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$p + 2 = P_2$ IN SHORT INTERVALS

SAVERIO SALERNO, ANTONIO VITOLO

Abstract. *Twin prime problem is well known in number theory. Sieve methods can only detect almost-primes because of parity phenomenon. Switching principle allowed Chen to get P_2. By a suitable weighted sieve, we prove that the representation $p + 2 = P_2$ occurs with the expected frequency in short intervals.*

1. INTRODUCTION AND STATEMENT OF THE RESULT

A famous conjecture in number theory states that there exist infinitely many twin primes, that is primes p such that $p + 2$ is also prime.

Sieve methods have been developed in order to check the expected number of primes in sequences having the property to be well distributed in the arithmetic progressions.

Selberg showed that, because of the so-called «parity phenomenon», sieve methods alone are not able to detect primes. A detailed account of this question is contained in a paper about the asymptotic sieve of Bombieri [B2].

In fact, when using sieve techniques, our realistic purpose is to show the existence of almost primes P_r, that is integers with at most r prime factors, counted according to multiplicity.

By using the «switching principle» together with a weighted sieve, Chen [C] succeeded in proving that $p + 2 = P_2$ for infinitely many primes p, and more precisely that for large x

$$\sum_{\substack{x \le p \le 2x \\ p+2=P_2}} \ge c\frac{x}{\log^2 x}. \tag{1.1}$$

It is also to be remarked that an improvement of the constant c has been obtained by Fouvry-Grupp [FG] by using the bilinear form of the error term in the linear sieve due to Iwaniec and the recent results of Bombieri-Friedlander-Iwaniec [BFI] on the distribution of the primes in the arithmetic pragressions.

In this paper we study the same question in short intervals with the aim of proving an analogous result, i.e.

$$\sum_{\substack{x \le p \le x+x^\vartheta \\ p+2=P_2}} 1 \gg \frac{x^\vartheta}{\log^2 x} \tag{1.2}$$

for $\vartheta < 1$. In order to deal with this problem we put

$$A \equiv A_\vartheta = \{p + 2; x \le p \le x + x^\vartheta, p \,\text{prime}\}$$

(1.3)
$$B = \{p > 2; p \,\text{prime}\}$$
$$P(z) = \prod_{\substack{p<z \\ p\in B}} p,$$

where $z > 2$.

Then we denote by $k(n)$ the well known Kuhn's weights, i.e.

(1.4)
$$k(n) = 1 - \lambda \sum_{\substack{z\leq p\leq y \\ p|n, p\in B}} 1,$$

where $\lambda > 0$ and $2 < z < y$.

In the case $\vartheta = 1$ the idea of Chen was to consider the weighted sum

(1.5)
$$\sum_{\substack{n\in A \\ (n,P(z))=1}} \{k_C(n) - s_C(n)\},$$

where the parameters in $k(n)$ were chosen in order to have

$$k_C(n) > 0 \Rightarrow n = P_3$$

whilst

(1.6)
$$s_C(n) = \begin{cases} k_C(n) & \text{if } n = p_1 p_2 p_3 \\ 0 & \text{otherwise} \end{cases} .$$

Hence, if $k_C(n) - s_C(n) > 0$, then $n = P_2$. Moreover, since $k_C(n) - s_C(n) \leq 1$, a positive lower bound of the right magnitude order for the sum (1.5) yields the result (1.1).

By continuity the above argument also provides estimates of the type (1.2), at least when ϑ is close to 1.

However an improvement is to be expected if we choose parameters λ, z and y in $k(n)$ such that

$$k(n) > 0 \Rightarrow n = P_4.$$

In this case we have to consider the sum

(1.7)
$$\sum_{\substack{n\in A \\ (n,P(z))=1}} \{k(n) - s(n) - r(n)\},$$

where $s(n)$ is again a weight of type (1.6), whilst

$$(1.8) \qquad r(n) = \begin{cases} k(n) & \text{if } n = p_1 p_2 p_3 p_4 \\ 0 & \text{otherwise} \end{cases} .$$

We observe that, for instance, if $n = p_1 p_2 p_3 p_4$, we have by construction

$$k_C(n) - s_C(n) < 0, \text{ whilst } k(n) - s(n) - r(n) = 0,$$

and so we take advantage on the integers with exactly four prime factors by considering sum (1.7) instead of (1.5).

Really, by choosing suitable values for the parameters in $k(n)$ and estimating the weighted sum in (1.7), we get the following

Theorem. *If* $\vartheta > 0,9729$, *we have*

$$\sum_{\substack{x \leq p \leq x - x^{\vartheta} \\ p+2=P_2}} 1 \gg \frac{x^{\vartheta}}{\log^2 x}.$$

2. THE WEIGHTS

We are concerned with the sequence A of (1.3).

Let u, v be positive real numbers with $u < v$, and put

$$(2.1) \qquad z = x^{\frac{1}{v}}, y = x^{\frac{1}{u}}.$$

Then consider $k(n)$ in (1.4) with $\lambda > 0$.

If we denote by $\Omega(n)$ the number of prime factors of an integer n, counted with multiplicity, we recall that in general, since the subset of not-squarefree integers is negligible,

$$(2.2) \qquad k(n) > 0 \Rightarrow \Omega(n) \leq \left[\frac{1}{\lambda}\right] + [u],$$

where $[t]$ is $t - 1$ if $t \in N$, the integer part of t if $t \in \mathbf{R}^+ - \mathbf{N}$.

We also recall that Chen's result ($\vartheta = 1$) was obtained by means of the weights $k(n) = k_C(n)$, coresponding to the following choice of the parameters:

$$(2.3) \qquad \lambda = \lambda_C = \frac{1}{2}, u = u_C = 3, v = v_C = 10,$$

whence, by (2.2),

$$k_C(n) > 0 \Rightarrow \Omega(n) \leq 3. \tag{2.4}$$

Now, by definition of $k(n)$, in view of the choice of λ_C and u_C, it turns out that

(*) an integer $n \in A$ cannot have 3 prime factors greatest than $x^{\frac{1}{u_C}}$;

(**) if an integer $n \in A$ has at least 2 prime factors between $x^{\frac{1}{v_C}}$ and $x^{\frac{1}{u_C}}$, then $k_C(n) \leq 0$.

