3' as complex numbers, then $|z_1' z_2' z_3'| = 1$, hence $z_1' z_2' z_3' = e^{i\theta}$ for some angle θ. Next, by rotating $\triangle z'_1 z'_2 z'_3$ by the angle $-\theta/3$ about the origin, we obtain a new triangle $\triangle z_1 z_2 z_3$ such that $z_1 z_2 z_3 = 1$ (since $z_i = z'_i e^{-i\theta/3}$). We codify what we have done so far in the following lemma.

Lemma 1. *Any triangle $\triangle ABC$ is similar to some triangle $\triangle z_1 z_2 z_3$ such that each z_i lies on the unit circle and $z_1 z_2 z_3 = 1$. Moreover, there are at most six such triangles which are similar to $\triangle ABC$.*

Proof. The first sentence has already been proved. To prove the second sentence, note that any other triangle $\triangle w_1 w_2 w_3 \sim \triangle ABC$ satisfying the two properties stated in the first sentence must also be similar to $\triangle z_1 z_2 z_3$. The similarity T which maps $\triangle z_1 z_2 z_3$ to $\triangle w_1 w_2 w_3$ must map the circumcenter of the former to the circumcenter of the latter. Hence $T(0) = 0$. Since $|z_1 - 0| = 1 = |T(z_1) - 0| = |T(z_1) - T(0)|$, the magnification factor of T is 1, so T is an isometry. It is easy to verify that the only isometries which fix the origin and preserve the property $z_1 z_2 z_3 = 1$ are the elements of the dihedral group of 6 elements, generated by complex conjugation (i.e. reflection about the x-axis) and rotation by $2\pi/3$. □

Recall that we wanted our «shape measure» to treat the data of our triangle symmetrically. The simplest function which is symmetric with respect to z_1, z_2, and z_3 is the sum $z_1 + z_2 + z_3$.

Definition 1. *The complex number a is a* shape invariant of $\triangle ABC$ *if $a = z_1 + z_2 + z_3$, where $\triangle z_1 z_2 z_3$ is one of the (at most six) triangles constructed in Lemma 1. We write this relation as follows: $a = \sigma(\triangle ABC)$.*

The somewhat clumsy nature of this definition is caused by the fact that $\sigma(\triangle ABC)$ is not uniquely defined – in general, it has 6 possible values. From the proof of Lemma 1, it is easy to see that if a is one of those values, then the others are $e^{2\pi i/3}a, e^{4\pi i/3}a, \overline{a}, e^{2\pi i/3}\overline{a}$, and $e^{4\pi i/3}\overline{a}$. While this non-uniqueness is annoying, it is a price that is worth paying because of the geometric interest of $\sigma(\triangle ABC)$, which will be explored in the next section.

To obtain a shape invariant which *is* uniquely defined, we start by observing that there are in general only two possible values for σ^3 (namely a^3 and $\overline{a}^3$), and only one of these lies in the upper half-plane. We denote this value of σ^3 by $Z(\triangle ABC)$. In particular, the imaginary part and norm of Z seem to convey the most important information about the shape of $\triangle ABC$, and this motivates the following definition.

Definition 2. *The $X-, Y-$, and R-*invariants *of $\triangle ABC$ are defined as follows:*

$$X = \Re(\sigma^3),\ Y = |\Im(\sigma^3)|,\ R = |\sigma|^2 = (X^2 + Y^2)^{1/3}. \tag{1}$$

The Y-invariant will also be called the scaleneness *and the R-invariant will be called the* obtuseness *of $\triangle ABC$.*

In the next sections we will explore the geometric and algebraic meaning of the invariants σ, X, Y and R.

FOUR EQUIVALENT PROBLEMS

A major step towards understanding σ is to answer the following question:

Problem 1. What are all the possible values of σ?

Perhaps the first point to be made is that not all complex numbers are possible values of σ. For example, since $|z_1| = |z_2| = |z_3| = 1$, we cannot have $z_1 + z_2 + z_3 = 4$. Readers who want to know right away what the answer to Problem 1 is may refer to Figure 2.

There are several interesting ways to rephrase Problem 1. We will proceed in a sequence from the more algebraic to the more geometric.

To begin, we note that any three complex numbers z_1, z_2, z_3 are roots of a complex polynomial:

$$P(z) = (z - z_1)(z - z_2)(z - z_3).$$

If z_1, z_2, z_3 satisfy the conclusions of Lemma 1, then the constant term of $P(z)$ equals -1. The z^2 coefficient is $-(z_1 + z_2 + z_3)$, which we recognize as the negative of the shape invariant σ. The z coefficient is the second symmetric function

$$\sigma_2(z_1, z_2, z_3) = z_1 z_2 + z_2 z_3 + z_3 z_1.$$

Since $z_1 z_2 z_3 = 1$ and $|z_i|^2 = z_i \overline{z_i} = 1$, we have

$$\sigma_2(z_1, z_2, z_3) = (1/z_1) + (1/z_2) + (1/z_3) = \overline{z_1} + \overline{z_2} + \overline{z_3}.$$

Thus $P(z)$ has the following form:

$$P(z) = z^3 - \sigma z^2 + \overline{\sigma} z - 1. \tag{2}$$

However, *not all polynomials of the form (2) have all their roots on the unit circle.* Hence one may think of Problem 1 as a converse to formula (2):

Problem 2. For which values of σ does the cubic polynomial $P(z) = z^3 - \sigma z^2 + \overline{\sigma} z - 1$ have all its roots on the unit circle?

