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ON EMBEDDINGS OF RELATIONAL CATEGORIES INTO ALGEBRAIC CATEGORIES

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Abstract. *We show that the fundamental relationship between relational systems and universal algebras, i. e. the relationship that universal algebras are special cases of relational systems, is valid also conversely: relational systems can be considered as special universal algebras. This result is formulated on the category theoretical level - for relational systems we define a special type of homomorphisms with respect to which relational categories can be embedded into algebraic categories.*

In [3], M. Novotný introduced and studied strong homomorphisms of binary relational systems. He proved that the category of binary relational systems with strong homomorphisms as morphisms can be embedded into the category of unary algebras (with usual homomorphisms as morphisms). An analogical result for ternary relational systems and groupoids has been proved by the same author in [4]. The aim of this note is to generalize the results from [3] and [4] to relational systems and algebras of arbitrary, not necessarily finite, arities.

In this paper, relations are considered in the general sense as introduced in [6]: if G, K are sets, then any set of maps $R \subseteq G^K$ is said to be a *relation* on G. The set K is called the *type* of R. By a *relational system* we understand a pair $\langle G, R\rangle$ where G is a set and R is a relation on G. The set G is called the *underlying set* of $\langle G, R\rangle$ and the type of R is said to be the *type* of $\langle G, R\rangle$. Throughout the paper, all relational systems are assumed to have non-empty types. Let $\langle G, R\rangle$, $\langle H, S\rangle$ be relational systems of the same type. A *homomorphism* of $\langle G, R\rangle$ into $\langle H, S\rangle$ is any map $\varphi : G \to H$ fulfilling $f \in R \Rightarrow \varphi\circ f \in S$ (see [7]). If card $K = n < \aleph_0$, then relations of type K coincide with the well-known n-ary relations, and homomorphisms of relational systems of type K coincide with the usual homomorphisms of n-ary relational systems.

Definition. *Let $\langle G, R\rangle$, $\langle H, S\rangle$ be relational systems of the same type $K, t \in K$ and let $\varphi : G \to H$ be a map. Then φ is called a strong homomorphism of $\langle G, R\rangle$ into $\langle H, S\rangle$ w.r.t. t whenever the following condition is fulfilled:*
for any map $f \in G^{K-\{t\}}$, if $g \in H^K$ is a map with $g(u) = \varphi(f(u))$ for each $u \in K - \{t\}$, then $g \in S$ iff there is an extension $\overline{f}$ of f onto K such that $\overline{f} \in R$ and $g(t) = \varphi(\overline{f}(t))$.

Obviously, any strong homomorphism is a homomorphism. The notion of a strong homomorphism introduced above for relational systems of the same type is a generalization of the notion of a strong homomorphism introduced for binary relational systems in [3] and for ternary relational systems in [4]. Clearly, if n is a positive integer, $\langle G, R\rangle$, $\langle H, S\rangle$ relational systems of type $K = \{1, 2, \ldots, n\}$ (i. e. n-ary relational systems) and $t \in K$, then a strong homomorphism of $\langle G, R\rangle$ into $\langle H, S\rangle$ w.r.t. t is a map $\varphi : G \to H$ for which the following is fulfilled: if $x_i \in G$ for each $i \in K - \{t\}$ and $y \in H$, then putting $y_i = \varphi(x_i)$ for every $i \in K - \{t\}$ and $y_t = y$ we have $(y_1, y_2, \ldots, y_n) \in S$ iff there is an $x \in G$ with both $(x_1, x_2, \ldots, x_n) \in R$ whenever $x_t = x$ and $\varphi(x) = y$. It can easily be shown that the composition of two strong homomorphisms w.r.t. t as well as any identity map is a strong homomorphism w.r.t. t (whenever $t \in K$ where K is the type considered). Consequently, for any set $K \neq \emptyset$ and any $t \in K$ there is a category whose objects are relational systems of type K and whose morphisms are strong homomorphisms w.r.t. t. This category is denoted by $\mathrm{Rel}_{K,t}$.

