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REMARKS CONCERNING THE SEPARABLE QUOTIENT PROBLEM

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1. INTRODUCTION

One of the famous unsolved problems of Functional Analysis is whether every infinite dimensional Banach space

(*) *E has a separable infinite-dimensional quotient.*

Likely this problem has been considered since 1932, but the first mentions connected with this question appear in [8] and [5]. It is known that every infinite-dimensional reflexive Banach space, or even every infinite-dimensional WCG Banach space, has an infinite-dimensional separable quotient, [12], Corollary 15.3.7.

For a Banach space E condition (*) is equivalent to each of the following conditions (cf. [1], [4], [9], [10]):

(a) E has a strictly increasing sequence of closed subspaces of E whose union is dense in E.

(b) E has a dense non-barrelled subspace.

(c) E has a dense proper subspace dominated by a Banach (or Frechet) space.

It is known that every non-normable Frechet space as well as every (LF)-space has an infinite-dimensional separable quotient, [6], [9], [11].

For F*-spaces,* i.e. metrizable and complete topological vector spaces (tvs), the separable quotient problem has been solved negatively: Popov, [7], showed that for every non-atomic measure space (Ω, Σ, μ) with $|\Sigma| > 2^{\aleph_o}$ there exists a subset Ω_1 of Ω with $\mu(\Omega_1) > 0$ such that the space $L^p(\Omega_1), 0 < p < 1$, has no infinite-dimensional separable quotient.

Because $L^p(\Omega_1)$ densely, properly and continuously includes $L^1(\Omega_1)$, the space $L^p(\Omega_1)$ has property (c) and property (b) (where barrelledness is replaced by ultrabarrelledness in the sense of Iyahen [2]). As easily seen $L^p(\Omega_1)$ satisfies also property (a).

Although problem (*) remains unsolved, we obtain several sufficient (and necessary) conditions for a tvs E to have an infinite-dimensional separable quotient E/M whose topological dual (E/M) separates points of E/M from zero. In this case we shall write: E has a *quotient.* Among the others we generalize a known result for Banach spaces by showing that every infinite dimensional dual-separating F-space which is WCG has a *quotient.* Developing ideas found in papers mentioned above we obtain the relationship between conditions (*), (a), (b), (c) but considered in the category of topological vector spaces.

The terminology and notations are, for the most part, those of [3]. Let (E, τ) be a tvs over the field $\mathbb{K}$ of either real or complex numbers. By E' we denote the *topological dual* of (E, τ). It is easy to see that for a subspace M of (E, τ) the following assertions are equivalent:

(i) M is weakly closed, i.e. closed in the weak topology $\sigma(E, E')$ of (E, τ).

(ii) The weak topology of the quotient space $(E/M, \tau/M)$ is Hausdorff.

(iii) The space $(E/M, \tau/M)$ is dual-separating, i.e. $(E/M)'$ separates points of E/M from zero.

We shall need several times the following variant of the Hahn-Banach theorem, cf. [3], Corollary 7.3.6:

Let V be a weakly closed absolutely convex subset of a tvs (E, τ). Then for every $x \notin V$ there exists $f \in V^0$ such that $f(x) > 1$, where $V^0 = \{g \in E' : |g(y)| \leq 1$ for all $y \in V\}$.

By $m = m(E, E')$ we denote the finest locally convex topology on E which is weaker than τ. The absolutely convex hulls of neighbourhoods of zero in (E, τ) form a fundamental system of neighbourhoods of zero for m. Clearly $\sigma(E, E') \leq m(E, E') \leq \tau$. If (E, τ) is a dual-separating ultrabarrelled space, then $(E, m(E, E'))$ is a (Hausdorff) barelled space.

Finally, we shall say that (E, τ) is *diminated* by an F-space if E admits a finer metrizable and complete vector topology.

2. RESULTS

As we mentioned a Banach space E has a *quotient* if E has property (a). Our first proposition extends this result to tvs.