It seems very unlikely that Problem 2 is new; in fact, a *sufficient* condition for all the roots of a *quartic* to lie on the unit circle is given in [7], and the corresponding sufficient condition for cubics can be derived as a special case. However, the solution to Problem 2 given below seems somewhat more enlightening.

To obtain a slightly more geometric phrasing of the Problems 1 and 2, we invoke the following lemma:

Lemma 2. *If* $|z_1| = |z_2| = |z_3|$, *then the orthocenter of* $\triangle z_1 z_2 z_3$ *is the point* $z = z_1 + z_2 + z_3$.

A proof of this lemma, which is not all difficult, can be found in [3]. Thus another way to phrase Problem 1 is

Problem 3. Describe the locus of the orthocenters of all triangles $\triangle z_1 z_2 z_3$ such that the vertices lie on the unit circle and $z_1 z_2 z_3 = 1$.

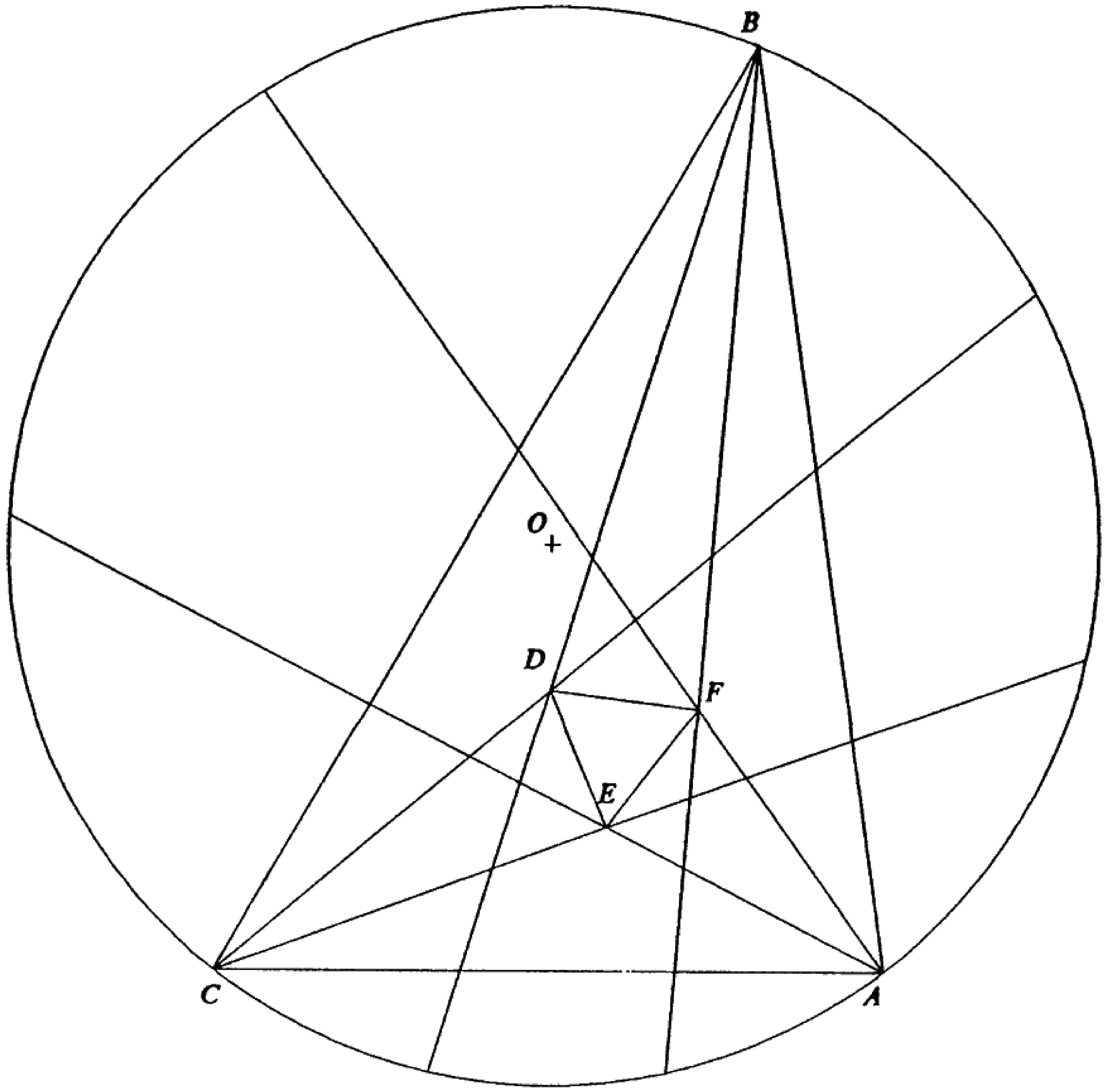

Figure 1. Morley's Theorem and Lemma 3. D, E, F **are the intersection points of «adjacent» trisectors of** $\angle A, \angle B, \angle C$.

