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ON PSEUDO-EINSTEIN RULED REAL HYPERSURFACES IN COMPLEX SPACE FORMS

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Abstract. *In this paper we define the new notion of* pseudo-Einstein ruled *real hypersurfaces, which are foliated by the leaves of pseudo-Einstein complex hypersurfaces in complex space forms* $M_n(c)$, $c\neq 0$. *Also we want to give a new characterization of this kind of* pseudo-Einstein ruled *real hypersurfaces in terms of the Ricci tensor and the certain integrability condition defined on the orthogonal distribution* T_0 *in* $M_n(c)$

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1 Introduction

A complex $n(\geq 2)$-dimensional Kaehler manifold of constant holomorphic sectional curvature c is called a complex space form, which is denoted by $M_n(c)$. A complete and simply connected complex space form is a complex projective space $P_n(\mathbb{C})$, a complex Euclidean space $\mathbb{C}^n$ or a complex hyperbolic space $H_n(\mathbb{C})$, according as $c>0$, $c=0$ or $c<0$. The induced almost contact metric structure of a real hypersurface M of $M_n(c)$ is denoted by (ϕ,ξ,η,g).

Until now several kinds of real hypersurfaces have been investigated by many differential geometers from different view points ([2],[3],[4],[7],[12]and [14]). Among them in a complex projective space $P_n(\mathbb{C})$ [3] Cecil-Ryan and [7] Kimura proved that they are realized as the tubes of constant radius over Kaehler submanifolds if the structure vector field ξ is principal. Also Berndt [2] showed recently that all real hypersurfaces with constant principal curvatures of a complex hyperbolic space $H_n(\mathbb{C})$ are realized as *horospheres* or the *tubes* of constant radius over certain submanifolds when the structure vector field ξ is principal. Nowadays in $H_n(\mathbb{C})$ they are said to be of type A_0,A_1,A_2, and B.

When the structure vector field ξ is not principal, Kimura [8] and Ahn, Lee and the present author [1] have constructed an example of ruled real hypersurfaces foliated by totally geodesic leaves, which are integrable submanifolds of the distribution T_0 defined by the subspace $T_0(x)=\{X\in T_xM : X\perp\xi\}$, $x\in M$, along the direction of ξ and *Einstein* complex hypersurfaces in $P_n(\mathbb{C})$ and $H_n(\mathbb{C})$ respectively. The expression of the Weingarten map is given by

$$A\xi=\alpha\xi+\beta U,\ AU=\beta\xi \text{ and } AX=0, \tag{1.1}$$

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where we have defined a unit vector U orthogonal to ξ in such a way that $\beta U = A\xi - \alpha\xi$ and β denotes the length of a vector field $A\xi - \alpha\xi$ and $\beta(x) \neq 0$ for any point x in M, and for any X in the distribution T_0 and orthogonal to ξ. Recently, several characterizations of such kind of ruled real hypersurfaces have been studied by the papers ([1],[8],[9],[10] and [15]). Moreover, among them there are so many ruled real hypersurfaces, which are foliated in *parallel* by the leaves of the distribution $T_0 = \{X \in T_xM : X \perp \xi\}$ along the integral curve of the structure vector ξ. Then in such a situation the vector field U defined in above is always *parallel* along the direction of ξ.

Now as a general extension of this fact we introduce a new kind of ruled real hypersurfaces in $M_n(c)$ foliated by pseudo-Einstein leaves, which are integrable submanifolds of the distribution T_0 defined by the subspace $\{X \in T_xM : X \perp \xi\}$, along the direction of ξ and *pseudo-Einstein* complex hypersurfaces in $M_n(c)$. Then such kind of ruled real hypersurfaces are said to be *pseudo-Einstein*, because its Ricci tensor of the integral submanifold $M(t)$ is given by

$$S^t = (\frac{n}{2}c - \mu)I + (\mu - \lambda)\{U \otimes U^* + \phi U \otimes (\phi U)^*\}.$$

Moreover, its expression of the Weingarten map is given by

$$AU = \beta\xi + \gamma U + \delta\phi U \text{ and } \quad A\phi U = \delta U - \gamma\phi U.$$

In Lemma 3.1 we know that the function λ in above is given by $\lambda = 2(\gamma^2 + \delta^2)$. When $\lambda = \mu$, ruled real hypersurfaces foliated by such kind of leaves are said to be *Einstein*. In particular, $\lambda = \mu = 0$, this kind of Einstein ruled real hypersurfaces are congruent to ruled real hypersurfaces in $M_n(c)$ foliated by totally geodesic Einstein leaves $M_{n-1}(c)$, which are said to be *totally geodesic* ruled real hypersurfaces in the sense of Kimura [8] for $c > 0$ and Ahn, Lee and the present author [1] for $c < 0$. In such a situation the function γ and δ both vanish identically.

