Note di Matematica Vol. XIII - n. 2, 269-276 (1993)

EXISTENCE RESULTS FOR A CLASS OF NONHOMOGENEOUS ELLIPTIC EQUATIONS WITH CRITICAL SOBOLEV EXPONENT

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Abstract. *In this paper we study the problem* $\Delta u + u|u|^{2^*-2} + f(x) = 0$ *in* $\Omega, u(x) = 0$ *on* $\partial\Omega$. *Since the embedding of* $H_0^1(\Omega)$ *in* $L^{2^*}(\Omega)$ *is not compact, classical variational and fixed-points approaches can not be applied to find solutions. We study that problem by making use of a fixed-point Theorem as well as one from approximation methods.*

1. INTRODUCTION

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$. In this paper we are concerned with the problem of finding $u \in H_0^1(\Omega)$ satisfying the nonlinear elliptic equation

$$\begin{aligned} \Delta u + u|u|^{2^*-2} + f(x) &= 0 \text{ in } \Omega \\ u(x) &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{1.1}$$

when $f \in L^\infty(\Omega), 2^* = \dfrac{2n}{n-2}, n \geq 3$.

The exponent 2^* is critical for the Sobolev embedding of $H_0^1(\Omega)$ in $L^{2^*}(\Omega)$. This embedding is not compact and therefore classical variational and fixed-points approaches can not be applied to find solutions of (1.1). In this paper we study this problem by making use of a fixed-point Theorem as well as one from approximation methods. This approach has already been used in [16] where the second author has studied a closely related problem. The problem (1.1) has already been studied in [14] and [5] since a variational point of view and in [10] with arguments of set valued functions.

In [14], if $f \neq 0$ and

$$\int_\Omega fu \leq c_n(\|u\|_{1,2})^{\frac{n+2}{2}}, \tag{1.2}$$

for all $u \in H_0^1(\Omega)$ and $\|u\|_{2^*} = 1$, where $c_n = \dfrac{4}{n-2}\left(\dfrac{n-2}{n+2}\right)^{\frac{n+2}{2}}$, then (1.1) has, at least, a weak solution. And if the inequality (1.2) is strict, (1.1) has, at least, two weak solutions. In particular (1.2) certainly holds if

$$\|f\|_{H^{-1}} \leq c_n S^{\frac{n}{4}}, \tag{1.3}$$

where S is the best Sobolev constant, (cf. [13]), and where H^{-1} denotes the dual Space $(H_0^1(\Omega))^*$.

For $f \geq 0$ it is know that (1.1) cannot admit positive solution when $||f||_{H^{-1}}$ is too large, see [7], [11] and [15].

It would seem that (1.3) is sharp, but here we give, in some instances, a better estimate than (1.3). For example in the case $f \in L^2(\Omega)$ and $n \geq 4$. We will prove that if $||f||_{2^+} \leq c_n S^{\frac{n+2}{4}}$, where $\frac{1}{2^+} + \frac{1}{2^*} = 1$, then (1.1) has a weak Solution. Notice that $||f||_{2^+} \leq |\Omega|^{\frac{1}{2^+} - \frac{1}{2}} ||f||_2 = ||f||_2$ (if we assume, whitout loss of generality, that $|\Omega| = 1$). Now, if $f \in L^2(\Omega)$ then $||f||_2 = ||f||_{H^{-1}}$. On the other hand $S \geq n\left(\frac{n-2}{n-1}\right)^2$. See [2], p. 41. Then $S > 1$ for $n \geq 4$. Thus we have that $c_n S^{\frac{n}{4}} < c_n S^{\frac{n+2}{4}}$, for $n \geq 4$. It shows us that our estimate is better than (1.3).