Unfortunately, this version is still not «fully geometric», since it involves the curious hypothesis that $z_1 z_2 z_3 = 1$. A surprising geometric interpretation of the meaning of this hypothesis is given in the next lemma.

Lemma 3. *If* $|z_1| = |z_2| = |z_3|$, *then the altitudes of the Morley triangle of* $\triangle z_1 z_2 z_3$ *are in the directions of the three cube roots of* $z_1 z_2 z_3$.

Here a little explanation is in order. The *Morley triangle* of $\triangle ABC$ is the triangle formed by the intersection points of the adjacent angle trisectors of $\angle A, \angle B, \angle C$. *Morley's theorem* (see [8], [9]; the former contains a bibliography of 150 articles pertaining to this theorem!) asserts that *the Morley triangle* ($\triangle DEF$ in Figure 1) *is equilateral.* In fact, Morley's theorem would be a corollary of Lemma 3 if we proved Lemma 3 first (since any triangle whose altitudes make angles of $2\pi/3$ with each other must be equilateral). However, it will be more convenient for us to assume Morley s theorem and make use of the calculations in the proof published by J. Hofmann [3]. (C. Lubin [6] has the identical formulas).

A second word of clarification: in Lemma 3, the altitudes should be understood as the *rays* from a vertex perpendicular to the opposite side and intersecting the line containing that side. We adopt this convention so that the direction of the altitudes will be unambiguously defined.

Proof of Lemma 3. By applying a dilation, we may assume that $|z_i| = 1$, so that we may write $z_k = e^{3i\theta_k}$ for $k = 1, 2, 3$. Further, we may assume that $\theta_2 < \theta_3 < \theta_1 < \theta_2 + 2\pi/3$. This amounts to the assumption that z_1, z_2, z_3 are labeled clockwise, with z_2 assumed to have the smallest argument. This convention agrees with Figure 12 of [3]. As in that article, we let u, v, w denote the coordinates of the vertices of the Morley triangle, viewed as complex numbers, and let $a = e^{i\theta_1}, b = e^{i\theta_2}, c^{i\theta_3}$. Hofmann calculates that

$$u = -\epsilon bc(\epsilon b + c) + a(\epsilon^2 b^2 + \epsilon bc + c^2),$$

$$v = -\epsilon ca(\epsilon c + a) + \epsilon b(\epsilon^2 c^2 + \epsilon ac + a^2),$$

$$w = -ab(a + b) + c(a^2 + ab + b^2),$$

where $\epsilon = e^{2i\pi/3}$. One of the altitudes of $\triangle DEF$ is represented by the vector $\frac{1}{2}(u+v) - w$. A straightforward calculation shows that

$$\frac{1}{2}(u + v) - w = \frac{\sqrt{3}}{2}(b - c)(a - c)(e^{-\pi/6}a + e^{-i\pi/6}b).$$

Because $\theta_2 < \theta_1 < \theta_2 + 2\pi/3$, the argument of $(e^{i\pi/6}a + e^{-i\pi/6}b)$ is the same as the argument of $a + b$. Hence

$$\operatorname{Arg}\left[\frac{1}{2}(u + v) - w\right] = \operatorname{Arg}(b - c)(a - c)(a + b).$$

We claim that $(b-c)(a-c)(a+b)$ has the same argument as abc, which will finish the proof, since abc is a cube root of $z_1 z_2 z_3$. Indeed,

$$(b-c)(a-c)(a+b)\overline{abc} = -2\Re(a-c)(\bar{b}-\bar{c}),$$

and the right-hand side is real and positive because the angle between the vectors $a-c$ and $b-c$ is greater than $2\pi/3$. □

Thanks to Lemma 3, we can restate Problem 1 in completely geometric terms.

Problem 4. Given a triangle with vertices on the unit circle, oriented so that one of the altitudes of the Morley triangle is in the direction of the positive x-axis, describe the region in which its orthocenter must lie.

THE SOLUTION TO PROBLEMS 1-4

Problem 2, being the most algebraic version, is the easiest to solve.

Theorem 4. *For all complex numbers σ, the discriminant of $P(z) = z^3 - \sigma z^3 + \overline{\sigma} z - 1$ is real. One root of $P(z)$ always lies on the unit circle. All three roots lie on the unit circle if and only if the discriminant is negative.*

For the definition of the discriminant of a cubic polynomial, see [4, p. 259]. The facts we will need can be stated very briefly. First, if the roots of the polynomial are z_1, z_2, and z_3, the discriminant Δ is defined to be $(z_1 - z_2)^2 (z_2 - z_3)^2 (z_3 - z_1)^2$. Second, if the polynomial is given by $P(z) = z^3 - c_1 z^2 + c_2 z - c_3$, the discriminant can be computed directly from the following equation:

$$\Delta = -4c_1^3 c_3 + c_1^2 c_2^2 + 18 c_1 c_2 c_3 - 4 c_2^3 - 27 c_3^2. \tag{3}$$

Finally, if c_1, c_2, c_3 are real, then all three roots of the polynomial are real if and only if the discriminant is positive.