On the other hand, Okumura [13] and Montiel and Romero [12] respectively have considered real hypersurfaces in $P_n(\mathbb{C})$ and in $H_n(\mathbb{C})$ satisfying the condition that the structure tensor ϕ and the shape operator A commute with each other, that is $\phi A = A\phi$, and have shown respectively that they are congruent to real hypersurfaces of type A_1, A_2 in $P_n(\mathbb{C})$ and of type A_0, A_1 and A_2 in $H_n(\mathbb{C})$. That is, we have the following

Theorem A. (Okumura [13], Montiel and Romero [12]) *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$, and $n \geq 3$. If it satisfies the condition*

$$A\phi - \phi A = 0, \tag{1.2}$$

then M is locally congruent to one of the following spaces:

(1) In case $M_n(c) = P_n(\mathbb{C})$

(A_1) a tube of radius r over a hyperplane $P_{n-1}(\mathbb{C})$, where $0 < r < \frac{\pi}{2}$,

(A_2) a tube of radius r over a totally geodesic $P_k(\mathbb{C})$ $(1 \leq k \leq n-2)$, where $0 < r < \frac{\pi}{2}$.

(2) In case $M_n(c) = H_n(\mathbb{C})$

(A_0) a horosphere in $H_n(\mathbb{C})$, i.e., a Montiel tube,
(A_1) a tube of radius $r > 0$ over a totally geodesic hyperplane $H_k(\mathbb{C})$ ($k = 0$ or $n-1$),
(A_2) a tube of radius $r > 0$ over a totally geodesic $H_k(\mathbb{C})$ ($1 \leq k \leq n-2$).

Let us consider a distribution T_0 defined by a subspace $T_0(x)$ of the tangent space T_xM of M at any point x in M such that $T_0(x) = \{u \in T_xM : g(u, \xi(x)) = 0\}$. Then such a distribution T_0 is said to be *holomorphic* in M, because it is invariant by the Kaehler structure J. Now we consider another condition on the distribution T_0 defined by

$$g((A\phi - \phi A)X, Y) = 0 \qquad (I)$$

for any X and Y in T_0, which is much more weaker than (1.2), that is, the structure tensor ϕ and the second fundamental tensor A commute with each other. Of course in the paper [1] and [8] we have shown that *totally geodesic* ruled real hypersurfaces in $M_n(c)$ satisfy the condition (1.1). So naturally they satisfy the formula (I).

On the other hand, the holomorphic distribution T_0 is said to be *integrable* when it satisfies

$$g((A\phi + \phi A)X, Y) = 0, \quad X, Y \in T_0. \qquad (II)$$

Now let us consider the restricted Ricci tensor defined on the distribution T_0 in such a way that

$$g((S\phi - \phi S)X, Y) = f g(AX, Y), \quad X, Y \in T_0, \qquad (III)$$

where f is a smooth function defined on M. When the function f vanishes on M identically and its structure vector ξ is principal, the formula (I) implies the formula (III). So naturally in such a situation real hypersurfaces of type A in Theorem A satisfy the formula (III). But its distribution T_0 can not be integrable.

On the other hand, in section 3 it will be shown that pseudo-Einstein ruled real hypersurfaces also satisfy the formula (III). Moreover, its distribution T_0 is integrable. Then as a characterization of this kind of ruled real hypersurfaces in $M_n(c)$ we assert the following:

Theorem B. *Let M be a real hypersurface in $M_n(c)$, $c \neq 0, n \geq 2$. If it satisfies the condition(III) provided with $f \neq 0$ and the holomorphic distribution T_0 is integrable, then M is locally congruent to a pseudo-Einstein ruled real hypersurface in $M_n(c)$.*

2 Preliminaries

First of all, we recall fundamental properties of real hypersurfaces of a complex space form. Let M be a real hypersurface of a complex n-dimensional complex space form $M_n(c)$ of constant holomorphic sectional curvature $c(\neq 0)$ and let C be a unit normal field on a neighborhood of a point x in M. We denote by J an almost complex structure of $M_n(c)$. For a local vector field X on a neighborhood of x in M, the transformation of X and C under J can be represented as

$$JX = \phi X + \eta(X)C, \qquad JC = -\xi,$$

where ϕ defines a skew-symmetric transformation on the tangent bundle TM of M, while η and ξ denote a 1-form and a vector field on a neighborhood of x in M, respectively. Moreover, it is seen that $g(\xi,X) = \eta(X)$,where g denotes the induced Riemannian metric on M. By properties of the almost complex structure J, the set (ϕ,ξ,η,g) of tensors satisfies

$$\phi^2 = -I + \eta\otimes\xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1,$$

where I denotes the identity transformation. Accordingly, the set is so called an *almost contact metric structure*. Furthermore the covariant derivative of the structure tensors are given by

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX,Y)\xi, \quad \nabla_X\xi = \phi AX, \tag{2.1}$$

where ∇ is the Riemannian connection of g and A denotes the shape operator with respect to the unit normal C on M.