Therefore an integer $n \in A$ having exactly 3 prime factors, say $n = p_1 p_2 p_3$, $p_1 < p_2 < p_3$, is counted with a positive weight $k_C(n)$ if and only if

$$x^{\frac{1}{v_C}} < p_1 < x^{\frac{1}{u_C}} < p_2 < p_3, \tag{2.5}$$

whence the contribution, to be subtracted in (1.5) in order to save only $P_2' s$, is more precisely

$$s_C(n) = \begin{cases} k_C(n) & \text{if } n = p_1 p_2 p_3 \text{ as in (2.5)} \\ 0 & \text{otherwise} \end{cases}.$$

As anticipated in the previous section, our argument leads to choose parameters λ and u in order to relax condition (2.4), by only requiring that

$$k(n) > 0 \Rightarrow \Omega(n) \leq 4. \tag{2.6}$$

This allows to take a smaller λ and a larger u, with respect to λ_C and u_C, provided that the right hand side of the implied inequality in (2.2) does not exceed 4. Then, in order to detect $P_2' s$, we shall subtract the contribution of the integers with exactly 3 or 4 prime factors.

First, we discuss the behaviour of (1.7) with respect to (1.5) when we take $\lambda < \lambda_C$, $u = u_C$ and $v = v_C$ in (1.4). Since $k(n)$ is a decreasing function of λ, then we gain in estimating $\sum k(n)$ instead of $\sum k_C(n)$. On the other hand, we loose in the estimate of the switching term, due to following facts: since we need an upper bound, this time the monotony of $k(n)$ as function of λ works in the opposite sense; there is no analogous of $\sum r(n)$ to be subtracted in (1.5); in substituting $\sum s_C(n)$ with $\sum s(n)$, a new contribution arises from those integers $n \in A$ with exactly 2 prime factors between $x^{\frac{1}{v_C}}$ and $x^{\frac{1}{u_C}}$, counted with $k_C(n) = 0$ in (1.5), but $k(n) > 0$ in (1.7), whence, together with the integers $n = p_1 p_2 p_3 \in A$ as in (2.5), in $\sum s(n)$ we also have to take account of the integers $n = p_1 p_2 p_3 \in A$ such that

$$x^{\frac{1}{v_C}} < p_1 < p_2 < x^{\frac{1}{u_C}} < p_3. \tag{2.7}$$

Analogously, if we take $u > u_C$, $\lambda = \lambda_C$ and $v = v_C$ in (1.5), the above considerations can be likewise repeated, but now in $\sum s(n)$ we have to take account of the integers $n = p_1p_2p_3 \in A$ such that

$$x^{\frac{1}{u_C}} < p_1 < p_2 < p_3,$$

whose cardinality is certainly less than the cardinality of the integers of the type (2.7): so the present choice of the parameters is convenient with respect to the previous one.

For our purpose ($\vartheta < 1$) the above discussion ($\vartheta = 1$) suggests the following weighted sum

$$(2.8) \qquad \sum_{\substack{n\in A\\(n,P(z))=1}} \left(1 - \frac{1}{2}\sum_{\substack{z\leq p\leq v\\ p|n,p\in B}} 1\right) - \sum_{n\in A} s(n) - \sum_{n\in A} r(n)$$

where we assume

$$(2.9) \qquad 3 \leq u \leq 4,\ \frac{1}{v} \leq 1 - \frac{3}{u},$$

$s(n) \neq 0 \Rightarrow n = p_1p_2p_3$ and

$$(2.10) \qquad s(n) = \begin{cases} 1 & \text{if } x^{\frac{1}{u}} < p_1 < p_2 < p_3 \\ \frac{1}{2} & \text{if } x^{\frac{1}{v}} < p_1 < x^{\frac{1}{u}} < p_2 < p_3, \\ 0 & \text{otherwise} \end{cases}$$

$r(n) \neq 0 \Rightarrow n = p_1p_2p_3p_4$ and

$$(2.11) \qquad r(n) = \begin{cases} \frac{1}{2} & \text{if } x^{\frac{1}{v}} < p_1 < x^{\frac{1}{u}} < p_2 < p_3 < p_4. \\ 0 & \text{otherwise} \end{cases}$$

Hence, from the above discussion, we shall have

$$\sum_{\substack{n\in A\\ n=P_2}} 1 \geq \sum_{\substack{n\in A\\(n,P(z))=1}} \left(1 - \frac{1}{2}\sum_{\substack{z\leq p\leq v\\ p|n,p\in B}} 1\right) \sum_{n\in A} s(n) - \sum_{n\in A} r(n).$$

So, in order to get our result, we have to show a positive lower bound of the correct magnitude order for the weighted sum on the right hand side.

We point out that a result can also be achieved when the second condition in (2.9) is not satisfied, as in the case $r(n) = 0$ and we reduce to the Chen's method different parameters.

We also remark that the idea of Chen can be carried on with the Richert's weights instead of Kuhn's ones, but with a worse result. This is essentially the approach of Ross [R2], but the result, contained in his paper, seems to be affected by computational errors, whilst the method actually leads to an exponent ϑ very close to 1, as pointed out by J. Wu in a recent paper [W].

In fact J. Wu [W] considers the problem of the distribution of the primes in the arithmetic progressions in short intervals, generalizing a result of Perelli-Pintz-Salerno [PPS1], and announces that, as an application, he will be concerned with the representation of P_2 in short intervals in a forthcoming paper, anticipating that he can obtain the exponent $\vartheta = 0,974$.

Anyway, the use of P_4 in place of P_3 seems to give an improvement. In principle this method can be iterated further on, but with increasing technical complications.