Proof of Theorem 4. Using formula (3), we can compute

$$\text{disc } P = -4\overline{\sigma}^3 - 4\sigma^3 + |\sigma|^4 + 18|\sigma|^2 - 27, \tag{4}$$

which is obviously real.

From complex analysis, the linear fractional transformation $f(w) = (iw+1)/(iw-1)$ is one-to-one and maps the extended real axis onto the unit circle. Hence the roots of $P(z)$

lie on the unit circle if and only if the solutions to the equation $P(f(w)) = 0$ lie on the extended real axis. By a straightforward computation, if we set $\sigma = x + iy$, then

$$P(f(w)) = \frac{-2yw^3 + (2x-6)w^2 - 2yw + (2x+2)}{(iw-1)^3} \tag{5}$$

Since the numerator of (5) is a real cubic polynomial, it must have a real root w_1, and hence $z_1 = f(w_1)$ is a root of P lying on the unit circle (if $y = 0$, then $w_1 = \infty$ and $z_1 = 1$).

Assuming that $y \neq 0$, all the roots of $P(z)$ lie on the unit circle if and only if all the roots of

$$Q(w) = w^3 - \left(\frac{x-3}{y}\right)w^2 + w - \left(\frac{x+1}{y}\right)$$

are real. Using equation (3) again, we have

$$\text{disc } Q = -4\frac{y^4 + 2(x^2+12x+9)y^2 + (x+1)(x-3)^3}{y^4} = -4\frac{\text{disc } P}{(\Im\sigma)^4}. \tag{6}$$

Hence the roots of $P(z)$ lie on the unit circle if and only if disc $P \leq 0$.

If $y = 0$ it is clear that the numerator of (5) has real roots if and only if $-1 \leq x \leq 3$, which is the same conclusion we get by setting disc $P \leq 0$. □

Corollary 5. *The roots of $P(z)$ lie on the unit circle if and only if $\sigma = x + iy$ lies in the region*

$$y^4 + 2(x^2+12x+9)y^2 + (x+1)(x-3)^3 \leq 0. \tag{7}$$

This region is the union of a deltoid and its interior.

Proof. This follows immediately from the expansion of disc P in equation (6). The fact that the boundary of this region is a deltoid can be verified by consulting the reference [5, p. 132]. □

Remark. There is one other well-known appearance of deltoids in the geometry of triangles: the theorem that the envelope of the Simson lines of a given triangle is a deltoid (see [2, Example 9.14.34.3D]). That deltoid, however, is circumscribed about the *nine-point circle* of a triangle, rather than about the circumcircle.

Theorem 6. *If σ satisfies inequality (7), then the roots of $P(z)$ form an acute triangle if and only if $|\sigma| < 1$, a right triangle if and only if $|\sigma| = 1$, and an obtuse triangle if and onli if*

$|\sigma| > 1$. *They form an isosceles triangle if and only if* $\text{Arg}(\sigma) = k\pi/3$ *for an integer* k, *and an equilateral triangle if and only if* $\sigma = 0$.

Proof. If $\triangle z_1 z_2 z_3$ is acute, the orthocenter lies in the interior of the triangle, hence in the interior of the circumcircle $|z| = 1$. If $\triangle z_1 z_2 z_3$ is right, the orthocenter is the vertex of the right angle, which lies on $|z| = 1$. If $\triangle z_1 z_2 z_3$ is obtuse, the orthocenter lies in the exterior of the circumcircle (we leave this to the reader to prove).

If $\triangle z_1 z_2 z_3$ is isosceles, we can write the three vertices as $e^{i\theta}, e^{i(\theta+\alpha)}$, and $e^{i(\theta-\alpha)}$. Then $z_1 z_2 z_3 = e^{3i\theta}$, so $\theta = 0, \frac{2\pi}{3}$, or $\frac{4\pi}{3}$. Moreover, $z_1 + z_2 + z_3 = e^{i\theta}(1 + 2\cos\alpha)$, so $\text{Arg}(z_1 + z_2 + z_3) = 0, 0 + \pi, \frac{2\pi}{3}, \frac{2\pi}{3} + \pi, \frac{4\pi}{3}$ or $\frac{4\pi}{3} - \pi$.