Since the ambient space is of constant holomorphic sectional curvature c, the equations of Gauss and Codazzi are respectively given as follows

$$\begin{aligned} R(Y,Z)U = \tfrac{c}{4}\{&g(Z,U)Y - g(Y,U)Z + g(\phi Z,U)\phi Y - g(\phi Y,U)\phi Z \\ &-2g(\phi Y,Z)\phi U\} + g(AZ,U)AY - g(AY,U)AZ, \end{aligned} \tag{2.2}$$

$$(\nabla_XA)Y - (\nabla_YA)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi\}, \tag{2.3}$$

where R denotes the Riemannian curvature tensor of M and ∇_XA denotes the covariant derivative of the shape operator A with respect to X.

Now let us suppose that the structure vector ξ is a principal vector with principal curvature α, that is, $A\xi = \alpha\xi$. Then, differentiating this, we have

$$(\nabla_XA)\xi = (X\alpha)\xi + \alpha\phi AX - A\phi AX, \tag{2.4}$$

where we have used (2.1). Then it follows

$$\begin{aligned} g((\nabla_XA)Y,\xi) = &(X\alpha)\eta(Y) + \alpha g(Y,\phi AX) \\ &-g(Y,A\phi AX) \end{aligned} \tag{2.5}$$

for any tangent vector fields X and Y on M. By the equation of Codazzi (2.3), we have

$$2A\phi AX - \frac{c}{2}\phi X = \alpha(\phi A + A\phi)X. \tag{2.6}$$

Therefore if a vector field X orthogonal to ξ is a principal vector with a principal curvature λ, then ϕX is also principal with principal curvature $\mu = \frac{(2\alpha\lambda+c)}{2(2\lambda-\alpha)}$, namely we have

$$A\phi X = \mu\phi X, \quad \mu = \frac{2\lambda\alpha + c}{2(2\lambda-\alpha)}. \tag{2.7}$$

Accordingly, the Ricci tensor S is given by

$$S = \frac{1}{4}c\{(2n+1)I - 3\eta\otimes\xi\} + hA - A^2 \tag{2.8}$$

where h is the trace of the second fundamental tensor A of M.

Now in order to get our results, we introduce a lemma due to Ki and the present author [5] as follows:

Lemma 2.1 *Let M be a real hypersurface in a complex space form $M_n(c)$, $n\geq 2$. If it satisfies*

$$A\phi+\phi A=0, \tag{2.9}$$

then we have $c=0$.

3 Pseudo-Einstein ruled real hypersurface

This section is concerned with the necessary properties about *pseudo-Einstein ruled* real hypersurfaces. Before going to give the notion of pseudo-Einstein ruled ones, we recall a ruled real hypersurface M of $M_n(c), c\neq 0$ which is defined in Kimura [7]. Let us denote by $\mathcal{D}$ a J-invariant integrable $(2n-2)$-dimensional distribution defined on $M_n(c)$ whose integral manifolds are holomorphic planes normal to the plane spanned by unit normals C and JC and let $\gamma : I\to M_n(c)$ be an integral curve for the vector $\xi=-JC$.

For any $t(\in I)$ let $M^{(t)}_{n-1}(c)$ be a totally geodesic complex hypersurface through the point $\gamma(t)$ of $M_n(c)$ which is orthogonal to a holomorphic plane spanned by $\gamma'(t)$ and $J\gamma'(t)$. Set $M=\{x\in M^{(t)}_{n-1}(c) : t\in I\}$. Then the construction of M asserts that M is a real hypersurface of $M_n(c)$, which is called a ruled real hypersurface. This means that there exists a ruled real hypersurfaces of $M_n(c)$ with the given distribution $\mathcal{D}$. This kind of ruled real hypersurface is foliated by leaves, which are totally geodesic complex hypersurfaces $M^{(t)}_{n-1}(c)$. Then from its construction it can be easily seen that the expression of the Weingarten map is given by

$$A\xi=\alpha\xi+\beta U,\ AU=\beta\xi \text{ and } AX=0, \tag{3.1}$$

where U is a unit vector orthogonal to ξ and α and β $(\beta\neq 0)$ denote certain differentiable function defined on M and for any X in $\mathcal{D}$ orthogonal to U. Moreover, it can be easily seen that the Ricci tensor S^t of the complex hypersurface $M(t)$ in $M_n(c)$ is propotional to its Riemannian metric such that $S^t=\frac{nc}{2}g$. That is, all of its leaves are Einstein complex hypersurfaces in $M_n(c)$. So such a ruled real hypersurface is naturally said to be *Einstein ruled.*