Proposition 1. *Let $E = (E, \tau)$ be a tvs. The following assertions are equivalent:*

(i) E has a quotient.

(ii) There exists a strictly increasing sequence (E_n) of weakly closed subspaces of E whose union is dense in E.

(iii) There exists a biorthogonal sequence $((x_n, f_n)) \subset E \times E'$ such that $(\mathrm{sp}(x_k : k \in \mathbf{N}) + \cap(f_k^{-1}(0) : k \in \mathbf{N}))$ is dense in E.

Proof. (i)$\Rightarrow$(ii): Assume that $Z := (E/M, \tau/M)$ is a *quotient.* Let (y_n) be a linearly independent sequence in Z such that $\mathrm{sp}(y_n : n \in \mathbf{N})$ is dense in Z. Since the quotient map $Q : E \to Z$ is weakly continuous and open, $E_n := Q^{-1}(\mathrm{sp}(y_i \dots y_n))$ is a weakly closed proper subspace of $E, n \in \mathbf{N}$, and $\bigcup_{n\geq 1} E_n$ is dense in E.

(ii)$\Rightarrow$(iii): We may assume that $\dim(E_{n+1}/E_n) \geq n$ for all $n \in \mathbf{N}$. We construct a biorthogonal sequence $((x_n, f_n)) \subset Ex\acute{E}'$ such that $x_n \in E_{n+1}$ and $f_n \in E_n^0$ for all $n \in \mathbf{N}$. Let $x_1 \in (E_2 \backslash E_1)$. Then there exists $f_1 \in E_1^0$ such that $f_1(x_1) = 1$. Suppose we have already found $(x_1, f_1) \dots (x_n, f_n)$ as claimed. Let

$$x_{n+1} \in (E_{n+2} \cap f_1^{-1}(0) \cap \dots \cap f_n^{-1}(0) \backslash E_{n+1}).$$

Then there exists $f_{n+1} \in E^0_{n+1}$ such that $f_{n+1}(x_{n+1}) = 1$.

Since $E_{n+1} \subset (\mathrm{sp}(x_1, x_2, \ldots, x_n) + \cap(f_k^{-1}(0) : k \in \mathbf{N}))$ for all $n \in \mathbf{N}$, so $(\mathrm{sp}(x_k : k \in \mathbf{N}) + \cap(f_k^{-1}(0) : k \in \mathbf{N}))$ is dense in E.

(iii)$\Rightarrow$(i): $E/\cap(f_k^{-1}(0) : k \in \mathbf{N})$ is a *quotient.*

In order to prove our next result we shall need the following simple Lemma; its proof is obvious.

Lemma 1. *Let V be a weakly closed absolutely convex subset of a tvs (E, τ) and L a finite-dimensional subspace of E. Then $(V + L)$ is weakly closed in (E, τ).*

Proposition 2. *Let (E, τ) be a tvs such that*

(i) E contains an increasing sequence (V_n) of weakly closed absolutely convex subsets of E whose union is linearly dense in (E, τ).

(ii) E contains a subspace E_0 dominated by an F-space (E_0, τ_0) such that $\dim(E_0/E_0 \cap \mathrm{sp}(V_n)) = \infty$ *for all $n \in \mathbf{N}$.*

Then (E, τ) has a quotient.

Proof. Let (U_n) be a basis of balanced neighbourhoods of zero in (E_0, τ_0) such that $U_{n+1} + U_{n+1} \subset U_n$, $n \in \mathbf{N}$. First we construct a biorthogonal sequence $((x_n, f_n)) \subset E \times E'$ such that $x_n \in U_n$ and $f_n \in V_n^0$ for all $n \in \mathbf{N}$. Choose $x_1 \in U_1 \cap (E_0 \setminus \mathrm{sp}(V_1))$. Then there exists $f_1 \in V_1^0$ with $f_1(x_1) = 1$. Suppose we have found $(x_1, f_1), \ldots, (x_n, f_n)$ as claimed. Choose