For a more detailed description of the weights, we remark that, according to the above notations, we can obviously suppose

$$p_2 \le \left(\frac{x}{p_1}\right)^{\frac{1}{2}} \qquad\qquad \text{if } s(n) > 0$$

$$p_2 \le \left(\frac{x}{p_1}\right)^{\frac{1}{3}}, \quad p_3 \le \left(\frac{x}{p_1 p_2}\right)^{\frac{1}{2}} \quad \text{if } r(n) > 0.$$

So we have for $n = p_1 p_2 p_3 \in A$

$$\frac{x}{p_1 p_2} \le p_3 \le \frac{x + x^{\vartheta}}{p_1 p_2}$$

and

$$s(n) = \begin{cases} 1 & \text{if } x^{\frac{1}{u}} < p_1 \le x^{\frac{1}{3}}, p_1 < p_2 \le \left(\frac{x}{p_1}\right)^{\frac{1}{2}} \\ \frac{1}{2} & \text{if } x^{\frac{1}{v}} < p_1 < x^{\frac{1}{u}}, x^{\frac{1}{u}} < p_2 \le \left(\frac{x}{p_1}\right)^{\frac{1}{2}} \\ 0 & \text{otherwise} \end{cases}$$

whilst for $n = p_1 p_2 p_3 p_4 \in A$

$$\frac{x}{p_1 p_2 p_3} \le p_4 \le \frac{x + x^{\vartheta}}{p_1 p_2 p_3}$$

and

$$r(n)=\begin{cases}\dfrac{1}{2} & \text{if } x^{\frac{1}{v}}<p_1\le x^{1-\frac{3}{u}},\, x^{\frac{1}{u}}<p_2\le\left(\dfrac{x}{p_1}\right)^{\frac{1}{3}},\, p_2<p_3\le\left(\dfrac{x}{p_1p_2}\right)^{\frac{1}{2}}.\\ 0 & \text{otherwise}\end{cases}$$

3. AUXILIARY LEMMAS AND NOTATIONS

Set

$$\mathrm{L}=\log x,$$

$$li(y)=\int_x^{x+y}\frac{dt}{\log t},$$

$$\pi(z,\mathcal{X})=\sum_{p<z,p\ prime}\mathcal{X}(p),\quad \pi(z;q,a)=\sum_{\substack{p<z,p\ prime\\ p\equiv a(\bmod q)}}1.$$

First we quote a large sieve inequality following from Lemma 2.4 of Gallagher [G] by partial summation.

Lemma 3.1. *Let $\sigma>0$, $s=\sigma+it, T>0$. If $\{a_n\}$ is a sequence of complex numbers such that*

$$\sum_n\frac{|a_n|^2}{n^{2\sigma-1}}<+\infty,$$

then

$$\sum_{q\le Q}\frac{q}{\varphi(q)}\sum_{\mathcal{X}\bmod q}{}^{*}\int_{-T}^{T}\left|\sum_n\frac{a_n\mathcal{X}(n)}{n^{it}}\right|^2\min\left(ky^{\sigma-1},\frac{y^\sigma}{|s|}\right)dt$$
$$\ll y^\sigma\sum_n\frac{|a_n|^2}{n^{2\sigma}}(ky^{-1}n+Q^2\log(T+2)),$$

where * denotes the restriction of the sum to primitive characters.

Next we recall a Siegel-Walfisz type theorem in short intervals, which follows frol zero-density estimates and informations about zero-free regions for $\zeta(s)$ (e.g., see Davenport [D]), as improved by means of a result of Huxley [H].

Lemma 3.2. *Let δ and N be positive constants, $Q\le\mathrm{L}^N$, $\vartheta\le1$. Then there is a positive constants C_N such that*

$$\sum_{q\le Q}\sum_{\mathcal{X}\bmod q}{}^{*}\max_{y\le x^\vartheta}|\pi(x+y,\mathcal{X})-\pi(x,\mathcal{X})|\ll x^\vartheta\exp\left(\frac{-C_N\mathrm{L}^{\frac{1}{4}}}{(\log\mathrm{L})^{\frac{3}{4}}}\right)+x^{\frac{7}{12}+\delta}.$$

Finally, we need a Bombieri's theorem for short intervals, which we state in the form proved by Perelli-Pintz-Salerno [PPS1] on using Heath-Brown extension of Vaughan identity.

Lemma 3.3. *Let $\frac{3}{5} < \vartheta \leq 1$. For every A there exists $B = B(A)$ such that*

$$\sum_{q\leq Q} \max_{y\leq x^{\vartheta}} \max_{(a,q)=1} \left| \pi(x+y;q,a) - \pi(x;q,a) - \frac{li(y)}{\varphi(q)} \right| \ll x^{\vartheta} \mathrm{L}^{-A},$$

if $Q \leq x^{\vartheta-\frac{1}{2}} \mathrm{L}^{-B}$.

In the estimate of our weighted sum (2.8) we shall use Selberg's linear sieve.

Let A be a sequence (finite) of integers, and $X = |A|$. We set

$$A_d = \{n \in A / a \equiv 0 (\mathrm{mod}\ d)\},\ R_d = |A_d| - \frac{\omega(d)}{d} X,$$

where $\omega(d)$ is a multiplicative function.

If B is a sequence of primes, we also set

$$P(z) = \prod_{\substack{p<z \\ p\in B}} p,\ V(z) = \prod_{\substack{p<z \\ p\in B}} \left(1 - \frac{\omega(p)}{p}\right). \tag{3.2}$$

In order to estimate the sifting function

$$S(A_d, B, z) = \sum_{\substack{n\in A \\ (n,P(z))=1}} 1, \tag{3.3}$$

we quote the following classical result (see [FG], Theorem 8.3).