Now let us consider more generalized notion than the above ones. We want to consider a generalized ruled real hypersurface M, which is foliated by *pseudo-Einstein* leaves. Here, the meaning of *pseudo-Einstein* leaves are integrable submanifolds of the distribution $\mathcal{D}$ which are *pseudo-Einstein* complex hypersurfaces in $M_n(c)$. Then in this case, this kind of generalized ruled real hypersurface is said to be *pseudo-Einstein ruled* real hypersurfaces.

For the construction of this, let us consider two shape operators A_C and A_ξ of any integral submanifold $M(t)=M^{(t)}_{n-1}(c)$ of $\mathcal{D}$ in $M_n(c)$ in the direction of C and ξ. For any unit vector field V along $\mathcal{D}$, let V^* be the corresponding 1-form defined by $V^*(V)=g(V,V)=1$. If the Ricci tensor of $M(t)$ is given by

$$S^t=(\frac{n}{2}c-\mu)I+(\mu-\lambda)\{V\otimes V^*+\phi V\otimes(\phi V)^*\}$$

for a certain vector field V, where λ and μ are smooth functions on M, then the real hypersurface M with the given distribution $\mathcal{D}$ of $M_n(c)$ is said to be *pseudo-Einstein ruled.* In particular, if $\lambda = \mu$, then it is said to be *Einstein ruled* and if $\lambda = \mu = 0$, then it is said to be *totally geodesic* and *Einstein ruled*, and is the ruled real hypersurface as discussed in above. Accordingly, we say that the real hypersurface M is *pseudo-Einstein ruled, Einstein ruled* or *totally geodesic ruled*, then it is easily seen that any integral submanifold of $\mathcal{D}$, which is a submanifold of real codimension 2 in $M_n(c)$, is *pseudo-Einstein, Einstein* or *totally geodesic*, respectively.

On the other hand, the distribution $T_0(= \mathcal{D})$ is integrable, we see

$$g((A\phi + \phi A)X, Y) = 0 \qquad (II)$$

for any vector fields X and Y in T_0.

Now from the notion of pseudo-Einstein ruled real hypersurfaces M in $M_n(c)$ we are going to give an expression of $A_\xi^2 + A_C^2$ of two shape operators A_ξ and A_C of the integral submanifold $M(t)$ of the distribution $\mathcal{D}$, which is a pseudo-Einstein submanifold of real codimension 2 in $M_n(c)$. Of course this expression will be useful to get a complete expression of the shape operator A of M (See Lemma 3.1). Since $M(t)$ is a submanifold of codimension 2, ξ and C are orthonormal vector fields on its leaf in $M_n(c)$. So we have the equation of Gauss

$$\begin{aligned} \bar{\nabla}_X Y &= \nabla_X Y + g(AX,Y)C \\ &= \nabla^t_X Y + g(A_\xi X,Y)\xi + g(A_C X,Y)C, \end{aligned}$$

where $\bar{\nabla}$ and ∇^t are the covariant derivatives in the ambient space $M_n(c)$ and in the submanifold $M(t)$, respectively and moreover A_C and A_ξ are the shape operators in the direction of C and ξ, respectively. Then we have

$$g(\bar{\nabla}_X Y, \xi) = g(\nabla_X Y, \xi) = -g(\nabla_X \xi, Y) = g(A_\xi X, Y),$$

for any $X, Y \in T_0$, from which it implies that

$$A_\xi X = -\phi A X, \ X \in T_0. \qquad (3.2)$$

On the other hand, by the equation of Gauss we have

$$g(AX,Y) = g(A_C X, Y), \ X, Y \in T_0$$

and therefore

$$A_C X = AX - \beta g(X,U)\xi, \ X \in T_0. \qquad (3.3)$$

By (II) we have

$$A\phi X = -\phi A X - \beta g(X, \phi U)\xi, \ X \in T_0. \qquad (3.4)$$

From this it can be easily seen that the traces of these two shape operators A_ξ and A_C are both equal to zero. Now the curvature tensor of the integral submanifold $M(t)$ is given by