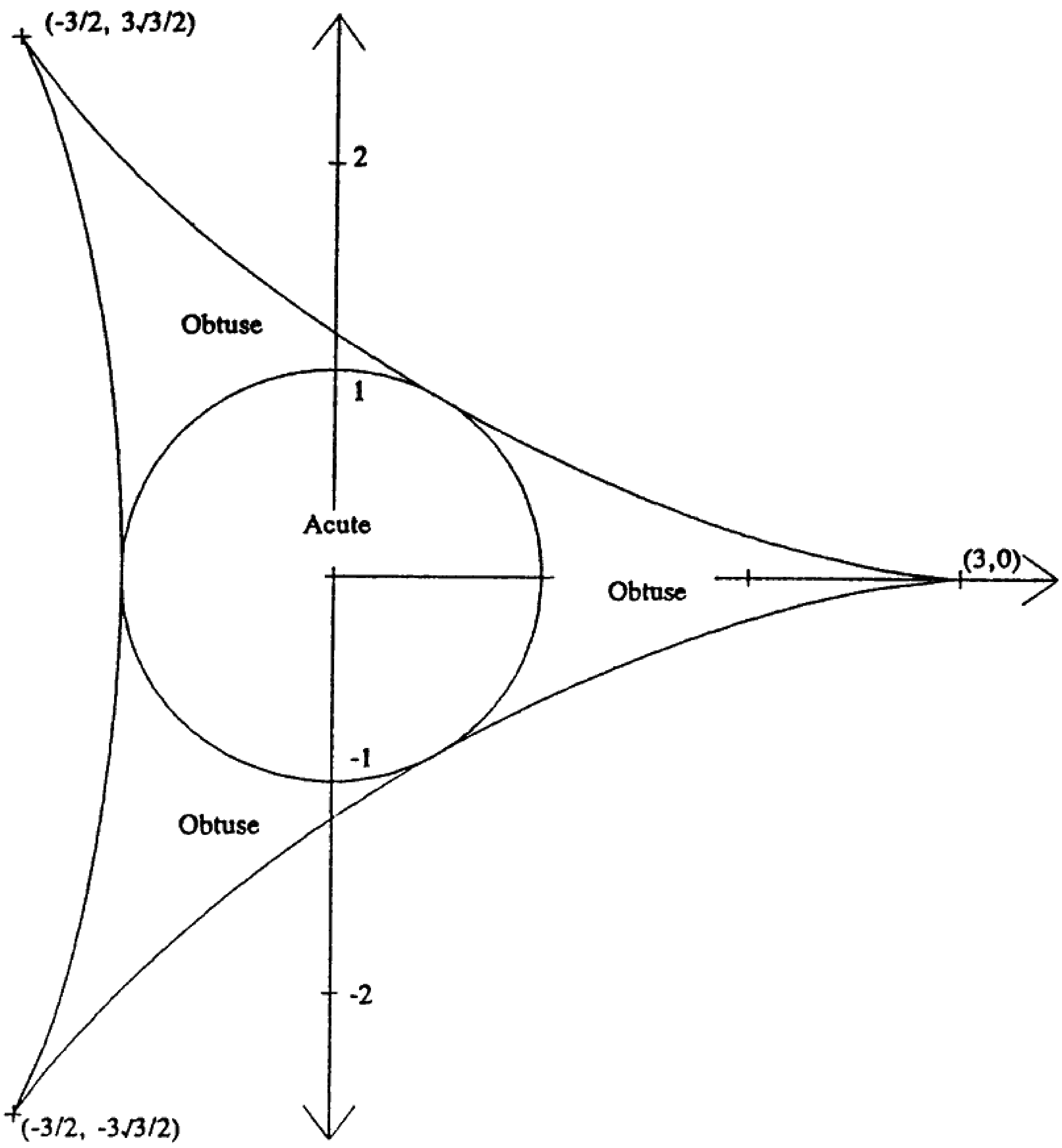

Figure 2. The deltoid described in Corollary 5. The shape invariant $\sigma(\triangle ABC)$ always lies in the interior of the deltoid; it lies in the unit disk if $\triangle ABC$ is acute, and in the region labeled «Obtuse» if $\triangle ABC$ is obtuse.

Conversely, if σ has one of the arguments listed above, we can write $\sigma = e^{i\theta}(1 + 2\cos\alpha)$ for some α, and for some $\theta = 0, 2\pi/3$ or $4\pi/3$. Then $P(z)$ factors as

$$(z - e^{i\theta})(z - e^{i(\theta+\alpha)})(z - e^{i(\theta-\alpha)}),$$

hence the roots form an isosceles triangle.

Finally, we leave it to the reader to prove that $\triangle z_1 z_2 z_3$ is equilateral if and only if its circumcenter and orthocenter coincide, hence $\sigma = 0$. □

Figure 2 illustrates the region described by inequality (7), along with the results of Theorem 7. As discussed in the previous section, these theorems answer all the other versions of Problems 1-4. Figure 3 illustrates, in particular, a triangle with its Morley triangle oriented as in Problem 4, showing that the orthocenter does indeed lie in the interior of the deltoid.