The problem (1.1) has been widely studied in the case $f = 0$. If for example Ω is star-shaped, (1.1) has no nonzero solutions. It is a consequence of Pohozaev's identity (cf. [12]). On the other hand, by a remarkable result of Bahri and Coron, (cf. [1]), the Topology of Ω plays an important role which may cancel Pohozaev's obstruction. They show that if $\Omega \subset \mathbb{R}^n, n \geq 3$, is a bounded domain with nontrivial Topology the problem (1.1), in the case $f = 0$, has a nonzero solution. For a survey and perspectives about the problem (1.1) we refer the reader to [3], [4] and [6].

2. NOTATIONS AND PRELIMINARIES

Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ defined as $g(x,u) = u|u|^{s-1} + f(x), 1 < s \leq N = 2^* - 1$. Then the operator of Nemytsky $G : L^{s+1}(\Omega) \to L^{\frac{s+1}{s}}(\Omega)$ defined as $G(u)(x) = g(x, u(x))$ is continuous and bounded, so that for every $\epsilon > 0$ there exists $r = r(\epsilon)$ such that if $||u||_{s+1} \leq r$ then

$$||g(x,u) - g(x,0)||_{\frac{s+1}{s}} \leq \epsilon$$

and we have the following inequality, cf. [8] p. 26,

$$||g(x,u)||_{\frac{s+1}{s}} \leq \left(\left(\frac{||u||_{s+1}}{r} \right)^{s+1} + 1 \right)^{\frac{s}{s+1}} \epsilon + ||f||_{\frac{s+1}{s}}. \tag{2.1}$$

We will indicate the norm in $L^p(\Omega)$ with $||\cdot||_p$. It is easy to see that for $g(x,u)$ a relationship between ϵ and r can be taken as $\epsilon = r^s$. It is well known that $||u||_{s+1} \leq K(s)||u||_{1,2}$, where

$$K(s) = \frac{1}{\sqrt{S}} |\Omega|^{\frac{1}{s+1} - \frac{1}{2^*}}. \tag{2.2}$$

Definition 2.1. *We say that $u \in H_0^1(\Omega)$ is a weak solution of (1.1) if for all $v \in H_0^1(\Omega)$, $s = N$,*

$$(2.3) \qquad \langle u, v\rangle_{1,2} = \int_\Omega g(x,u)v.$$

For $u \in H_0^1(\Omega)$ fixed, the right side of (2.3) defines a linear continuous functional, then by Riesz's Theorem there exists $F_s : H_0^1(\Omega) \to H_0^1(\Omega)$ such that

$$(2.4) \qquad \langle F_s(u), v\rangle_{1,2} = \int_\Omega g(x,u)v.$$

Then $u \in H_0^1(\Omega)$ is a weak solution of (1.1) if and only if u is a fixed point of F_N. It is well known that only for $s < 2^* - 1 = N, F_s$ is compact.

3. THE MAIN RESULTS

Our first result is the following

Theorem 3.1. *Assume that $f \in L^\infty(\Omega)$ and suppose that*

$$(3.1) \qquad \|f\|_{2^*} \leq c_n S^{\frac{n+2}{4}},$$

then the problem (1.1) has, at least, a weak solution if we assume that $\partial\Omega$ is sufficiently smooth.

Proof. First we will consider the following problem

$$(3.2) \qquad \begin{aligned} \Delta u + u|u|^{s-1}f(x) &= 0 \text{ in } \Omega \\ u(x) &= 0 \text{ on } \partial\Omega, \end{aligned}$$

where $1 < s < N = 2^* - 1$. By (2.1), (2.4) and by using $\|u\|_{s+1} \leq K(s)\|u\|_{1,2}$ we see that (3.2) has a weak solution if there exists $\alpha > 0$ such that for all $\|u\|_{1,2} = \alpha$

$$(3.3) \qquad \left(\left(\left(K(s)\frac{\|u\|_{1,2}}{r}\right)^{s+1} + 1\right)^{\frac{s}{s+1}} \epsilon + \|f\|_{\frac{s+1}{s}}\right) K(s)\|u\|_{1,2} \leq \|u\|_{1,2}^2.$$

