Note di Matematica Vol. XIII - n. 2, 291-294 (1993)

A COMBINATORIAL LEMMA ON THE CARTESIAN PRODUCT OF FINITE SETS

U. BARTOCCI, A. GIRARDI

Summary. *A sufficient condition for the cartesian product of a finite family of equipotent finite sets to have elements without repetitions is given, and some application is discussed.*

As it is well known, graph theory furnishes a necessary and sufficient condition for the cartesian product of *any* finite family of finite sets to have elements without repetitions (this is Hall's Theorem; see for instance *Graph Theory with Applications*, J.A. Bondy & U.S.R. Murty, MacMillan, 1977, p. 72). In this paper we give a different simple sufficient condition in the case of a family of sets with the same number of elements. The proof, obtained by means of an easy combinatorial argument, has the advantage of not requiring graph theory.

Lemma. *Let A be a finite set of n elements $(n \geq 1)$, and F a family of r subsets $A_1, \ldots, A_r$ of A, all with the same cardinal number $k(1 \leq k \leq n)$, such that, when $r \geq k+1$, the intersection of any $(k+1)$-tuple of them is empty. Then, in the cartesian product $A_1 x \ldots x A_r$ there exists at least one element $(x_1, \ldots, x_r)$ without repetitions, i.e., such that $x_i \neq x_j$ for all $i, j = 1, \ldots, r, i \neq j$. In particular, it necessarily follows $r \leq n$.*

Proof. The statement is obviously true when $r \leq k$, and in this case one can even find a matrix M of type $k x r$ with elements in A, let us call it:

$$\begin{pmatrix} a_{11} & \cdots & a_{1r} \\ a_{21} & \cdots & a_{2r} \\ a_{k1} & \cdots & a_{kr} \end{pmatrix},$$

such that, all rows of M are elements without repetitions in the cartesian product of the family F, and all columns of M are permutations of the subsets $A_1, \ldots, A_r$ respectively.

Let us then proceed by induction with respect to $r \geq k$, starting from such a matrix M, and building a similar one of type $k x(r+1)$.

Let the elements of A_{r+1} be $y_1, \ldots, y_k$, then add the first element y_1 at the end of all k rows of M. It is obvious that, in the actual condition, at least one of the following k $(r+1)$-tuples:

$$(a_{11} \ldots a_{1r} y_1), \ldots,$$

$$(a_{k1} \ldots a_{kr} y_1),$$

will have no repetitions. As a matter of fact, y_1 cannot coincide in distinct $(r+1)$-tuples with elements a_{ij} which belong to the same column; moreover, y_1 cannot appear in all columns since the subset $A_1 \cap \ldots \cap A_r \cap A_{r+1}$ is empty.

Let us suppose the first row to be one of these particular $(r+1)$-tuples, and to have added in a similar way the first t elements $y_1, \ldots, y_t$ of A_{r+1}, for some $t, 1 \leq t \leq k$, respectively to the 1st, the 2nd, and the t-th row of M. We can proceed again by induction, but now with respect to t: we shall prove that one can add, if $t < k$, the element y_{t+1} to one of the remaining $(k-t)$ rows of M, whence the conclusion.

If all $(k-t)$ $(r+1)$-tuples

$$(a_{t+1,1} \ldots a_{t+1,r} y_{t+1}), \ldots,$$
$$(a_{k1} \ldots a_{kr} y_{t+1})$$

have repetitions, then - for the same reasons as before - at least one of the analogous $(r+1)$-tuples relative to an index between 1 and t does not have repetitions. Without any loss of generality, we can suppose that it is again the first $(r+1)$-tuple $(a_{11} \ldots a_{1r} y_{t+1})$ one of those which have no repetitions, as $(a_{11} \ldots a_{1r} y_1)$. Let us further suppose that, for instance, $a_{t+1,1} = y_{t+1}$.