Lemma 3.4. *Suppose that*

$$0 \leq \frac{\omega(p)}{p} \leq 1 - \frac{1}{A_1} \tag{3.4}$$

for some constant $A_1 > 1$, and that

$$\left| \sum_{\substack{w\leq p\leq z \\ p\ prime}} \frac{\omega(p)}{p} \log p - \log \frac{z}{w} \right| \leq A_2, \tag{3.5}$$

with A_2 independent of w and z.
For $z \ll D^\rho$ $(\rho > 0)$ we have

$$S(A_d, B, z) \leq \frac{\omega(d)}{d} X V(z) \left\{ F\left(\frac{\log D}{\log z}\right) + B_1(\rho) \frac{A_2}{(\log D)^{\frac{1}{14}}} \right\} + \sum_{\substack{m<D \\ m|P(z)}} 3^{\nu(m)} |R_{dm}|$$

and

$$S(A_d, B, z) \geq \frac{\omega(d)}{d} X V(z) \left\{ f\left(\frac{\log D}{\log z}\right) - B_2 \frac{A_2}{(\log D)^{\frac{1}{14}}} \right\} - \sum_{\substack{m<D \\ m|P(z)}} 3^{\nu(m)} |R_{dm}|,$$

where F and f are the continuous functions, resp. monotonically decreasing and increasing towards 1 at infinity, such that

$$(3.6) \qquad F(s) = \frac{2e^\gamma}{s}, \; f(s) = 0, \; 0 < s \leq 2,$$

and

$$(3.7) \qquad (sF(s))' = f(s-1), \; (sf(s))' = F(s-1), \; s > 2.$$

4. THE STANDARD TERMS

In this section we are concerned with the estimate of the sum

$$\sum_{\substack{n \in A \\ (n, P(z)) = 1}} k(n),$$

where A, B and $P(z)$ are defined in (1.3), $k(n)$ in (1.4) with $\lambda = \frac{1}{2}$, $z = x^{\frac{1}{v}}$ and $y = x^{\frac{1}{u}}$ with u and v subject to conditions (2.9).

First we apply Lemma 3.4 with

$$X \cong li(x^\vartheta), \qquad \omega(p) = \frac{p}{p-1},$$

and so

$$(4.1) \qquad V(z) \cong \frac{2\sigma e^{-\gamma} v}{\log x},$$

where

$$\sigma = \prod_{\substack{p>2 \\ p \text{ prime}}} \left(1 - \frac{1}{(p-1)^2}\right).$$

Let $\varepsilon > 0$. By standard calculations, there exists $\delta > 0$ such that

$$\sum_{\substack{n\in A \\ (n,P(z))=1}} k(n) \geq \left\{ f(\alpha v) - \frac{1}{2}\int_u^v F\left(\alpha v - \frac{v}{t}\right)\frac{dt}{t} - \frac{\varepsilon}{3} \right\} \tag{4.2}$$
$$- \sum_{\substack{d<D \\ d|P(z)}} 3^{\nu(d)}|R_d| - \sum_{\substack{z\leq p\leq y \\ p\in B}} \sum_{\substack{pd<D \\ d|P(z)}} 3^{\nu(d)}|R_{pd}|$$

with $D = x^{\alpha-\delta}$, where

$$R_d = \pi(x + x^{\vartheta}; d, -2) - \pi(x; d, -2) - \frac{li(x^{\vartheta})}{\varphi(d)}. \tag{4.3}$$

Let $\alpha = \vartheta - \dfrac{1}{2}$. By using Cauchy-Schwarz inequality, Brun-Titchmarsh theorem and Lemma 3.3 with a suitable A, we get

$$\sum_{d<D} \mu^2(d) 3^{\nu(d)}|R_d| \ll \left(\sum_{d<D} \mu^2(d) 9^{\nu(d)}|R_d|\right)^{\frac{1}{2}} \left(\sum_{d<D} |R_d|\right)^{\frac{1}{2}}$$
$$\ll x^{\frac{\vartheta}{2}} \left(\sum_{d<D} \frac{\mu^2(d) 9^{\nu(d)}}{d}\right)^{\frac{1}{2}} \left(\sum_{d<D} |R_d|\right)^{\frac{1}{2}} \ll x^{\vartheta} L^{-3},$$

and similarly

$$\sum_{\substack{z\leq p\leq y \\ p\in B}} \sum_{\substack{pd<D \\ d|P(z)}} 3^{\nu(d)}|R_{pd}| \ll \sum_{q<D} \mu^2(q) 3^{\nu(q)} \nu(q) |R_q| \ll x^{\vartheta} L^{-3}.$$

Then, if we choose ν such that

$$4 \leq \alpha\nu \leq 6, \tag{4.4}$$

by means of the last inequalities, from (4.2), by (3.6) and (3.7), we get

$$\sum_{\substack{n\in A \\ (n,P(z))=1}} k(n) \geq \frac{4\sigma x^{\vartheta}}{\alpha \log^2 x} \left(k(\vartheta; u, v) - \frac{\varepsilon}{3}\right), \tag{4.5}$$

with

$$k(\vartheta; u, v) = \log(\alpha v - 1) - \frac{1}{2} \log \frac{\alpha v - 1}{\alpha u - 1} \tag{4.6}$$
$$+ \int_2^{\alpha v - 2} \log(t-1) \log \frac{\alpha v - 1}{1+t} \frac{dt}{t}$$
$$- \frac{1}{2} \int_2^{\alpha v - 2} \log(t-1) \log \frac{(\alpha v - 1)(\alpha v - 1 - t)}{1+t} \frac{dt}{t}.$$

5. THE SWITCHING TERMS

In this section we are concerned with sieving subsequences of squarefree integers of A, which all have the same number of prime factors.

Let $\mathscr{D}$ be the set of the integers $n = p_1 \dots p_h (h \geq 3)$, with $p_1, \dots, p_h$ primes such that

$$x^\rho < p_1 \leq x^\sigma,\ a_1(x, p_1) < p_2 < b_1(x, p_1),$$
$$\dots, a_{h-2}(x, p_1, \dots, p_{h-2}) < p_{h-1} < b_{h-2}(x, p_1, \dots, p_{h-2}),$$

where $\frac{1}{v} \leq \rho < \sigma$ and $p_i \leq a_i(x, p_1, \dots, p_i) < b_i(x, p_1, \dots, p_i) \leq p_{i+2}, i = 1, \dots, h-2$.