The following algebraic concepts are taken from [5]. Given sets G, K, by an *operation* of type K on G we mean a map $p : G^K \to G$. The pair $\langle G, p\rangle$ is then called an *algebra* of type K. The set G is said to be *underlying set* of $\langle G, p\rangle$. Let $\langle G, p\rangle$, $\langle H, q\rangle$ be two algebras of the same type K and $\varphi : G \to H$ a map. Then φ is called a *homomorphism* of $\langle G, p\rangle$ into $\langle H, q\rangle$ if $\varphi(p(f)) = q(\varphi \circ f)$ for each $f \in G^K$. If card $K = n < \aleph_0$, then algebras of type K coincide with the well-known n-ary algebras, and homomorphisms of algebras of type K coincide with the usual homomorphisms of n-ary algebras. Obviously, the class of all algebras of the same given type K together with homomorphisms forms a category. This category is denoted by Alg_K.

Let $\langle G, p\rangle$ be an algebra of type K and let t be an element with $t \notin K$. We set

$$T_{p,t} = \left\{ f \in G^{K \cup \{t\}} ; f(t) = p(f|K) \right\}$$

where $f|K$ denotes the restriction of f onto K. The relational system $\langle G, T_{p,t}\rangle$ of type $K \cup \{t\}$ is said to be associated with the algebra $\langle G, p\rangle$. The reader can easily prove the following

Proposition 1. *Let $\langle G, p\rangle$, $\langle H, q\rangle$ be algebras of the same type K, $t \notin K$ an element and let $\varphi : G \to H$ be a map. Then the following statements are equivalent:*
(i) φ is a homomorphism of $\langle G, p\rangle$ into $\langle H, q\rangle$.
(ii) φ is a homomorphism of $\langle G, T_{p,t}\rangle$ into $\langle H, T_{q,t}\rangle$.
(iii) φ is a strong homomorphism of $\langle G, T_{p,t}\rangle$ into $\langle H, T_{q,t}\rangle$ w.r.t. t.

Strong homomorphisms can also be used for determining the relational systems that are associated with some algebras. More precisely, we shall prove the following

Proposition 2. *Let $\langle G, R\rangle$, $\langle H, S\rangle$ be two relational systems of the same type K, $t \in K$ and let $\varphi : G \to H$ be an injective strong homomorphism of $\langle G, R\rangle$ into $\langle H, S\rangle$ w.r.t. t. If there is an algebra $\langle H, q\rangle$ of type $K - \{t\}$ with $T_{p,t} = S$, then there is an algebra $\langle G, p\rangle$ of type $K - \{t\}$ with $T_{p,t} = R$.*

Proof. Let the assumptions of the statement be fulfilled and let $f \in G^{K-\{t\}}$. Put $g(u) = \varphi(f(u))$ for any $u \in K - \{t\}$ and let $\overline{g} \in H^K$ be the extension of g onto K with $\overline{g}(t) = q(g)$. Then $\overline{g} \in T_{q,t} = S$ and hence there is an extension $\overline{f}$ of f onto K with $\overline{f} \in R$ and $\overline{g}(t) = \varphi(\overline{f}(t))$. As φ is an injection, $\overline{f}(t) \in G$ is uniquely determined. Thus, putting $p(f) = \overline{f}(t)$ we have defined an operation p of type $K - \{t\}$ on G. Clearly, $T_{p,t} \subseteq R$. To prove the converse inclusion, let $\overline{f} \in R$ be an arbitrary map and let $f = \overline{f}|K - \{t\}$. Then $\varphi \circ \overline{f} \in S = T_{q,t}$, hence $\varphi(\overline{f}(t)) = q(\varphi \circ f) = \overline{g}(t)$ where $\overline{g}$ is defined in the same way as in the first part of the proof. Consequently, $\overline{f}(t) = p(f)$ and thus $\overline{f} \in T_{p,t}$. Therefore $R \subseteq T_{p,t}$ and the proof is complete.