$$x_{n+1} \in U_{n+1} \cap [(E_0 \cap f_1^{-1}(0) \cap \ldots \cap f_n^{-1}(0) \setminus (E_0 \cap \mathrm{sp}(V_{n+1} \cup \{x_1, \ldots, x_n\}))];$$

this is possible since $E_0 \cap f_1^{-1}(0) \cap \ldots \cap f_n^{-1}(0)$ has finite codimension in E_0 and $E_0 \cap \mathrm{sp}(V_{n+1} \cap \mathrm{sp}(x_1, \ldots, x_n))$ is a subspace of E_0 of infinite-codimension. Then there exists $f_{n+1} \in (V_{n+1} + \mathrm{sp}(x_1, x_2, \ldots, x_n))^0$ with $f_{n+1}(x_{n+1}) = 1$.

For every $m \in \mathbf{N}$ and $y \in V_m$ the series $\sum_{n=1}^{\infty} f_n(y)x_n$ converges in (E_0, τ_0), hence also in $(E.\tau)$. If $y_0 = \sum_{n=1}^{\infty} f_n(y)x_n$, then $f_n(y - y_0) = 0$ for all $n \in \mathbf{N}$, so $(y - y_0) \in M$, where $M = \bigcap_{k \geq 1} f_k^{-1}(0)$, and $y \in \overline{M + \mathrm{sp}(x_n : n \in \mathbf{N})}^{\tau}$. Hence $\mathrm{sp}\left(\bigcup_{k \geq 1} V_k\right) \subset \overline{M + \mathrm{sp}(x_n : n \in \mathbf{N})}^{\tau}$, so the space $M + \mathrm{sp}(x_n : n \in \mathbf{N})$ is dense in (E, τ). Therefore $\mathrm{sp}(x_n + M : n \in \mathbf{N})$ is dense in $(E/M, \tau/M)$. We proved that E/M is a *quotient.*

Corollary 1. *Let* (E, τ) *be a tvs which contains a sequence* (V_n) *as in Proposition 2. If* E *contains a compact absolutely convex subset* K *such that* $\dim(\mathrm{sp}(K)/\mathrm{sp}(K \cap V_n) = \infty$, $n \in \mathbf{N}$, *then* (E, τ) *has a quotient.*

We shall say that a subset V of a tvs (E, τ) is a *(proper) generator of* (E, τ) if V is an absolutely convex set whose linear span is a (proper) dense subspace of (E, τ).

Lemma 2. *Every separable* F*-space contains a compact generator.*

Proof. Let (y_n) be a sequence in E such that $Z := \mathrm{sp}(y_n : n \in \mathbf{N})$ is dense in E and $\Sigma \|y_n\| < \infty$, where $\| \ \|$ denotes and F-norm defining the original topology of E. It is enough to see that $T : l^1 \to E$, $T((x_k)) := \sum_{j=1}^{\infty} x_j y_j$ is a compact linear map.

Proposition 3. *Let* (E, τ) *be a tvs dominated by an* F*-space* (E, τ_0). *Then the space* (E, τ) *has a quotient if* E *contains a weakly closed generator* V *such that* $\dim(E/\mathrm{sp}(V)) = \infty$.

Proof. Let $H := (E/M, \tau/M)$ be a *quotient* and S a dense $\aleph_0$-dimensional subspace of H. By S_0 we denote the (τ_0/M)-closure of S. Since $(S_0, (\tau_0/M)|S_0)$ is a separable F-space, it contains a compact generator K. Hence K is (τ_0/M)-compact, and K is weakly closed in H. Clearly K is also linearly dense in H. Moreover, $\dim(S_0/\mathrm{sp}(K)) = \infty$; otherwise K would be a compact neighbourhood of zero in $(\mathrm{sp}(K), (\tau_0/M)|\,\mathrm{sp}(K))$ (see [2], Proposition 3.1). Therefore $V := Q^{-1}(K)$, where $Q : E \to E/M$ is the quotient map, is a weakly closed generator of (E, τ) such that $\dim(E/\mathrm{sp}(V)) = \infty$.