$$\begin{aligned} g(R^t(X,Y)Z,W) = \ & \tfrac{c}{4}\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W) + g(\phi Y,Z)g(\phi X,W) \\ & - g(\phi X,Z)g(\phi Y,W) - 2g(\phi X,Y)g(\phi Z,W)\} \\ & + g(A_\xi Y,Z)g(A_\xi X,W) + g(A_C Y,Z)g(A_C X,W) \\ & - g(A_\xi X,Z)g(A_\xi Y,W) - g(A_C X,Z)g(A_C Y,W) \end{aligned}$$

for any vector fields X,Y,Z and W in $\mathcal{D}$. Since the traces of the above two shape operators A_ξ and A_C are both equal to zero, its Ricci tensor S^t of $M(t)$ in $M_n(c)$ is given by

$$\begin{aligned} g(S^tY,Z) &= \textstyle\sum_{i=1}^{2n-2} g(R^t(e_i,Y)Z,e_i) \\ &= \tfrac{n}{2}cg(Y,Z) - g((A_\xi^2+A_C^2)Y,Z) \end{aligned} \tag{3.5}$$

for any Y,Z in $\mathcal{D}$. In such a situation we can define the Ricci tensor S^t of the pseudo-Einstein submanifold $M(t)$ in such a way that

$$S^t = (\frac{n}{2}c-\mu)I + (\mu-\lambda)\{U\otimes U^* + \phi U\otimes(\phi U)^*\}.$$

Then by (3.5) it can be easily checked that the expression of the Ricci tensor S^t is equivalent to the expression of the tensor $A_\xi^2 + A_C^2$ of $M(t)$ given by

$$\begin{cases} (A_\xi^2+A_C^2)U = & \lambda U, \\ (A_\xi^2+A_C^2)\phi U = & \lambda\phi U, \\ (A_\xi^2+A_C^2)X = & \mu X, \quad X\in\mathcal{D}\perp U,\phi U, \end{cases} \tag{3.6}$$

where λ and μ are smooth functions on $M(t)$.

Now we give some examples of pseudo-Einstein ruled real hypersurfaces in complex projective space $P_n(\mathbb{C})$.

Example 1 Let M be a ruled real hypersurface in $P_n(\mathbb{C})$ foliated by complex hyperplane $P_{n-1}(\mathbb{C})$. Then the expression (3.1) implies that

$$A_\xi X = 0 \text{ and } A_C X = 0$$

for any $X\in\mathcal{D}$, where $\mathcal{D}$ denotes the distribution of $P_{n-1}(\mathbb{C})$. This implies $A_\xi^2 + A_C^2 = 0$ on the distribution $\mathcal{D}$. Then its Ricci tensor is given by $S^t = \frac{nc}{2}g$. So we know that M is a totally geodesic Einstein ruled real hypersurface in $P_n(\mathbb{C})$.

Example 2 Let M be a real hypersurface in $P_n(\mathbb{C})$ foliated by complex quadric Q^{n-1}. Then it is known that in Kimura [10] the shape operator A_C defined on the distribution of the complex quadric Q^{n-1} satisfies

$$A_C^2 = \lambda^2 I.$$

Moreover, we know that $A_\xi X = -\phi AX$ for $X\in\mathcal{D}$. Then we know

$$\begin{aligned} A_\xi^2 X &= \phi A\phi AX \\ &= \phi A\phi A_C X \\ &= -\phi^2 AA_C X \\ &= -\phi^2\{A_C^2 X + \beta g(A_0X,U)\xi\} \\ &= -\phi^2\{\lambda^2 X\} \\ &= \lambda^2 X, \end{aligned}$$

where in the third equality we have used the integrability of the distribution $\mathcal{D}$. So it follows that $(A_\xi^2 + A_C^2)X = 2\lambda^2 X$ for any $X\in\mathcal{D}$. Then the Ricci tensor S^t is given by $S^t = \{\frac{n}{2}c - 2\lambda\}g$. From this we conclude that M is not totally geodesic *Einstein* ruled real hypersurface.

Example 3 Let Γ be a complex curve in $P_n(\mathbb{C})$. Now let us consider

$$\phi_{\frac{\pi}{2}}(\Gamma) = \cup_{x\in\Gamma}\{exp_x\frac{\pi}{2}v|v \text{ is a unit normal vector of } \Gamma \text{ at } x\}.$$

Then $\phi_{\frac{\pi}{2}}(\Gamma)$ is an $(n-1)$-dimensional complex hypersurface in $P_n(\mathbb{C})$ (See [8],[9]), which is a submanifold of real codimension 2 in $P_n(\mathbb{C})$. Moreover, it is a *pseudo-Einstein* complex hypersurface in $P_n(\mathbb{C})$. Then we construct a real hypersurface M in $P_n(\mathbb{C})$ foliated by such kind of leaves along the integral curve of the normal vector field $\xi = -JC$.