Then, by switching,

$$\sum_{\substack{n \in A \cap D \\ (n, P(z)) = 1}} 1 = \sum_{\substack{n \in D, x+2 \leq n \leq x + x^\vartheta + 2 \\ n-2 \text{ prime}}} 1 \leq S(\mathscr{U}, B, x^{\frac{1}{2}}) + O(x^{\frac{1}{2}}),$$

where

$$\mathscr{U} = \{mp - 2 / mp \in \mathscr{D}, p' | m \Rightarrow p' < p, x + 2 \leq mp \leq x + x^\vartheta + 2\}. \tag{5.2}$$

By applying Lemma 3.4 to sieve $\mathscr{U}$, we get

(5.3)

$$S(\mathscr{U}, B, x^{\frac{1}{2}}) \leq |\mathscr{U}| V(x^{\frac{1}{2}}) \left\{ F\left(\frac{\log D}{\log x^{\frac{1}{2}}}\right) + B_2(1) \frac{A_2}{(\log D)^{\frac{1}{14}}} \right\} + \sum_{\substack{d < D \\ d | P(D)}} 3^{\nu(d)} |R_d|,$$

where $V(z)$ is given by (3.2) and

$$R_d = \sum_{\substack{x+2 \leq mp \leq x + x^\vartheta + 2, mp \equiv 2 (\bmod d) \\ mp \in D, p' | m \Rightarrow p' < p}} - \frac{1}{\varphi(d)} \sum_{\substack{x+2 \leq mp \leq x + x^\vartheta + 2. \\ mp \in D, p' | m \Rightarrow p' < p}} l. \tag{5.4}$$

First we deal with the main term.

Lemma 5.1. *We have*

$$|\mathscr{U}| = \eta \frac{x^{\vartheta}}{\log x}(1 + O(L^{-1})),$$

where

$$\eta = \int_{\rho}^{\sigma} \frac{ds_1}{s_1} \int_{\alpha_1(s_1)}^{\beta_1(s_1)} \frac{ds_2}{s_2} \cdots \int_{\alpha_{h-2}(s_1,\dots,s_{h-2})}^{\beta_{h-2}(s_1,\dots,s_{h-2})} \frac{ds_{h-1}}{s_{h-1}(1 - s_1 - \dots - s_{h-1})}$$

with

$$\alpha_i(s_1,\dots,s_i) = \frac{\log a_i(x, x^{s_1},\dots,x^{s_i})}{\log x}, \beta_i(s_1,\dots,s_i) = \frac{\log b_i(x, x^{s_1},\dots,x^{s_i})}{\log x}.$$

Proof. By the prime numbers theorem we have

$$|\mathscr{U}| = \sum_{p_1,\dots,p_{h-1}} \sum_{\frac{x+2}{p_1\cdots p_{h-1}} \le p_h \le \frac{x+x^{\vartheta}+2}{p_1\cdots p_{h-1}}}$$

$$= \frac{x^{\vartheta}}{\log x}(1 + O(L^{-1})) \sum_{p_1,\dots,p_{h-1}} \frac{1}{p_1 \cdots p_{h-1} \cdot \left(1 - \frac{\log p_1 \cdots p_{h-1}}{\log x}\right)}$$

Then, using the fact that

$$\sum_{t \le p \le w} \frac{1}{p} = \log \frac{\log w}{\log t} + O(\log^{-1} t), 2 \le t \le w,$$

by partial summation we get the result. ∎

Hence, by (3.6) and (4.1), we have the upper bound

$$S(\mathscr{U}, B, x^{\frac{1}{2}}) \le \frac{4\eta}{\alpha'} \frac{\sigma x^{\vartheta}}{\log^2 x}(1 + O(L^{-\frac{1}{14}})) + \sum_{\substack{d<D \\ d|P(D)}} 3^{\nu(d)} |R_d|, \tag{5.5}$$

where $D = x^{\alpha'}$.

Our aim is to show that the error term is negligible for any $\alpha' < \vartheta - \dfrac{1}{2}$. For this purpose we make the following assumption:

$$\text{if } mp \in \mathscr{D} \text{ and } x + 2 \le mp \le x + x^{\vartheta} + 2, \text{ then } x^{1-\vartheta+\varepsilon} \le m \le x^{\frac{12}{5}\vartheta - \frac{7}{5} - \varepsilon}, \tag{5.6}$$

for some $\varepsilon > 0$.

As in the previous section, by Cauchy-Schwarz inequality and Brun-Titchmarsh theorem, we get

$$\sum_{\substack{d<D \\ d|P(D)}} 3^{\nu(d)}|R_d| \ll \left(x^{\vartheta}L^9\right)^{\frac{1}{2}} \left(\sum_{d<D} \mu^2(d)|R_d|\right)^{\frac{1}{2}}. \tag{5.7}$$

Then we express R_d by means of the multiplicative characters mod d, setting

$$\psi(\mathcal{X}) = \sum_{\substack{x+2\leq mp\leq x+x^{\vartheta}+2 \\ mp\in D, p'|m\Rightarrow p'<p}} \mathcal{X}(mp). \tag{5.8}$$

By separating the contribution of the principal character $\mathcal{X}_0$, we have

$$\sum_{d<D} \mu^2(d)|R_d| \leq \sum_{d<D} \frac{\mu^2(d)}{\varphi(d)} \left| \sum_{\substack{\mathcal{X} \bmod d \\ \mathcal{X}\neq\mathcal{X}_0}} \overline{\mathcal{X}}(2)\psi(\mathcal{X}) \right| + \sum_{d<D} \frac{\mu^2(d)}{\varphi(d)} \sum_{\substack{x+2\leq mp\leq x+x^{\vartheta}+2 \\ mp\in D, p'|m\Rightarrow p'<p, (mp,d)>1}} l = S_1 + S_2, \text{ say.} \tag{5.9}$$