For any set G we denote by $\mathscr{P}(G)$ the power set of G (i. e. the set of all subsets of G). If G, H are sets and $\varphi : G \to H$ a map, then by $\widehat{\varphi}$ we denote the map $\widehat{\varphi} : \mathscr{P}(G) \to \mathscr{P}(H)$ given by $\widehat{\varphi}(X) = \{\varphi(x); x \in X\}$ for each $X \subseteq G$.

Let K be a set and $t \in K$. For any relational system $\langle G, R\rangle$ of type K we set $\mathscr{F}_t(\langle G, R\rangle) = \langle \mathscr{P}(G), p\rangle$ where p is the operation of type $K - \{t\}$ on $\mathscr{P}(G)$ given by

$$p(r) = \{x \in G;\ \text{there is an } f \in R \text{ such that } f(u) \in r(u) \text{ for each } u \in K - \{t\} \text{ and } f(t) = x\}.$$

Next, if $\langle G, R\rangle$, $\langle H, S\rangle$ are relational systems of type K, then for any strong homomorphism φ of $\langle G, R\rangle$ into $\langle H, S\rangle$ w.r.t. t we set $\mathscr{F}_t(\varphi) = \widehat{\varphi}$.

Theorem 1. *Let K be a set and $t \in K$. Then $\mathscr{F}_t$ is an embedding of the category* $\mathrm{Rel}_{k,t}$ *into* $\mathrm{Alg}_{K-\{t\}}$.

Proof. Obviously, $\mathscr{F}_t$ assigns an object of $\mathrm{Alg}_{K-\{t\}}$ to any object of $\mathrm{Rel}_{K,t}$. Let $\langle G, R\rangle$, $\langle H, S\rangle$ be relational systems of type K and $\varphi : G \to H$ a strong homomorphism w.r.t. t. Let $\langle \mathscr{P}(G), p\rangle = \mathscr{F}_t(\langle G, R\rangle)$ and $\langle \mathscr{P}(H), q\rangle = \mathscr{F}_t(\langle H, S\rangle)$. We shall show that $\widehat{\varphi}$ is a homomorphism of $\langle \mathscr{P}(G), p\rangle$ into $\langle \mathscr{P}(H), q\rangle$, i.e. that $\widehat{\varphi}(p(r)) = q(\widehat{\varphi} \circ r)$ for any $r \in (\mathscr{P}(G))^{K-\{t\}}$. To this end, let $r \in (\mathscr{P}(G))^{K-\{t\}}$ and $y \in \widehat{\varphi}(p(r))$. Then there is an $x \in p(r)$ with $y = \varphi(x)$. Consequently, there is an $f \in R$ such that $f(u) \in r(u)$ for each $u \in K - \{t\}$ and $f(t) = x$. Putting $g = \varphi \circ f$ we get $g \in S$ since φ is a homomorphism. It follows that $g(u) = \varphi(f(u)) \in \widehat{\varphi}(r(u))$ for each $u \in K - \{t\}$ and $g(t) = \varphi(f(t)) = y$. Hence $y \in q(\widehat{\varphi} \circ r)$. We have proved the inclusion $\widehat{\varphi}(p(r)) \subseteq q(\widehat{\varphi} \circ r)$. Conversely, let $y \in q(\widehat{\varphi} \circ r)$. Then there is a $g \in S$ such that $g(u) \in \widehat{\varphi}(r(u))$ for each $u \in K - \{t\}$ and $g(t) = y$. Consequently, for each $u \in K - \{t\}$ there is an $x_u \in r(u)$ with $g(u) = \varphi(x_u)$. Let $f \in G^{K-\{t\}}$ be the map defined by $f(u) = x_u$ for each $u \in K - \{t\}$. Then $g(u) = \varphi(f(u))$ for each $u \in K - \{t\}$. As φ is a strong homomorphism, there is an extension $\overline{f}$