For this, we consider a regular curve $\gamma : I \to M_n(c)$. Then we can construct a ruled real hypersurface M foliated by pseudo-Einstein complex hypersurfaces in such a way that

$$\begin{aligned} M &= \cup_t \gamma(t)\times\phi_{\frac{\pi}{2}}(\Gamma) \\ &= \cup_t \phi^{(t)}_{\frac{\pi}{2}}(\Gamma). \end{aligned}$$

Moreover, let us take a structure vector ξ such that $\xi(\gamma(t)) = \gamma'(t)$ orthogonal to the tangent space of $\phi_{\frac{\pi}{2}}(\Gamma)$ at $\gamma(t)$. The vector $\xi(\gamma(t))$ can be smoothly extended to any point in $\phi^{(t)}_{\frac{\pi}{2}}(\Gamma)$ by parallel displacement P in such a way that $P\xi(\gamma(t))\perp T_x\phi^{(t)}_{\frac{\pi}{2}}(\Gamma)$ for any x in $\phi^{(t)}_{\frac{\pi}{2}}(\Gamma)$. Then in this case we call such a real hypersurface in $P_n(\mathbb{C})$ *pseudo-Einstein ruled* real hypersurface. Now let us show that its leaves are pseudo-Einstein complex hypersufaces in $P_n(\mathbb{C})$.

In fact, if we consider the principal curvatures of the shape operator A_C defined on the distribution of $\phi_{\frac{\pi}{2}}(\Gamma)$, it is given by

$$cot(\frac{\pi}{2}+\theta) \text{ with multiplicity } 1,$$

$$cot(\frac{\pi}{2}-\theta) \text{ with multiplicity } 1,$$

$$0 \text{ with multiplicity } 2n-4.$$

Then from this expression of the shape operator A_C we can put

$$A_C U = cot(\frac{\pi}{2}+\theta)U,\ A_C\phi U = cot(\frac{\pi}{2}-\theta)\phi U, \text{and } A_C X = 0$$

for a certain vector field $U \in \mathcal{D}$ and any vector field $X \in \mathcal{D}$ orthogonal to U and ϕU, where $\mathcal{D}$ denotes the distribution of $\phi_{\frac{\pi}{2}}(\Gamma)$ orthogonal to the structure vector ξ. Then it can be easily seen that

$$\begin{aligned} A_C^2 U &= cot^2(\tfrac{\pi}{2}+\theta)U = \tfrac{\lambda}{2}U, \\ A_C^2\phi U &= cot^2(\tfrac{\pi}{2}-\theta)\phi U = \tfrac{\lambda}{2}\phi U, \\ A_C^2 X &= 0 \end{aligned}$$

for any X orthogonal to $U, \phi U$. Also if we apply the same method as in Example 2, the shape operator A_ξ can be calculated. So naturally it follows that

$$\begin{aligned} (A_\xi^2 + A_C^2)U &= \lambda U, \\ (A_\xi^2 + A_C^2)\phi U &= \lambda\phi U, \\ (A_\xi^2 + A_C^2)X &= 0 \end{aligned}$$

for any X orthogonal to U and ϕU. Accordingly, we have our assertion.

Now from the formula (3.6) it follows

Lemma 3.1 *Let M be a proper pseudo-Einstein ruled real hypersurfaces in $M_n(c)$, $c\neq 0, n\geq 3$. Then we have*

$$\begin{cases} AU = & \beta\xi+\gamma U+\delta\phi U, \\ A\phi U = & \delta U-\gamma\phi U, \quad \lambda=2(\gamma^2+\delta^2). \end{cases} \tag{3.7}$$

In particular, if it is totally geodesic, we have $\gamma=\delta=0$.

Proof. Naturally let us put

$$\begin{aligned} A\xi &= \alpha\xi+\beta U, \\ AU &= \beta\xi+\gamma U+\delta\phi U+\varepsilon X, \\ A\phi U &= -\gamma\phi U+\delta U-\varepsilon\phi X, \end{aligned} \tag{3.8}$$

for some vector field X orthogonal to ξ, U and ϕU where in the third equation we have used the condition (II), because the distribution $\mathcal{D}$ is integrable. Since M is supposed to be proper pseudo-Einstein, we may put $\lambda\neq\mu$. In order to prove $\varepsilon=0$, firstly let us prove the following

$$A^2U=(\alpha+\gamma)\beta\xi+(\beta^2+\frac{\lambda}{2})U. \tag{3.9}$$

Indeed, (3.2), (3.3) and the first formula of (3.6) imply

$$\begin{aligned} \lambda U &= -A_\xi\phi AU+A_C(AU-\beta\xi) \\ &= \phi A\phi AU+A(AU-\beta\xi)-\beta g(AU-\beta\xi,U)\xi \\ &= 2\{A^2U-\beta A\xi-\beta g(AU,U)\xi\}, \end{aligned}$$

where in the third equality we also have used the condition (II).