Lemma 5.2. *We have*

$$S_2 \ll \max\left(x^{\vartheta-\rho} x^{\vartheta-\frac{1}{h}}\right) L.$$

Proof. We observe that

$$S_2 \leq \sum_{d<D} \frac{\mu^2(d)}{\varphi(d)} \left(\sum_{\substack{x+2\leq mp\leq x+x^{\vartheta}+2 \\ mp\in D, p'|m\Rightarrow p'<p, (m,d)>1}} 1 + \sum_{\substack{x+2\leq mp\leq x+x^{\vartheta}+2 \\ mp\in D, p'|m\Rightarrow p'<p, p|d}} 1 \right).$$

Since $mp \geq x+2$ and mp has h prime factors such that $p'|m \Rightarrow p' < p$, we have

$$S_2 \ll \sum_{p>x^{\frac{1}{h}}} \sum_{\frac{x+2}{p}\leq m\leq\frac{x+x^{\vartheta}+2}{p}} \left(\sum_{p'|m} \sum_{\substack{d<D \\ d\equiv 0(\bmod p')}} \frac{1}{\varphi(d)} + \sum_{\substack{d<D \\ d\equiv 0(\bmod p)}} \frac{1}{\varphi(d)} \right)$$

$$\ll \sum_{p>x^{\frac{1}{h}}} \sum_{\frac{x+2}{p}\leq m\leq\frac{x+x^{\vartheta}+2}{p}} \left(\sum_{p'|m} \frac{1}{p'} \sum_{q<D/p'} \frac{1}{\varphi(q)} + \frac{1}{p} \sum_{q<D/p} \frac{1}{\varphi(q)} \right),$$

and therefore, since $p'|m \Rightarrow p' \geq x^{\rho}$,

$$S_2 \ll \left(\sum_{p > x^{\frac{1}{k}}} \sum_{\frac{x+2}{p} \leq m \leq \frac{x+x^{\vartheta}+2}{p}} l \right) \left(x^{-\rho} + x^{-\frac{1}{k}} \right) L,$$

whence our lemma. ∎

Next we restrict to primitive characters.

It is known that every non-principal character $\mathcal{X}(\bmod\ d)$ is induced by a primitive character $\mathcal{X}^*(\bmod\ q)$ where $q|d$, and $\mathcal{X}(n) = \mathcal{X}^*(n)$ if $\left(n, \frac{d}{q}\right) = 1$.

We set

$$S_3 = \sum_{d<D} \frac{\mu^2(d)}{\varphi(d)} \left| \sum_{\mathcal{X} \bmod d}{}^{*} \overline{\mathcal{X}}(2) \psi(\mathcal{X}) \right|. \tag{5.10}$$

Lemma 5.3. *Let $\varepsilon > 0$. We have*

$$S_1 \ll LS_3 + \max\left(x^{\vartheta - \rho + \varepsilon}, x^{\vartheta - \frac{1}{k} + \varepsilon} \right).$$

Proof. From (5.9), as remarked above when passing to primitive characters, we get

$$\begin{aligned} S_1 \leq & \sum_{d<D} \frac{\mu^2(d)}{\varphi(d)} \sum_{1<q|d} \left| \sum_{\mathcal{X} \bmod q}{}^{*} \overline{\mathcal{X}}(2) \psi(\mathcal{X}) \right|. \\ & + \sum_{d<D} \frac{\mu^2(d)}{\varphi(d)} \sum_{1<q|d} \left| \sum_{\mathcal{X} \bmod q}{}^{*} \overline{\mathcal{X}}(2) \sum_{\substack{x+2 \leq mp \leq x+x^{\vartheta}+2 \\ mp \in D, p'|m \Rightarrow p' < p, (mp, d/q) > 1}} \mathcal{X}(mp) \right| \end{aligned}$$

Putting $d = qr$, by recalling definition (5.10), we have

$$\begin{aligned} S_1 \leq & \sum_{q<D} \frac{\mu^2(q)}{\varphi(q)} \left| \sum_{\mathcal{X} \bmod q}{}^{*} \overline{\mathcal{X}}(2) \psi(\mathcal{X}) \right| \sum_{r<D/q} \frac{\mu^2(r)}{\varphi(r)} \\ & + \sum_{qr<D} \frac{\mu^2(qr)}{\varphi(qr)} \sum_{\substack{x+2 \leq mp \leq x+x^{\vartheta}+2 \\ mp \in D, p'|m \Rightarrow p' < p, (mp, r) > 1}} \left| \sum_{\mathcal{X} \bmod q}{}^{*} \overline{\mathcal{X}}(2) \mathcal{X}(mp) \right| \\ \ll & LS_3 + \sum_{r<D} \frac{\mu^2(r)}{\varphi(r)} \sum_{\substack{x+2 \leq mp \leq x+x^{\vartheta}+2 \\ mp \in D, p'|m \Rightarrow p' < p, (mp, r) > 1}} \sum_{q<D} \frac{\mu^2(q)}{\varphi(q)} (mp-2, q), \end{aligned}$$

where

$$\sum_{q<D}\frac{\mu^2(q)}{\varphi(q)}(mp-2,q)=\sum_{q<D}\frac{\mu^2(q)}{\varphi(q)}\sum_{k|mp-2,k|q}\varphi(k)$$

$$=\sum_{k|mp-2}\mu^2(k)\varphi(k)\sum_{q<D,q\equiv 0(\mathrm{mod}\ k)}\frac{\mu^2(q)}{\varphi(q)}\ll x^{\varepsilon}.$$

Hence, from (5.9) and Lemma 5.2 we get the proof. ∎

Now we are in position to estimate small moduli.