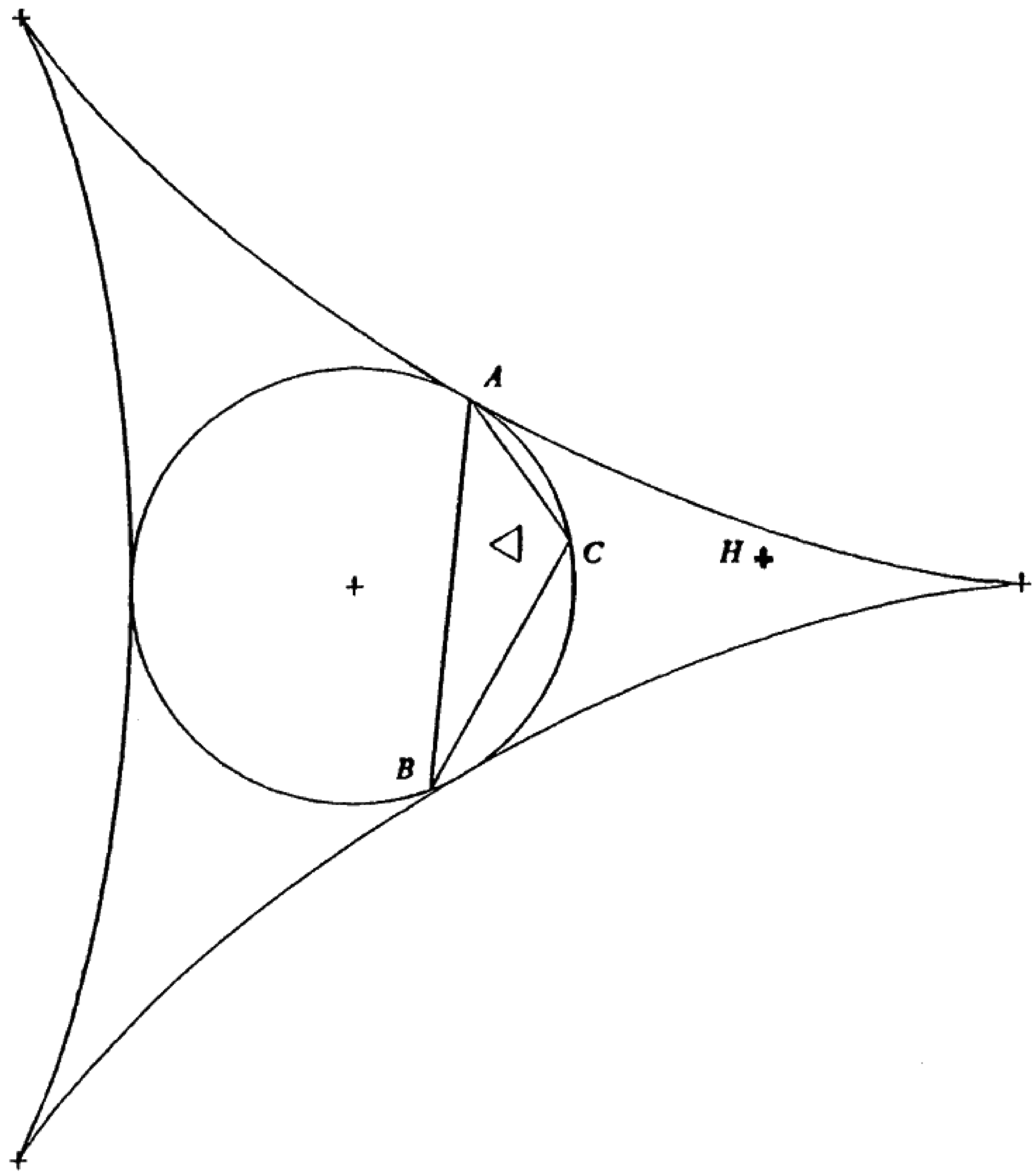

Figure 3. An obtuse triangle $\triangle ABC$, its Morley triangle, and its orthocenter H. Note that one altitude of the Morley triangle points in the positive x-direction, and that H lies in the region labeled «Obtuse» in Figure 2.

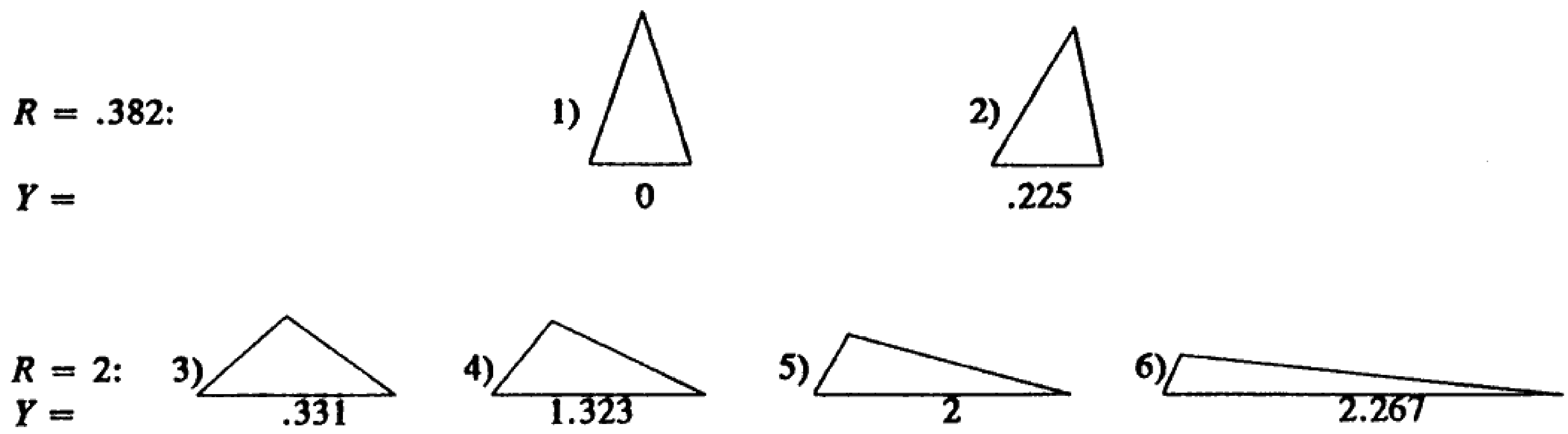

Figure 4. The triangles 1-6 listed in Table 1, along with their obtuseness R and scaleneness Y (to three decimal places).