See [9] p. 107. Since $\epsilon = r^s$ and we can take ϵ sufficiently small we conclude that (3.2) has a weak solution if there exists $\alpha > 0$ such that

$$(3.4) \qquad (K(s))^{s+1}\alpha^s + K(s)\|f\|_{\frac{s+1}{s}} < \alpha.$$

It is easy to see that (3.4) has the solution

$$\alpha_s = \left(\frac{1}{s(K(s))^{s+1}}\right)^{\frac{1}{s-1}}, \tag{3.5}$$

if

$$\|f\|_{\frac{s+1}{s}} < \frac{s-1}{s^{\frac{s}{s-1}}(K(s))^{\frac{2s}{s-1}}}. \tag{3.6}$$

Now, since $\|f\|_{\frac{s+1}{2}} \to \|f\|_{2^*}$ if $s \to N = 2^* - 1$ and the right side of (3.6) converges to the right side of (3.1) we conclude the existence of $s_0 < N$ such that for all $s \in (s_0, N)$ the inequality (3.6) holds and there exists α_s, as in (3.5), satisfying (3.4). That is, for all $s \in (s_0, N)$ the problem (3.2) has a weak solution u_s. Also we know that $\|u_s\|_{1,2} \leq \alpha_s$, and since $\{\alpha_s, s \in (s_0, N)\}$ is bounded, it follows that $\{u_s\}$ is bounded as well. Then there exists a subsequence of $\{u_s\}$, that we have labeled in the same form, such that $u_s \rightharpoonup u_N$ for some $u_N \in H_0^1(\Omega)$. Furthermore, there exists a subsequence of $\{u_s\}$, labeled in the same form, such that $u_s \to u_N$ strongly in $H_0^1(\Omega)$. In fact, let $t \in (s_0, N)$ fixed and let A be a bounded subset of $H_0^1(\Omega)$. Then for $s > t$, s near t, and for $u, v \in A$ we have

$$\begin{aligned}
\|F_s(u) - F_t(v)\|_{1,2} &= \sup_{\|\phi\|_{1,2}=1} \{|\langle u|u|^{s-1} - v|v|^{t-1}, \phi\rangle_2|\} \\
&\leq c_1 \|u|u|^{s-1} - v|v|^{t-1}\|_{\frac{s+1}{s}} \\
&\leq c_1 c_2 \|u|u|^{s-1} - v|v|^{t-1}\|_{\frac{t+1}{t}} \\
&= c_1 c_2 \left\| u|u|^{\frac{s-t}{t}} \left(\left|u|u|^{\frac{s-t}{t}}\right|\right)^{t-1} - v|v|^{t-1} \right\|_{\frac{t+1}{t}} \\
&\leq c_1 c_2 \left\{ \frac{\left\| u|u|^{\frac{s-t}{t}} - v \right\|_{t+1}^{t+1}}{r^{t+1}} + 1 \right\}^{\frac{t}{t+1}} \epsilon.
\end{aligned}$$

Since $\epsilon = r^t$ and we can take r arbitrarily small, we obtain

$$\|F_s(u) - F_t(v)\|_{1,2} \leq c_1 c_2 \left\| u|u|^{\frac{s-t}{t}} - v \right\|_{t+1}^{t} \leq c_1 c_2 \|u - v\|_{t+1}^{t} + o(1) \tag{3.7}$$

as $s \to t$, where

$$o(1) = \sup_{v,u \in A} c_1 c_2 \left(\left\| u|u|^{\frac{s-t}{t}} - v \right\|_{t+1}^{t} - \|u - v\|_{t+1}^{t} \right).$$