Lemma 5.4. *Assume that (5.6) holds. Then for every $N>0$ there exists a positive constant C_N such that*

$$\sum_{q<L^N}\frac{\mu^2(q)}{\varphi(q)}\sum_{\mathcal{X}\bmod q}{}^{*}|\psi(\mathcal{X})|\ll x^{\vartheta}\exp\left(-C_N L^{\frac{1}{3}}\right).$$

Proof. By definition of $\pi(z,\mathcal{X})$ (cf. Section 3) and $\psi(\mathcal{X})$ (see (5.8)) we have

$$|\psi(\mathcal{X})|\le\sum_{m\le x^{\frac{12}{5}\vartheta-\frac{7}{5}-\varepsilon}}\left|\pi\left(\frac{x+x^{\vartheta}+2}{m},\mathcal{X}\right)-\pi\left(\frac{x+2}{m},\mathcal{X}\right)\right|,$$

in view of assumption (5.6).

Since $\left(\frac{x}{m}\right)^{\frac{7}{12}}\le\frac{x^{\vartheta}}{m}$, from Lemma 3.2 we deduce that

$$\sum_{q<L^N}\frac{\mu^2(q)}{\varphi(q)}\sum_{\mathcal{X}\bmod q}{}^{*}|\psi(\mathcal{X})|\ll\sum_{m\le x^{\frac{12}{5}\vartheta-\frac{7}{5}-\varepsilon}}\frac{x^{\vartheta}}{m}\exp\left(-C_N\frac{L^{\frac{1}{4}}}{(\log L)^{\frac{3}{4}}}\right)$$

$$+\sum_{m\le x^{\frac{12}{5}\vartheta-\frac{7}{5}-\varepsilon}}\left(\frac{x}{m}\right)^{\frac{7}{12}+\delta},$$

whence, by (5.6), our lemma. ∎

Finally, we consider large moduli.

For this purpose we define

$$\Delta(m)=\begin{cases}1 & \text{if } mp\in\mathscr{D}\text{ for some prime } p \text{ such that } p'|m\Rightarrow p'<p\\ 0 & \text{otherwise}\end{cases}$$

and observe that the characteristic function of $\mathscr{D}$ is the convolution product of $\Delta(m)$ with the characteristic function $\Pi(p)$ of the prime numbers.

Hence, by (5.8), we have

$$\psi(\mathcal{X}) = \sum_{x+2\le n\le x+x^{\vartheta}+2} \mathcal{X}(n)(\Pi * \Delta)(n). \tag{5.11}$$

Then we set

$$E(w) = \sum_{d<w} \frac{\mu^2(d)}{\varphi(d)} \sum_{\mathcal{X} \bmod d}{}^{*} |\psi(\mathcal{X})|, \ 1 \le w \le x^{\vartheta-\frac{1}{2}}.$$

In the next lemma, in order to estimate $E(w)$, we shall use the Dirichlet series defined in the complex half-plane $\sigma > 1$ by

(5.12)

$$H(s) = \sum_{p\le\xi} \mathcal{X}(p)\Pi(p)p^{-s}, \ I(s) = \sum_{p>\xi} \mathcal{X}(p)\Pi(p)p^{-s}, \ J(s) = \sum_{m} \mathcal{X}(m)\Delta(m)m^{-s},$$

where $\xi = w^2 x^{1-\vartheta}$.

Lemma 5.5. *Suppose that (5.6) holds. We have*

$$E(w) \ll L^{\frac{3}{2}} x^{\vartheta} + L^2 x^{\vartheta-\frac{5}{2}} w + L^2 x^{\frac{1}{2}} w^2.$$

Proof. Using Mellin's transform to pick out coefficients of a Dirichlet series and the well known fact that the Dirichlet series of the convolution product of two sequences is the product of the Dirichlet series of each sequence, from (5.11) and (5.12) we get

$$\psi(\mathcal{X}) = \frac{1}{2\pi i} \int_{a-iT}^{a+iT} (H+I)J(s) \frac{(x+x^{\vartheta})^s - x^s}{s} ds + O(xLT^{-1}),$$

with $a = 1 + L^{-1}$.

Next, H and J being regular for $\sigma > 0$, we can shift the integration line from $\sigma = a$ to $\sigma = \frac{1}{2}$ for $HJ(s)$.

Since

$$\frac{(x+x^{\vartheta})^s - x^s}{s} \ll \min\left(x^{\sigma} x^{\vartheta-1}, \frac{x^{\sigma}}{|s|}\right),$$

on taking $T = x^2$ we have

$$\psi(\mathcal{X}) \ll \int_{-T}^{T} |IJ(a+it)| \min\left(x^{\vartheta+a-1}, \frac{x^a}{|a+it|}\right) dt$$
$$+ \int_{-T}^{T} \left|HJ\left(\frac{1}{2}+it\right)\right| min\left(x^{\vartheta-\frac{1}{2}}, \frac{x^{\frac{1}{2}}}{|\frac{1}{2}+it|}\right) dt + O(Lx^{-1}).$$

If we set

$$\Sigma(\sigma, K) = \sum_{d<w} \frac{d}{\varphi(d)} \sum_{\chi \bmod d} \int_{-T}^{T} |K(\sigma+it)| \min\left(x^{\vartheta+\sigma-1}, \frac{x^{\sigma}}{|\sigma+it|}\right) dt,$$

then

$$E(w) \ll \Sigma(a, IJ) + \Sigma\left(\frac{1}{2}, HJ\right). \tag{5.13}$$

By Lemma 3.1, in view of assumption (5.6) and our choice of ξ in (5.12), we have

$$\begin{aligned}
&\Sigma(a, I^2) \ll Lx^{\vartheta} + L^2 x\xi^{-1} w^2 \ll L^2 x^{\vartheta};\\
&\Sigma(a, J^2) \ll Lx^{\vartheta} + L^2 x x^{-(1-\vartheta+\varepsilon)} w^2 \ll Lx^{\vartheta} + L^2 x^{\vartheta-\varepsilon} w^2;\\
&\Sigma\left(\frac{1}{2}, H^2\right) \ll x^{\vartheta-\frac{1}{2}}\xi + Lx^{\frac{1}{2}} w^2 \ll L^2 x^{\frac{1}{2}} w^2;\\
&\Sigma\left(\frac{1}{2}, J^2\right) \ll x^{\vartheta-\frac{1}{2}} x^{\frac{12}{5}\vartheta-\frac{7}{5}-\varepsilon} + L^2 x^{\frac{1}{2}} w^2 \ll x^{\frac{17}{5}\vartheta-\frac{19}{10}-\varepsilon} + L^2 x^{\frac{1}{2}} w^2.
\end{aligned}$$