of f onto K such that $\overline{f} \in R$ and $g(t) = \varphi(\overline{f}(t))$. Since $\overline{f}(u) = x_u \in r(u)$ for each $u \in K - \{t\}$, putting $x = \overline{f}(t)$ we get $x \in p(r)$. As $y = g(t) = \varphi(\overline{f}(t)) = \varphi(x)$, we have $y \in \widehat{\varphi}(p(r))$. Hence $q(\widehat{\varphi} \circ r) \subseteq \widehat{\varphi}(p(r))$ and the equality $\widehat{\varphi}(p(r)) = q(\widehat{\varphi} \circ r)$ is proved. Since $\mathscr{F}_t$ clearly preserves both compositions of morphisms and identity morphisms, $\mathscr{F}_t$ is a (covariant) functor from $\mathrm{Rel}_{K,t}$ into $\mathrm{Alg}_{K-\{t\}}$. It is evident that $\mathscr{F}_t$ is one-to-one (both on objects and morphisms). Therefore $\mathscr{F}_t$ is an embedding of $\mathrm{Rel}_{K,t}$ into $\mathrm{Alg}_{K-\{t\}}$ and the proof is complete.

Now we are aiming to describe the subcategory of $\mathrm{Alg}_{K-\{t\}}$ that is isomorphic with $\mathrm{Rel}_{K,t}$. For this purpose we use the following two concepts (see e.g. [3]):

Let G, H be sets. A map $F : \mathscr{P}(G) \to \mathscr{P}(H)$ is called *totally additive* if $F\left(\bigcup_{i\in I} X_i\right) = \bigcup_{i\in I} F(X_i)$ for any system $\{X_i; i \in I\}$ of subsets of G, and it is called *atom-preserving* if for any $x \in G$ there is a $y \in H$ such that

$$F(\{x\}) = \{y\}.$$

Obviously, a map $F : \mathscr{P}(G) \to \mathscr{P}(H)$ is totally additive and atom-preserving iff there is a map $\varphi : G \to H$ such that $F = \widehat{\varphi}$ (the map φ is then given by $\varphi(x) = y$ where $y \in H$ is the element with $F(\{x\}) = \{y\}$).

Let G, K be sets and p an operation of type K on $\mathscr{P}(G)$. The operation p is said to be *totally additive* if $p(r) = \bigcup\{p(\tilde{r}); \tilde{r} \in (\mathscr{P}(G))^K$ is a map such that for each $u \in K$ there is an $x_u \in r(u)$ with $\tilde{r}(u) = \{x_u\}\}$ whenever $r \in (\mathscr{P}(G))^K$. For unary or binary operations this concept of total additivity coincides with that introduced in [3] or [4], respectively.

Lemma. *Let K be a set and $t \in K$. If $\langle G, R\rangle \in \mathrm{Rel}_{K,t}$ is an object and $\langle \mathscr{P}(G), p\rangle = \mathscr{F}_t(\langle G, R\rangle)$, then the operation p is totally additive.*

Proof. Let the assumptions of the statement be fulfilled. Let $r \in (\mathscr{P}(G))^{K-\{t\}}$ and $x \in p(r)$. Then there is an $f \in R$ such that $f(u) \in r(u)$ for each $u \in K - \{t\}$ and $f(t) = x$. For any $u \in K - \{t\}$ put $\tilde{r}(u) = \{f(u)\}$ and $x_u = f(u)$. Then $\tilde{r} \in (\mathscr{P}(G))^{K-\{t\}}$ and $x_u \in r(u)$, $\tilde{r}(u) = \{x_u\}$ for each $u \in K - \{t\}$. Since $f(u) \in \tilde{r}(u)$ for each $u \in K - \{t\}$ and $f(t) = x$, we have $x \in p(\tilde{r})$. Consequently, let $x \in p(\tilde{r})$ where $\tilde{r} \in (\mathscr{P}(G))^K$ is a map such that for each $u \in K - \{t\}$ there is an $x_u \in r(u)$ with $\tilde{r}(u) = \{x_u\}$. From $x \in p(\tilde{r})$ it follows that there is an $f \in R$ such that $f(u) \in \tilde{r}(u)$ for each $u \in K - \{t\}$ and $f(t) = x$. We have $f(u) \in \tilde{r}(u) = \{x_u\} \subseteq r(u)$ for each $u \in K - \{t\}$. Therefore $x \in p(r)$. The total additivity of p is proved.