Theorem 7 explains our choice of terminology for the R-invariant and Y-invariant discussed earlier. Since $\triangle z_1 z_2 z_3$ becomes increasingly obtuse as R increases, it is natural to call R the obtuseness; similarly we call Y the scaleneness because $Y = 0$ if $\triangle z_1 z_2 z_3$ is isosceles and $Y > 0$ if it is scalene. To give an intuitive sense of the meaning of these invariants, we illustrate various triangles along with their obtuseness and scaleneness in Figure 4. The angles of these triangles are given in Table 1.

The geometry of the invariant $Z = \sigma^3(\triangle z_1 z_2 z_3)$ is not quite as nice as the geometry of σ. It turns out that the region of possible values of Z is also described by a quartic inequality:

$$Y^4 + 2(X^2 + 216X + 3645)Y^2 + (X + 1)(X - 27)^3 \leq 0 .$$

The curve which bounds this region is described more easily by the parametric formula

$$(X, Y) = (12 + 14 \cos\theta + \cos 2\theta, 2 \sin\theta - \sin 2\theta) .$$

Table 1. Some triangles with rational angles and their shape invariants.

Number in Fig. 4	*Angles (in degrees)*	R	Y
1.	$36 - 72 - 72$	ϕ^{-2}	0
2.*	$42 - 60 - 78$	ϕ^{-2}	$\phi^{-5/2} 5^{1/4}/2$
3.*	$36 - 42 - 102$	2	$3^{1/2}\phi^{-2}/2$
4.	$\frac{180}{7} - \frac{360}{7} - \frac{720}{7}$	2	$7^{1/2}/2$
5.	$15 - 60 - 105$	2	2
6.*	$6 - 66 - 108$	2	$3^{1/2}\phi^2/2$

Note: An asterisk (*) indicates those triangles for which R and Y have been calculated only by computer. ϕ denotes the number $(1 + \sqrt{5})/2$.