Therefore, by Cauchy-Schwarz inequality,

$$\begin{aligned}
&\Sigma(a, IJ) \ll L^{\frac{3}{2}} x^{\vartheta} + L^2 x^{\vartheta-\frac{\varepsilon}{2}} w\\
&\Sigma\left(\frac{1}{2}, HJ\right) \ll Lx^{\vartheta-\frac{\varepsilon}{2}} w + L^2 x^{\frac{1}{2}} w^2,
\end{aligned}$$

whence, by (5.13), the result. ∎

From Lemma 5.5, by partial summation, we deduce that

$$\sum_{L^N \le q < x^{\vartheta-\frac{1}{2}} L^{-N}} \frac{\mu^2(q)}{\varphi(q)} \sideset{}{^*}\sum_{\chi \bmod q} |\psi(\chi)| \ll x^{\vartheta} L^{2-N},$$

which, combined with Lemma 5.4, yields

$$S_3 \ll x^{\vartheta} L^{2-N},$$

when $D = x^{\vartheta-\frac{1}{2}} L^{-N}$ in (5.10).

Hence, by (5.7), (5.9), Lemma 5.2 and Lemma 5.3, it suffices to choose $N > 18$ to have

$$\sum_{\substack{d<D \\ d|P(D)}} 3^{\nu(d)}|R_d| \ll x^{\vartheta}L^{-3}, \tag{5.14}$$

where $D = x^{\alpha'}$, for any $\alpha' < \vartheta - \frac{1}{2}$.

Let $\varepsilon > 0$. From (5.5) and (5.14) we obtain

$$S(\mathscr{U}, B, x^{\frac{1}{2}}) \leq \frac{4\,\sigma x^{\vartheta}}{\alpha \log^2 x}\left(\eta + \frac{\varepsilon}{3}\right), \tag{5.15}$$

where, as like as in (4.5), $\alpha = \vartheta - \frac{1}{2}$.

6. CONCLUSIONS

First look at the sets typified by $\mathscr{D}$ as in (5.1) with $h = 4$. If $p_1p_2p_3p_4 \in \mathscr{D} \cap A$, then $p_1 < p_2 < p_3 < p_4$ and $x + 2 \leq p_1p_2p_3p_4 \leq x + x^{\vartheta} + 2$. So all that we can say is that $p_4 > x^{\frac{1}{4}}$ and $m \ll x^{\frac{3}{4}}$. Hence, in order to satisfy (5.6), we must have

$$\vartheta > \frac{43}{48}, \tag{6.1}$$

and this turns out to be a limit of our argument.

From (5.5) we easily deduce the two following results.

Corollary 6.1. *If (2.9) and (6.1) hold, then*

$$\sum_{n \in A} r(n) \leq \frac{4\,\sigma x^{\vartheta}}{\alpha \log^2 x}\left(r(u,v) + \frac{\varepsilon}{3}\right),$$

where

$$r(u,v) = \frac{1}{2}\int_{\frac{1}{u}}^{1-\frac{3}{u}} \int_{\frac{1}{u}}^{\frac{1}{3}(1-t)} \frac{1}{st(1-s-t)} \log\left(\frac{1-2s-t}{s}\right) dsdt.$$

Corollary 6.2. *If (2.9) and (6.1) hold, then*

$$\sum_{n \in A} s(n) \leq \frac{4\,\sigma x^{\vartheta}}{\alpha \log^2 x}\left(s(u,v) + \frac{\varepsilon}{3}\right),$$

where

$$s(u,v) = \int_{\frac{1}{u}}^{\frac{1}{3}} \frac{dt}{t(1-t)} \log \frac{1-2t}{t} + \frac{1}{2} \int_{\frac{1}{v}}^{\frac{1}{u}} \frac{dt}{t(1-t)} \log(u-1-ut).$$

Then, if (2.9), (4.4) and (6.1) are satisfied, from (4.5), (4.6), Corollary 6.1 and Corollary 6.2, we get for any $\varepsilon > 0$

$$\sum_{\substack{n\in A\\ (n,P(z))=1}} \left(1 - \frac{1}{2} \sum_{\substack{z\le p\le y\\ p/n, p\in B}} 1 \right) - \sum_{n\in A} s(n) - \sum_{n\in A} r(n) \ge \frac{4\,\sigma x^{\vartheta}}{(\vartheta - \frac{1}{2}) \log^2 x} (c(\vartheta; u, v) - \varepsilon),$$

where

$$c(\vartheta; u, v) = k(\vartheta; u, v) - s(u,v) - r(u,v).$$

Finally, by numerical integeration we check for $\vartheta = 0,9729$

$$c\left(\vartheta; \frac{8}{5} \frac{1}{\vartheta - \frac{1}{2}}, \frac{21}{4} \frac{1}{\vartheta - \frac{1}{2}} \right) > 0,$$

and this concludes the proof of our Theorem.

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S. Salemo
D.I.I.M.A.
Facoltà di Ingegneria
Univ. di Salemo
84084 Fisciano (SA)
Italy

A. Vitolo
D.I.I.M.A.
Facoltà di Scienze
Univ. di Salemo
84081 Baronissi (SA)
Italy