Obviously, identity maps of power sets are totally additive and atom preserving, and the composition of two totally additive atom-preserving maps is also totally additive atom-preserving. For this reason, given a set K, we can define a subcategory Pal_K of Alg_K as follows: the objects of Pal_K are precisely the algebras $\langle \mathscr{P}(G), p\rangle$ of type K where G is a set and p is a totally additive operation of type K on $\mathscr{P}\langle G\rangle$, and the morphism in Pal_K are precisely the totally additive atom-preserving homomorphisms.

Theorem 2. *Let K be a set and $t \in K$. Then $\mathscr{F}_t$ is an isomorphism of* $\mathrm{Rel}_{K,t}$ *onto* $\mathrm{Pal}_{K-\{t\}}$.

Proof. By virtue of Theorem 1 and Lemma it is sufficient to prove that the functor $\mathscr{F}_t$: $\mathrm{Rel}_{K,t} \to \mathrm{Pal}_{K-\{t\}}$ is surjective both on objects and morphisms. Therefore, let $\langle \mathscr{P}(G), p\rangle \in \mathrm{Pal}_{K-\{t\}}$ be an object. Put $R = \{f \in G^K; f(t) \in p(s)$ where $s \in (\mathscr{P}(G))^{K-\{t\}}$ is the map given by $s(u) = \{f(u)\}$ for each $u \in K - \{t\}\}$. Then $\langle G, R\rangle \in \mathrm{Rel}_{K,t}$ and we shall show that $\mathscr{F}(\langle G, R\rangle) = \langle \mathscr{P}(G), p\rangle$, i.e. that $p(r) = \{x \in G;$ there is an $f \in R$ such that $f(u) \in r(u)$ for each $u \in K - \{t\}$ and $f(t) = x\}$ for any $r \in (\mathscr{P}(G))^{K-\{t\}}$. To this end, let $r \in (\mathscr{P}(G))^{K-\{t\}}$ and $x \in p(r)$. As p is totally additive, there is a map $\tilde{r} \in (\mathscr{P}(G))^{K-\{t\}}$ with $x \in p(\tilde{r})$ such that for each $u \in K - \{t\}$ there is an $x_u \in r\{u\}$ fulfilling $\tilde{r}(u) = \{x_u\}$. Let $f \in G^K$ be the map given by $f(u) = x_u$ for each $u \in K - \{t\}$ and $f(t) = x$. Then $f(t) \in p(\tilde{r})$ and $\tilde{r}(u) = \{f(u)\}$ for each $u \in K - \{t\}$. Hence $f \in R$. We have found an $f \in R$ such that $f(u) \in r(u)$ for each $u \in K - \{t\}$ and $f(t) = x$. Conversely, let $x \in G$ be an element such that there is an $f \in R$ with $f(u) \in r(u)$ for each $u \in K - \{t\}$ and $f(t) = x$. From $f \in R$ it follows that $f(t) \in p(s)$ where $s \in (\mathscr{P}(G))^{K-\{t\}}$ is the map given by $s(u) = \{f(u)\}$ for each $u \in K - \{t\}$. Hence $x \in p(s)$ and putting $x_u = f(u)$ for each $u \in K - \{t\}$ we get $x_u \in r(u)$ and $s(u) = \{x_u\}$. This yields $x \in p(r)$ because p is totally additive. The equality $\mathscr{F}_t(\langle G, R\rangle) = \langle \mathscr{P}(G), p\rangle$ is proved and therefore the functor $\mathscr{F}_t : \mathrm{Rel}_{K,t} \to \mathrm{Pal}_{K-\{t\}}$ is surjective on objects. Further, let $F : \langle \mathscr{P}(G), p\rangle \to \langle \mathscr{P}(H), q\rangle$ be a morphism in $\mathrm{Pal}_{K-\{t\}}$. Let $\langle G, R\rangle$, $\langle H, S\rangle \in \mathrm{Rel}_{K,t}$ be the objects with $\mathscr{F}_t(\langle G, R\rangle) = \langle \mathscr{P}(G), p\rangle$ and $\mathscr{F}_t(\langle H, S\rangle) = \langle \mathscr{P}(H), q\rangle$. Denote by φ the map $\varphi : G \to H$ given by $\varphi(x) = y \Leftrightarrow F(\{x\}) = \{y\}$. Then $F = \widehat{\varphi} = \mathscr{F}_t(\varphi)$ and we are to prove that φ is a strong homomorphism of $\langle G, R\rangle$ into $\langle H, S\rangle$ w.r.t. t.