Let us now look at the corresponding element a_{11} in the first row, and remark that if one could exchange in M the two elements a_{11} and $a_{t+1,1}$, then everything would be all right: because the $(t+1)-th$ row so obtained would be «good» for y_{t+1}, while the first would still be good for y_1. If this exchange would not be possible, it would mean that a_{11} appears somewhere in the $(t+1)-th$ row, say for instance $a_{t+1,r_1} = a_{11}$. Observe that r_1 is of course different from 1.

Let us now look at the pair

$$(a_{t+1,1}, a_{t+1,r_1} = a_{11})$$

and to its corresponding pair in the first row:

$$(a_{11}, a_{1r_1}).$$

If one could exchange these two pairs, in a similar way to the one just discussed for single elements, then everything would again be all right: because the new $(t+1)-th$ row would be good for y_{t+1}, while the first would be still good for y_1.

If this exchange would not be possible too, it would mean that even the element a_{1r_1} appears in the $(t+1)-th$ row. Moreover, it certainly does not coincide neither with $a_{t+1,1}$, nor with a_{t+1,r_1}. In this case we can go on with the same construction, considering two corresponding triplets, one in the first, the other in the $(t+1)-th$ row, and check again the possibility of the desired exchange. It's obvious that one can go on until the exchange can be done - in the extreme case, one will exchange the entire first row of M with the entire $(t+1)-th$ row.

In conclusion, one can define a new matrix M', of the same type $k\,x\,r$ as M, such that y_1 can be added to the first row, y_2 to the second, $\dots, y_{t+1}$ to the $(t+1)-th$, q.e.d.. ■

Let us add some more remarks.

First of all, as a consequence of the previous proof one obtains that in the cartesian product of the family F there exist indeed at least k different elements without repetitions; furthermore, for each element $a_i \in A_i (1 \leq i \leq r)$ there exists at least one such element with a_i as its $i-th$ component.

Combining Hall's Theorem and the previous Lemma, one gets immediately:

Corollary 1. *In the same hypotheses as before, the union of any t sets of the family F, $1 \leq t \leq r$, has cardinality greater or equal to t.*

As direct applications of the Lemma, one has the following:

Corollary 2. *Let A, B be two finite sets with the same number of elements $n \geq 1$, and $\Phi : A \to P(B)$, the set of all subsets of B, any k-regular multifunction, i.e. Φ is such that, for any $a \in A$, the set $\Phi(a)$ has cardinal number equal to k. If for any $b \in B$ one has $\Phi^{-1}(b) \leq k$ (and then, as one sees immediately, $\Phi^{-1}(b) = k$), then there exists a bijection $f : A \to B$ which is a restriction of Φ, i.e. f is such that $f(a) \in \Phi(a)$ for all $a \in A$. (This is the so called «Marriage Theorem»; Bondy & Murty, loc. cit., p. 73).*

Corollary 3. *Let G be any finite bipartite k-regular graph, i.e. a finite graph such that each of its vertices is incident with exactly k edges, and whose set of vertices V admits a partition $V = V' \cup V''$ such that each edge of G has one end in V' and the other in V''. It is possible to extend (restrict, if $k \geq 1$) G to a bipartite $(k+1)$-regular graph ($(k-1)$-regular), which has the same vertex set V and the same partition $V = V' \cup V''$, just by adding (subtracting) enough edges to the set of edges of G.*

Corollary 4. *Let V be any finite dimensional vector space over a field K ($dim(V) = n \geq 1$), $B = (v_1, \dots, v_n)$ one basis of V, and T the exterior product $T_1 \wedge T_2 \wedge \dots \wedge T_r$ of any family T_i of coordinated k-dimensional subspaces of V ($1 \leq k \leq n$), i.e., subspaces such that any of them is generated by a subset of k elements of B. T is certainly a non-zero space when the subspaces of the given family are at $(k+1)$ at $(k+1)$ disjoint in $V - \{0\}$ (this condition is required, of course, only in the case $r \geq (k+1)$).*

Received July 2, 1993
U. Bartocci, A. Girardi
Dipartimento di Matematica
Università degli studi
06100 Perugia
Italy