Now, since $\{u_s\}$, $u_s = F(u_s)$, is a bounded sequence in $H_0^1(\Omega)$ thus by (3.7) we have

$$\|u_s - u_t\|_{1,2} \le c_1 c_2 \|u_s - u_t\|_{t+1}^{t} + o(1). \tag{3.8}$$

Now, we do not lose generality by assuming $u_s \to u_t$ in $L^{t+1}(\Omega)$. In fact, since $\{u_s\}$ is bounded and the embedding $H_0^1(\Omega) \hookrightarrow L^{t+1}(\Omega)$ is compact $u_s \to \hat{u}_t$ in $L^{t+1}(\Omega)$. Then, by (3.7) we get $\hat{u}_t = F(\hat{u}_t)$. Thus we can take $u_t = \hat{u}_t$. By using (3.8) we get that $\{u_s\}$ has a Cauchy's subsequence, then $u_s \to u_N$ strongly. Our next step is to show that u_N is a weak solution of (1.1). First let us notice that since $f \in L^\infty(\Omega)$, by an iterative argument (bootstrapping procedure) we can see that $u_s \in C^{0,\beta}(\Omega), \beta \in (0,1)$, and since $\partial\Omega$ is smooth, u_s is, in particular, continuous on $\overline{\Omega}$, see [2] p. 50. We will prove that u_N is a weak solution of (1.1) making use of the following diagram

$$\begin{array}{ccc}
\langle u_s|u_s|^{r-1} + f, v\rangle_2 & \xrightarrow[r\to N]{A} & \langle u_s|u_s|^{N-1} + f, v\rangle_2 \\
s \to N \big\downarrow C & r, s \to N \searrow D & B \big\downarrow s \to N \\
\langle u_N|u_N|^{r-1} + f, v\rangle_2 & \xrightarrow[r\to N]{E} & \langle h + f, v\rangle_2 \\
 & r \to N \searrow F & \| H \\
 & & \langle u_N|u_N|^{N-1} + f, v\rangle_2 \\
 & & \| I \\
 & & \langle u_N, v\rangle_{1,2}
\end{array}$$

The identity I shows us that $u_N \in H_0^1(\Omega)$ is a weak solution of (1.1). We proceed now to establish the displayed convergences.

VERIFICATION OF CONVERGES

A) We see that the convergence A holds by using the following two facts: a) for all $v \in H_0^1(\Omega)$ fixed, $\langle ., v\rangle_2$ defines a continuous functional on $L^{2^*}(\Omega)$ and b) since u_s is continuous we have, by Lebesgue's dominated convergence Theorem, that

$$u_s|u_s|^{r-1} \to u_s|u_s|^{N-1}, \; r \to N,$$

on $L^{2^+}(\Omega)$.

B) The sequence $\{u_s\}$ is bounded in $H_0^1(\Omega)$ and thus is bounded in $L^{2^*}(\Omega)$ as well. Furthermore, Nemytsky's operator $L^{2^*}(\Omega) \to L^{2^+}(\Omega)$ defined by $u|u|^{N-1}$ is bounded, so that $\{u_s|u_s|^{N-1}, s \in (s_0, N)\}$ is bounded in $L^{2^+}(\Omega)$ and therefore there is a subsequence, labeled in the same form, of $\{u_s|u_s|^{N-1}\}$ and $h \in L^{2^+}(\Omega)$ such that B holds.

C) Since $u_s \rightharpoonup u_N$, we have that $u_s|u_s|^{r-1} \to u_N|u_N|^{r-1}, s \to N$, in $L^{\frac{r+1}{r}}(\Omega)$. Then the convergence C holds.