To this end, let $f \in G^{K-\{t\}}$ and let $g \in H^K$ be a map with $g(u) = \varphi(f(u))$ for each $u \in K - \{t\}$. Assume $g \in S$. Denote by s the map $s \in (\mathscr{P}(H))^{K-\{t\}}$ given by $s(u) = \{\varphi(f(u))\} = \widehat{\varphi}(\{f(u)\})$ for each $u \in K - \{t\}$. Then $g(u) \in s(u)$ for each $u \in K - \{t\}$ and thus $g(t) \in q(s)$. Let $r \in (\mathscr{P}(G))^{K-t}$ be the map defined by $r(u) = \{f(u)\}$ for each $u \in K - \{t\}$. Then $s = \widehat{\varphi} \circ r$, hence $q(s) = q(\widehat{\varphi} \circ r) = \widehat{\varphi}(p(r))$. Thus $g(t) \in \widehat{\varphi}(p(r))$, i.e. there is an $x \in p(r)$ with $g(t) = \varphi(x)$. It follows that there is an $h \in R$ such that $h(u) \in r(u)$ for each $u \in K - \{t\}$ and $h(t) = x$. For any $u \in K - \{t\}$ we have $h(u) = f(u)$, i.e. h is an extension of f onto K, and $g(t) = \varphi(h(t))$. Conversely, assume that there is an extension $\overline{f}$ of f onto K such that $\overline{f} \in R$ and $g(t) = \varphi(\overline{f}(t))$. Let $r \in (\mathscr{P}(G))^{K-\{t\}}$ be the map defined by $r(u) = \{f(u)\}$ for each $u \in K - \{t\}$. Then $\overline{f}(u) = f(u) \in r(u)$ whenever $u \in K - \{t\}$, hence $\overline{f}(t) \in p(r)$. Consequently, $\varphi(\overline{f}(t)) \in \widehat{\varphi}(p(r)) = q(\widehat{\varphi} \circ r)$. Let $s \in (\mathscr{P}(H))^{K-\{t\}}$ be the map given by $s(u) = \{\varphi(f(u))\}$ for each $u \in K - \{t\}$. Then $s = \widehat{\varphi} \circ r$ and thus $\varphi(\overline{f}(t)) \in q(s)$. Hence there is an $h \in S$ such that $h(u) \in s(u)$ for each $u \in K - \{t\}$ and $h(t) = \varphi(\overline{f}(t))$. This results in $g = h$ which implies $g \in S$. We have shown that φ is a strong homomorphism of $\langle G, R\rangle$ into $\langle H, S\rangle$ w.r.t. t. Therefore the functor $\mathscr{F}_t : \mathrm{Rel}_{K,t} \to \mathrm{Pal}_{K-\{t\}}$ is surjective on morphisms. The proof is complete.