Secondly, we calculate the following

$$A^2\phi U=\beta\delta\xi+\frac{\lambda}{2}\phi U. \tag{3.10}$$

In fact, (3.2), (3.3) and the second formula of (3.6) give

$$\begin{aligned} \lambda\phi U &= (A_\xi^2+A_C^2)\phi U \\ &= \phi A^2U+A^2\phi U-\beta^2\phi U-\beta g(A\phi U,U)\xi. \end{aligned}$$

So by (3.8) we get the above (3.10).

Finally we give the following for any X orthogonal to ξ, U and ϕU.

$$A^2X=\beta\varepsilon\xi+\frac{\mu}{2}X, \tag{3.11}$$

because the third formula of (3.6) and the condition (II) imply that

$$\begin{aligned} \mu X &= -A_\xi\phi AX+A_C\{AX-\beta g(X,U)\xi\} \\ &= 2(A^2X-\beta g(AX,U)\xi). \end{aligned}$$

Now let us apply the shape operator A to the second formula of (3.8) and use also (3.8) and (3.9). Then

$$\begin{aligned}\varepsilon AX &= (\tfrac{\lambda}{2} - \gamma^2 - \delta^2)U - \gamma\varepsilon X + \delta\varepsilon\phi X \\ &= \varepsilon^2 U - \gamma\varepsilon X + \delta\varepsilon\phi X,\end{aligned}$$

where we have used

$$\begin{aligned}\|A\phi U\|^2 &= \gamma^2 + \delta^2 + \varepsilon^2 \\ &= \tfrac{\lambda}{2},\end{aligned} \tag{3.12}$$

which can be obtained from (3.8) and (3.10). So let us assume $\varepsilon \neq 0$, then $AX = \varepsilon U - \gamma X + \delta\phi X$. This implies

$$\begin{aligned}A^2X &= \varepsilon AU - \gamma AX + \delta A\phi X \\ &= (\beta\xi + \gamma U + \delta\phi U + \varepsilon X) - \gamma(\varepsilon U - \gamma X + \delta\phi X) \\ &\quad -(\varepsilon\phi U - \gamma\phi X - \delta X) \\ &= \varepsilon\beta\xi + (\varepsilon^2 + \gamma^2 + \delta^2)X.\end{aligned}$$

From this together with (3.11) it follows

$$\mu = 2(\gamma^2 + \delta^2 + \varepsilon^2).$$

Then by (3.12) we have $\lambda = \mu$, which makes a contradiction. So we should have $\varepsilon = 0$. It completes the proof of Lemma 3.1. □

Now let us suppose that the coefficients α, β, γ and δ of the vector $A\xi$ and AU satisfy $\beta^2\gamma = 2\alpha(\gamma^2 + \delta^2)$. A smooth function f is defined by $f\gamma = 2\alpha\delta$, and then it satisfies

$$f\delta = \beta^2 - 2\alpha\gamma.$$

Moreover, if we put $AX = \lambda X$ for any $X \in T'$, where T' denotes the orthogonal complement of $L(\xi, U, \phi U)$, then (3.4) gives $A\phi X = -\lambda\phi X$. From this and (3.7) it follows $h = TrA = \alpha$. Thus (3.7) together with these formulas imply

$$\begin{aligned}2hA\phi U + \beta^2\phi U - fAU &\equiv 0 \ (mod\ \xi), \text{and} \\ -2hAU + \beta^2 U - fA\phi U &\equiv 0 \ (mod\ \xi).\end{aligned}$$

where in the second equation we have used the condition (II). When the function μ in (3.6) vanishes, then (3.11) implies $\|AX\|^2 = 0$ for any $X \in T'$, where T' denotes the orthogonal complement of $L(\xi, U, \phi U)$. So it follows

$$2hA\phi X - fAX \equiv 0 \ (mod\ \xi),\ X \in T'.$$

Consequently, if the coefficients α, β and γ satisfy $\beta^2\gamma = 2\alpha(\gamma^2 + \delta^2)$, then they satisfy

$$2hA\phi X - fAX + \beta^2\{g(X, \phi U)U + g(X, U)\phi U\} \equiv 0 \ (mod \xi), \tag{3.13}$$