D) We claim that for each $\epsilon > 0$ there exists $s_1 = s_1(\epsilon)$ such that

$$|\langle u_s|u_s|^{r-1} + f, v\rangle_2 - \langle u_N|u_N|^{r-1} + f, v\rangle_2| < \frac{\epsilon}{2}, \tag{3.9}$$

for all $s \in (s_1, N)$ and for all $r \in [s_0, N]$. In fact:

$$\begin{aligned} |\langle u_s|u_s|^{r-1} - u_N|u_N/^{r-1}, v\rangle_2| &\leq \|u_s|u_s|^{r-1} - u_N|u_N|^{r-1}\|_{\frac{r+1}{r}}\|v\|_{r+1} \\ &\leq c\|u_s|u_s|^{r-1} - u_N|u_N|^{r-1}\|_{\frac{r+1}{r}}\|v\|_{N+1} \\ &\leq \left(\frac{\|u_s - u_N\|_{r+1}^{r+1}}{\delta^{r+1}} + 1\right)^{\frac{r+1}{r}} \theta c\|v\|_{N+1} \\ &= (\|u_s - u_N\|_{r+1}^{r+1} + \delta^{r+1})^{\frac{r+1}{r}} c\|v\|_{N+1}, \end{aligned} \tag{3.10}$$

where $\theta = \delta^r$ and $c = |\Omega|^{\frac{1}{s_0+1} - \frac{1}{N+1}}$. Since δ can be taken arbitrarly small we obtain from (3.10)

$$|\langle u_s|u_s|^{r-1} - u_N|u_N|^{r-1}, v\rangle_2| \leq \|u_s - u_N\|_{N+1}^{r}\|v\|_{N+1}(c^{N+1} + d), \tag{3.11}$$

for some $d > 0$. Let $t_0 < N$ such that $\|u_s - u_N\|_{N+1} < 1$ for $s \geq t_0$, then by (3.11) we have

$$|\langle u_s|u_s|^{r-1} - u_N|u_N|^{r-1}, v\rangle_2| \leq \|u_s - u_N\|_{N+1}^{s_0}\|v\|_{N+1}(c^{N+1} + d), \tag{3.12}$$

for $s \geq t_0$. By (3.12) and since $u_s \to u_N$ in $H_0^1(\Omega)$ we get (3.9). Now, by using (3.9), the convergence B and that $\{u_s\}$ are continuous it is easy to see that the double limit D holds.

E) Follows from C, D and F.

F) Since $u_N|u_N|^{r-1}v \leq (|u_N|^N + 1)|v|$ if $|u_N| \geq 1$, Lebesgue's dominated convergence Theorem tells us that F holds.

H) It is a consequence of D and the fact that

$$\langle u_s|u_s|^{s-1} + f, v\rangle_2 = \langle F_s(u_s), v\rangle_{1,2} = \langle u_s, v\rangle_{1,2} \to \langle u_N, v\rangle_{1,2},$$

if $s \to N$. The proof is complete. □

The following Theorem takes on the same steps of Theorem 3.1 and then its proof will be left out

Theorem 3.2. *Suppose that*

$$2^{\frac{n+2}{2n}}|\lambda||\Omega|^{\frac{2}{n}} < S. \tag{3.13}$$

Then if $f \in L^\infty(\Omega)$ *satisfies*

$$\|f\|_{2^*} \leq c_n \left(1 - \frac{2^{\frac{n+2}{2n}}}{S}|\lambda||\Omega|^{\frac{2}{n}}\right)^{\frac{n+2}{2n}} \left(2^{\frac{n+2}{2n}}\left(\frac{1}{S}\right)^{\frac{n+2}{n-2}}\right)^{\frac{2-n}{4}}, \tag{3.14}$$

for $n \geq 3$, *the following problem has a week solution*

$$\begin{aligned} \Delta u + \lambda u + u|u|^{2^*-2} + f(x) &= 0 \ \text{ in } \Omega \\ u(x) &= 0 \ \text{ on } \partial\Omega. \end{aligned} \tag{3.15}$$

Remark. For $\lambda = 0$ the estimate (3.14) does not coincide with (3.1), therefore (3.14) is not sharp. For $f \geq 0$ and $\lambda \geq 0$ the solutions of the problems (1.1) and (3.15) are positive solutions. This is a consequence of maximum principle.

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Received July 8, 1992 and in revised form January 20, 1993

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