Example 1. We denote by N the set of all non-negative integers and by N^+ the set of all positive integers. Let G be a set and $\pi : G \to \mathscr{P}(G^{N^+})$ a map. The elements of G^{N^+} are called sequences and the pair $\langle G, \pi\rangle$ is called a sequential space. We say that a sequence $\{x_i\}_{i\in N} \in G^{N^+}$ converges to $x \in G$ in $\langle G, \pi\rangle$ if $\{x_i\}_{i\in N^+} \in \pi(x)$. A sequential space $\langle G, \pi\rangle$ is said to be strict if each sequence $\{x_i\}_{i\in N^+} \in G^{N^+}$ fulfils the inequality card $\{x \in G; \{x_i\}_{i\in N^+} \in \pi(x)\} \leq 1$. If $\langle G, \pi\rangle$, $\langle H, \varrho\rangle$ are sequential spaces, then a homomorphism of $\langle G, \pi\rangle$ into $\langle H, \varrho\rangle$ is any map $\varphi : G \to H$ fulfilling $\{x_i\}_{i\in N^+} \in \pi(x) \Rightarrow \{\varphi(x_i)\}_{i\in N^+} \in \varrho(f(x))$ for each $x \in G$. By a strong homomorphism of $\langle G, \pi\rangle$ into $\langle H, \varrho\rangle$ we mean a map φ: $G \to H$ such that for any sequence $\{x_i\}_{i\in N^+} \in G^{N^+}$ and any element $y \in H$ the condition $\{\varphi(x_i)\}_{i\in N^+} \in \varrho(y)$ is valid iff there is an $x \in G$ with $\{x_i\}_{i\in N^+} \in \pi(x)$ and $\varphi(x) = y$. Hence each strong homomorphism is a homomorphism. We denote by Seq the category of sequential spaces as objects and strong homomorphisms as morphisms. The full subcategory of Seq whose objects are precisely the strict sequential spaces is denoted by Str. For any object $\langle G, \pi\rangle \in$ Seq we set $\mathscr{G}(\langle G\pi\rangle) = \langle G, R\rangle$ where $R \subseteq G^N$ is the relation defined by $R = \{f \in G^N; \{f(i)\}_{i\in N^+} \in \pi(f(0))\}$. For any morphism φ in Seq we set $\mathscr{G}(\varphi) = \varphi$. Then, clearly, $\mathscr{G}$ is an isomorphism of Seq onto $\mathrm{Rel}_{N,0}$. Further, for any object $\langle \mathscr{P}(G), p\rangle \in \mathrm{Pal}_{N^+}$ we set $\mathscr{H}(\langle \mathscr{P}(G), p\rangle) = \langle \mathscr{P}(G), \sigma\rangle$ where the operation $\sigma : \mathscr{P}(G) \to \mathscr{P}((\mathscr{P}(G))^{N^+})$ is defined by $\{X_i\}_{i\in N^+} \in \sigma(X)$ iff $X = p(r)$ whenever $r \in (\mathscr{P}(G))^{N^+}$ is the map given by $r(i) = X_i$ for each $i \in N^+$. For each morphism F in Pal_{N^+} we set $\mathscr{H}(F) = F$. Clearly, $\mathscr{H}$ is an embedding of Pal_{N^+} into Str. Thus, there is an embedding $\mathscr{E} = \mathscr{H} \circ \mathscr{F}_0 \circ \mathscr{G}$ of the category Seq into Str. Obviously, for any sequential space $\langle G, \pi\rangle$ the strict sequential space $\langle \mathscr{P}(G), \sigma\rangle = \mathscr{E}(\langle G, \pi\rangle)$ is given by $\{X_i\}_{i\in N^+} \in \sigma(X)$ iff $X = \{x \in G$; there is an $x_i \in X_i$ for each $i \in I$ such that $\{x_i\}_{i\in N^+} \in \pi(x)\}$. Of course, $\mathscr{E}(\varphi) = \widehat{\varphi}$ for each morphism φ in Seq.