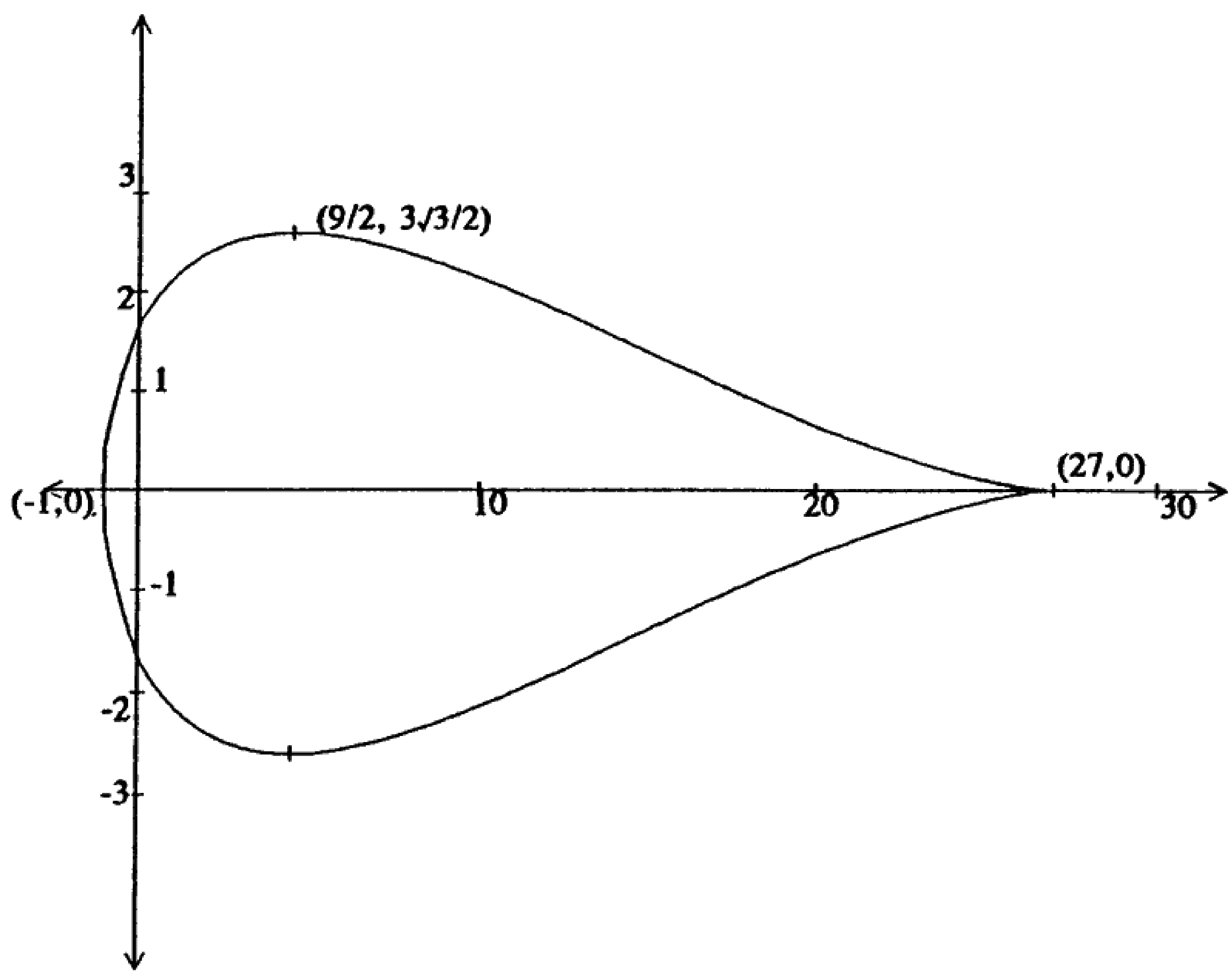

Figure 5. The curve obtained by applying the transformation $f(z) = z^3$ to the deltoid in Figure 2. The shape invariant $Z(\triangle ABC)$ always lies in the interior of this curve.

This curve is sketched in Figure 5; to the best of my knowledge it is not a «named curve» (in particular, it is not congruent to a piriform, though it resembles one).

FORMULAS FOR THE SHAPE INVARIANTS

Now that we have defined the shape invariants σ, X, Y, and R and have some idea of their geometric meaning, it is reasonable to ask how one can compute them for a given triangle. We will discuss this question from two points of view; first, if the side lengths a, b, c are given; second, if the angles $\angle A, \angle B, \angle C$ are given.

We begin by examining the case of triangles $\triangle z_1 z_2 z_3$ satisfying the conclusion of Lemma 1. The basic strategy is to find a formula for our invariants which is symmetric with respect to the squared side lengths

$$a^2 = |z_2 - z_3|^2, b^2 = |z_3 - z_1|^2, c^2 = |z_1 - z_2|^2$$

and is invariant under similarities. From the latter fact, in particular, it will follow that the same formula holds for arbitrary triangles.

Using the definition $\sigma = z_1 + z_2 + z_3$, it is easy to verify that

$$a^2 + b^2 + c^2 = 9 - |\sigma|^2 = 9 - R, \tag{8}$$

and hence the obtuseness $R = 9 - a^2 - b^2 - c^2$. Note that this formula is not, however, invariant under similarity.

The computation of Y is quite tedious; hence we describe the general procedure and omit the details. Since $Y = 0$ if and only if two of the numbers a^2, b^2, c^2 are equal, it is reasonable to think that the discriminant $\Delta = (a^2 - b^2)^2(b^2 - c^2)^2(c^2 - a^2)^2$ may have something to do with Y. We can compute the discriminant by using formula (3), where $c_1 = a^2 + b^2 + c^2$, $c_2 = a^2b^2 + a^2c^2 + b^2c^2$, and $c_3 = a^2b^2c^2$. Formula (8) already gives an expression for c_1 in terms of R. The corresponding formulas for c_2 and c_3 are

$$c_2 = 27 - 9R + 2X, \tag{9}$$

$$c_3 = 8X + 27 - 18R - R^2. \tag{10}$$

(The latter follows from Theorem 4 and equation (4)). Substituting (9) and (10) into (3), we compute

$$\Delta = 4(R^3 - X^2)(8X + 27 - 18R - R^2) = 4Y^2a^2b^2c^2.$$

Thus we arrive at an elegant formula for Y :

$$Y = \frac{|(a^2 - b^2)(b^2 - c^2)(c^2 - a^2)|}{2abc}. \tag{11}$$

Formula (11), like (8), is not scale-invariant, which is not surprising since it was derived under the assumption that the circumradius r of $\triangle z_1z_2z_3$ is 1. To obtain scale-invariant formulas, valid for all triangles $\triangle ABC$, we simply multiply or divide by the appropriate power of r :

$$R = 9 - \frac{a^2 + b^2 + c^2}{r^2},$$

$$Y = \frac{|(a^2 - b^2)(b^2 - c^2)(c^2 - a^2)|}{2abcr^3}$$

Finally, using the law of sines, we obtain formulas for Y and R in terms of the angles of $\triangle ABC$:

$$R = 9 - 4(\sin^2 A + \sin^2 B + \sin^2 C), \tag{12}$$

$$Y = 4\frac{|(\sin^2 A - \sin^2 B)(\sin^2 B - \sin^2 C)(\sin^2 C - \sin^2 A)|}{\sin A \sin B \sin C} \tag{13}$$

Equation (12) may also be written as

$$R = 1 - 8 \cos A \cos B \cos C, \tag{14}$$

which makes the fact that $R = 1$ when $\triangle ABC$ is a right triangle much more evident.

Formulas (13) and (14) were used in generating the values in Table 1. For example, since

$$\cos(\pi/7)\cos(2\pi/7)\cos(4\pi/7) = -\frac{1}{8},$$

the obtuseness of triangle (4) in Figure 4 is 2. To derive similar formulas for the other triangles requires a very good understanding of cyclotomic fields, and the reader may note that I do not have computer-free proofs for some of the required identities. An interesting problem for further exploration would be to determine whether there are other triangles with rational angles (expressed in degrees) with particularly simple values of R and Y.

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Received March 18, 1992 and in revised form June 20, 1992
D.N. Mackenzie
Kenyon College,
Gambier, OH 43022,
U.S.A.