For the converse, it is enough to use Proposition 2 (we put $V_n = V$, $n \in \mathbf{N}$).

It is easy to see that a Banach (or barrelled) space E contains a dense non-barrelled subspace if E contains a proper closed generator.

Our Proposition 3 and Iyahen's Proposition 3.1 lead to the following generalization of the equivalence between (*) and (b).

Corollary 2. *An* F*-space* E *has a quotient if* E *has a proper weakly closed generator.*

Corollary 3. *If a dual-separating* F*-space* (E, τ) *contains a proper dense subspace* L *dominated by a Frechet space* (L, τ_0), *then* (E, τ) *has a* quotient.

Proof. Let $m := m(E, E')$. Then (E, m) is barrelled. Since $m_0 := m|L$ is strictly weaker than τ_0, the space (L, m_0) is not barrelled (by the open mapping theorem, [3], 11.3.5). Hence (L, m_0) contains a barrel W which is not a neigbourhood of zero in (L, m_0). Then $V := \overline{W}^m$ is a proper weakly closed generator of (E, τ). Now Corollary 2 completes the proof.

Remark. The assumption that (E, τ) is dual-separating cannot be removed; cf. Introduction, remark concerning Popov's result.

A tvs (E, τ) is called *weakly compact generated (WCG)* if E has a weakly compact generator.

The following corollary generalizes a known result for Banach spaces.

Corollary 4. *If (E, τ) is an infinite-dimensional dual-separating F-space which is WCG, then (E, τ) has a* quotient.

Proof. Let V be a weakly compact generator of (E, τ). By Corollary 2 it is enough to consider the case when $\mathrm{sp}(V) = E$. But then $(E, m(E, E'))$ is a reflexive Banach space, and consequently $\tau = m(E, E')$.

A tvs (E, τ) will be called an $(LF)_{tv}$-space if (E, τ) is the inductive limit of a strictly increasing sequence (E_n, τ_n) of F-spaces such that $\tau_{n+1}|E_n \leq \tau_n, n \in \mathbf{N}$; the sequence (E_n, τ_n) will be called a *defining sequence* for (E, τ).

Proposition 4. *Let (E, τ) be an $(LF)_{tv}$-space with a proper weakly closed generator V. Then (E, τ) has a* quotient.

Proof. If $Q : E \to E/L$ is the quotient map, where $L : \cap(f^{-1}(0) : f \in E')$, then (as easily seen) $Q(V)$ is a proper weakly closed generator of the ultrabarrelled space $(E/L, \tau/L)$. Hence $\mathrm{sp}(Q(V))$ is a non-barrelled subspace of the barrelled space $(E/L, m(E/L, (E/L)'))$. By Theorem 11.3.2 of [3] $\dim((E/L)/\mathrm{sp}(Q(V)) > \aleph_0$, and consequently $\dim(E/\mathrm{sp}(V)) > \aleph_0$. Therefore there exists $n_0 \in \mathbf{N}$ such that $\dim(E_{n_0}/E_{n_0} \cap \mathrm{sp}(V)) = \infty$, where (E_n, τ_n) is a defining sequence for (E, τ). Now the conclusion follows from Proposition 2.

Corollary 5 ([9], Theorem 3). *Every (LF)-space (E, τ) has a* quotient.

Proof. Let (E_n, τ_n) be a defining sequence for (E, τ). If no space E_n is τ-dense we can apply Proposition 1. Now suppose that E_k is τ-dense for some $k \in \mathbf{N}$. Since the identity map $(E_k, \tau_k) \to (E, \tau)$ is not almost open (by the open mapping theorem), (E, τ) contains a proper weakly closed generator.

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