Example 2. A relational system $\langle G, R\rangle$ of type K is said to be reflexive if R contains all constant maps of K into G. If $\langle G, R\rangle$ is a reflexive relational system of type K, $t \in K$ and

$\langle\mathscr{P}(G), p\rangle = \mathscr{F}_t(\langle G, R\rangle)$, then it can easily be seen that for any subset $X \subseteq G$ the constant map $r : K - \{t\} \to \mathscr{P}(G)$ given by $r(u) = X$ for each $u \in K - \{t\}$ fulfils $X \subseteq p(r)$.

In addition to the reflexivity the following important property of relational systems is introduced in [7]: a relational system $\langle G, R\rangle$ of type K is called diagonal if for any $g, h \in R^K$ with $g(u)(v) = h(v)(u)$ for all $u, v \in K$ the map $f \in G^K$ given by $f(u) = g(u)(u)$ for each $u \in K$ fulfils $f \in R$. This concept of diagonality is due to V. Novák who introduced it for n-ary relational systems (n a positive integer) in [2]. Clearly, an n-ary relational system $\langle G, R\rangle$ is diagonal iff for each $n \times n$-matrix M consisting of elements of G such that all rows and all columns of M belong to R it follows that also the diagonal of M belongs to R. Particularly, a binary relational system $\langle G, R\rangle$ is diagonal iff R is transitive. We shall show that for any diagonal relational system $\langle G, R\rangle$ of type K and any $t \in K$ the algebra $\langle\mathscr{P}(G), p\rangle = \mathscr{F}_t(\langle G, R\rangle)$ has the following property:

if $q \in (\mathscr{P}(G))^{(K-\{t\})\times(K-\{t\})}$ is a map such that for arbitrary $u, v \in K - \{t\}, u \neq v$, there is an $x \in G$ with $q(u,v) = \{x\} = q(v,u)$, then the map $r \in (\mathscr{P}(G))^{K-\{t\}}$ given by $r(u) = p(q_u)$ for each $u \in K - \{t\}$, where $q_u \in (\mathscr{P}(G))^{K-\{t\}}$ is the map defined by $q_u(v) = q(u,v)$ for every $v \in K - \{t\}$, fulfils $p(r) \subseteq p(s)$ whenever $s \in (\mathscr{P}(G))^{K-\{t\}}$ is the map given by $s(u) = q(u,u)$ for all $u \in K - \{t\}$.

To prove this, let $y \in p(r)$. Then there is an $f \in R$ such that $f(u) \in r(u)$ for each $u \in K - \{t\}$ and $f(t) = y$. As $f(u) \in r(u) = p(q_u)$, there is a $g_u \in R$ such that $g_u(v) \in q_u(v)$ for each $v \in K - \{t\}$ and $g_u(t) = f(u)$. Let $g \in R^K$ be the map given by $g(u) = g_u$ for each $u \in K - \{t\}$ and $g(t) = f$. Then $g(u)(v) \in q(u,v), g(v)(u) \in q(v,u)$, and since $q(u,v) = q(v,u)$ is a singleton, we have $g(u)(v) = g(v)(u)$ whenever $u, v \in K - \{t\}$. Next, $g(u)(t) = f(u) = g(t)(u)$ for any $u \in K - \{t\}$. It follows that $g(u)(v) = g(v)(u)$ for every pair $u, v \in K$. Consequently, the map $h \in G^K$ given by $h(u) = g(u)(u)$ for each $u \in K$ fulfils $h \in R$. We have $h(t) = g(t)(t) = f(t) = y$. Thus, as $h(u) = g(u)(u) \in q(u,u) = s(u)$ for each $u \in K - \{t\}$, $y \in p(s)$ and the proof is complete.

Particularly, let $\langle G, R\rangle$ be a diagonal ternary relational system (i.e. of type $\{1,2,3\}$) and $t \in \{1,2,3\}$. By the result proved, the groupoid $\langle\mathscr{P}(G), \cdot\rangle = \mathscr{F}_t(\langle G, R\rangle)$ has the following property: $(X \cdot \{y\}) \cdot (\{y\} \cdot Z) \subseteq X \cdot Z$ whenever $X, Z \subseteq G$ and $y \in G$.

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