A COMBINATORIAL LEMMA ON THE CARTESIAN PRODUCT
OF FINITE SETS

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Summary. A sufficient condition for the cartesian product of a finite family of equipotent
finite sets to have elements without repetitions is given, and some application is discussed.

As it is well known, graph theory furnishes a necessary and sufficient condition for the
cartesian product of any finite family of finite sets to have elements without repetitions (this
is Hall's Theorem; see for instance Graph Theory with Applications, J.A. Bondy & U.S.R.
Murty, MacMillan, 1977, p. 72). In this paper we give a different simple sufficient condition
in the case of a family of sets with the same number of elements. The proof, obtained by
means of an easy combinatorial argument, has the advantage of not requiring graph theory.

Lemma. Let \( A \) be a finite set of \( n \) elements \((n \geq 1)\), and \( F \) a family of \( r \) subsets \( A_1, \ldots, A_r \)
of \( A \), all with the same cardinal number \( k (1 \leq k \leq n) \), such that, when \( r \geq k + 1 \), the
intersection of any \((k+1)\)-tuple of them is empty. Then, in the cartesian product \( A_1 \times \ldots \times A_r \)
there exists at least one element \((x_1, \ldots, x_r)\) without repetitions, i.e., such that \( x_i \neq x_j \) for
all \( i, j = 1, \ldots, r, i \neq j \). In particular, it necessarily follows \( r \leq n \).

Proof. The statement is obviously true when \( r \leq k \), and in this case one can even find a
matrix \( M \) of type \( k \times r \) with elements in \( A \), let us call it:

\[
\begin{pmatrix}
a_{11} & \cdots & a_{1r} \\
a_{21} & \cdots & a_{2r} \\
a_{k1} & \cdots & a_{kr}
\end{pmatrix},
\]

such that, all rows of \( M \) are elements without repetitions in the cartesian product of the family
\( F \), and all columns of \( M \) are permutations of the subsets \( A_1, \ldots, A_r \) respectively.

Let us then proceed by induction with respect to \( r \geq k \), starting from such a matrix \( M \),
and building a similar one of type \( k \times (r + 1) \).

Let the elements of \( A_{r+1} \) be \( y_1, \ldots, y_k \), then add the first element \( y_1 \) at the end of all \( k \)
rows of \( M \). It is obvious that, in the actual condition, at least one of the following \( k \) \((r+1)\)-
tuples:

\[
(a_{11} \ldots a_{1r}, y_1), \ldots, \\
(a_{k1} \ldots a_{kr}, y_1),
\]

will have no repetitions. As a matter of fact, \( y_1 \) cannot coincide in distinct \((r+1)\)-tuples with
elements \( a_{ij} \) which belong to the same column; moreover, \( y_1 \) cannot appear in all columns
since the subset \( A_1 \cap \ldots \cap A_r \cap A_{r+1} \) is empty.
Let us suppose the first row to be one of these particular \((r + 1)\)-tuples, and to have added in a similar way the first \(t\) elements \(y_1, \ldots, y_t\) of \(A_{r+1}\), for some \(t, 1 \leq t \leq k\), respectively to the 1st, the 2nd, and the \(t\)-th row of \(M\). We can proceed again by induction, but now with respect to \(t\): we shall prove that one can add, if \(t < k\), the element \(y_{t+1}\) to one of the remaining \((k - t)\) rows of \(M\), whence the conclusion.

If all \((k - t)\) \((r + 1)\)-tuples

\[
(a_{t+1,1} \cdots a_{t+1,r} y_{t+1}), \ldots, \\
(a_{k+1} \cdots a_{k,r} y_{t+1})
\]

have repetitions, then - for the same reasons as before - at least one of the analogous \((r + 1)\)-tuples relative to an index between 1 and \(t\) does not have repetitions. Without any loss of generality, we can suppose that it is again the first \((r + 1)\)-tuple \((a_{11} \cdots a_{1,r} y_{t+1})\) one of those which have no repetitions, as \((a_{11} \cdots a_{1,r} y_1)\). Let us further suppose that, for instance, \(a_{t+1,1} = y_{t+1}\).

Let us now look at the corresponding element \(a_{11}\) in the first row, and remark that if one could exchange in \(M\) the two elements \(a_{11}\) and \(a_{t+1,1}\), then everything would be all right: because the \((t + 1) - th\) row so obtained would be «good» for \(y_{t+1}\), while the first would still be good for \(y_1\). If this exchange would not be possible, it would mean that \(a_{11}\) appears somewhere in the \((t + 1) - th\) row, say for instance \(a_{t+1,r_1} = a_{11}\). Observe that \(r_1\) is of course different from 1.

Let us now look at the pair

\[(a_{t+1,1}, a_{t+1,r_1} = a_{11})\]

and to its corresponding pair in the first row:

\[(a_{11}, a_{1,r_1}).\]

If one could exchange these two pairs, in a similar way to the one just discussed for single elements, then everything would again be all right: because the new \((t + 1) - th\) row would be good for \(y_{t+1}\), while the first would be still good for \(y_1\).

If this exchange would not be possible too, it would mean that even the element \(a_{1,r_1}\) appears in the \((t + 1) - th\) row. Moreover, it certainly does not coincide neither with \(a_{t+1,1}\), nor with \(a_{t+1,r_1}\). In this case we can go on with the same construction, considering two corresponding triplets, one in the first, the other in the \((t + 1) - th\) row, and check again the possibility of the desired exchange. It's obvious that one can go on until the exchange can
be done - in the extreme case, one will exchange the entire first row of $M$ with the entire $(t + 1) - th$ row.

In conclusion, one can define a new matrix $M'$, of the same type $k \times r$ as $M$, such that $y_1$ can be added to the first row, $y_2$ to the second, $\ldots$, $y_{t+1}$ to the $(t + 1) - th$, q.e.d.

Let us add some more remarks.

First of all, as a consequence of the previous proof one obtains that in the cartesian product of the family $F$ there exist indeed at least $k$ different elements without repetitions; furthermore, for each element $a_i \in A_i$ ($1 \leq i \leq r$) there exists at least one such element with $a_i$ as its $i - th$ component.

Combining Hall's Theorem and the previous Lemma, one gets immediately:

**Corollary 1.** In the same hypotheses as before, the union of any $t$ sets of the family $F$, $1 \leq t \leq r$, has cardinality greater or equal to $t$.

As direct applications of the Lemma, one has the following:

**Corollary 2.** Let $A, B$ be two finite sets with the same number of elements $n \geq 1$, and $\Phi : A \to P(B)$, the set of all subsets of $B$, any $k$-regular multifunction, i.e. $\Phi$ is such that, for any $a \in A$, the set $\Phi(a)$ has cardinal number equal to $k$. If for any $b \in B$ one has $\Phi^{-1}(b) \leq k$ (and then, as one sees immediately, $\Phi^{-1}(b) = k$), then there exists a bijection $f : A \to B$ which is a restriction of $\Phi$, i.e. $f$ is such that $f(a) \in \Phi(a)$ for all $a \in A$. (This is the so called «Marriage Theorem»; Bondy & Murty, loc. cit., p. 73).

**Corollary 3.** Let $G$ be any finite bipartite $k$-regular graph, i.e. a finite graph such that each of its vertices is incident with exactly $k$ edges, and whose set of vertices $V$ admits a partition $V = V' \cup V''$ such that each edge of $G$ has one end in $V'$ and the other in $V''$. It is possible to extend (restrict, if $k \geq 1$) $G$ to a bipartite $(k + 1)$-regular graph $((k - 1)$-regular), which has the same vertex set $V$ and the same partition $V = V' \cup V''$, just by adding (subtracting) enough edges to the set of edges of $G$.

**Corollary 4.** Let $V$ be any finite dimensional vector space over a field $K$ ($\dim(V) = n \geq 1$), $B = (v_1, \ldots, v_n)$ one basis of $V$, and $T$ the exterior product $T_1 \wedge T_2 \wedge \ldots \wedge T_r$ of any family $T_i$ of coordinated $k$-dimensional subspaces of $V$ ($1 \leq k \leq n$), i.e., subspaces such that any of them is generated by a subset of $k$ elements of $B$. $T$ is certainly a non-zero space when the subspaces of the given family are at $(k + 1)$ at $(k + 1)$ disjoint in $V - \{0\}$ (this condition is required, of course, only in the case $r \geq (k + 1)$).
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