for a smooth function f by $2\alpha\delta/\gamma$. Then (3.13) is equivalent to

$$g((S\phi - \phi S)X, Y) = fg(AX, Y),\ X, Y \in T_0. \tag{3.14}$$

4 Proof of the Theorem

Let M be a real hypersurface of $M_n(c)$, $c{\neq}0$, $n{\geq}2$. Assume that it satisfies

$$g((A\phi+\phi A)X,Y)=0, X,Y{\in}T_0, \qquad (II)$$

$$g((S\phi-\phi S)X,Y)=fg(AX,Y),\ X,Y{\in}T_0, \qquad (III)$$

where f is a smooth function on M. The condition (III) is equivalent to

$$g(\left[h(A\phi-\phi A)-(A^2\phi-\phi A^2)-fA\right]X,Y)=0 \tag{4.1}$$

for any vector fields X and Y in T_0.

Without loss of generality, we may suppose that ξ is not principal. Then we can put $A\xi=\alpha\xi+\beta U$, where α and β are smooth functions on M and β does not vanish identically on M. Let M_0 be an open subset in M consisting of points x at which $\beta(x){\neq}0$. Since ξ is supposed to be not principal, M_0 is not empty.

From the condition (II) it follows that

$$(A\phi+\phi A)X=\beta g(\phi X,U)\xi, X{\in}T_0 \tag{4.2}$$

from which together with (4.1) it follows that

$$2hg(A\phi X,Y)-\beta^2\{g(X,U)g(\phi Y,U)+g(Y,U)g(\phi X,U)\}=fg(AX,Y), \tag{4.3}$$

for any vector fields X,Y in T_0.

In fact we have

$$g(A\phi X,Y)=-g(\phi AX,Y)$$

for any vector fields X and Y in T_0 by (II). So it follows

$$\begin{aligned} g(A^2\phi X,Y) &= g(A\phi X,AY) \\ &= g(A\phi X,(AY)_0)+\beta g(Y,U)g(A\phi X,\xi) \\ &= -g(\phi AX,(AY)_0)+\beta^2 g(Y,U)g(\phi X,U) \\ &= -g(A\phi AX,Y)+\beta^2 g(Y,U)g(\phi X,U) \\ &= -g(A\phi(AX)_0,Y)+\beta^2 g(Y,U)g(\phi X,U) \\ &= g(\phi A^2X,Y)+\beta^2\{g(X,U)g(\phi Y,U)+g(Y,U)g(\phi X,U)\} \end{aligned}$$

for any vector fields X and Y in T_0. Then substituting these formulas into (4.1), we have the formula (4.3).

Now let us take U in place of X in (4.3), we have

$$2hA\phi U-fAU+\beta^2\phi U{\equiv}0 \pmod{\xi}. \tag{4.4}$$

Assume that the holomorphic distribution T_0 is integrable. Namely, we assume

$$g((A\phi+\phi A)X,Y)=0,\ X,Y{\in}T_0. \qquad (II)$$

Suppose that ξ is principal. Then by the condition (II) we see

$$A\phi+\phi A=0.$$

By Lemma 2.1 due to Ki and the present author implies that $c = 0$, a contradiction. Hence we may suppose that ξ is not principal. Then we can put $A\xi = \alpha\xi + \beta U$, where α and β are smooth functions on M and β does not vanish identically on M. Let M_0 be an open subset in M consisting of points x at which $\beta(x) \neq 0$. Since ξ is supposed to be not principal, M_0 is not empty. On the open subset M_0 we put $AU = \beta\xi + \gamma U + \delta V$, where ξ, U and V are orthonormal, where γ and δ are smooth functions on M_0. We denote by $L(\xi, U)$ or $L(\xi, U, V)$ a distribution spanned by ξ, U and V in the tangent bundle TM, respectively.

Now let us put

$$\begin{aligned} AU &= \beta\xi + \gamma U + \delta V \\ A\phi U &= -\phi AU = -\gamma\phi U - \delta\phi V \end{aligned} \tag{4.5}$$

Substituting these into (4.4), we have

$$f\gamma U + f\delta V + (2h\gamma - \beta^2)\phi U + 2h\delta\phi V \equiv 0 \ (\mathrm{mod}\ \xi). \tag{4.6}$$

Now firstly we assert that an information for the distribution $L(\xi, U, \phi U)$ is given.

Lemma 4.1 *If it satisfies (II) and (III) and $f \neq 0$, then we have*

$$\begin{cases} A\xi = \alpha\xi + \beta U; \\ AU = \beta\xi + \gamma U + \delta\phi U; \qquad \beta^2\gamma = 2\alpha(\gamma^2 + \delta^2),\ h = \alpha, \\ A\phi U = \delta U - \gamma\phi U; \end{cases} \tag{4.7}$$

on the open subset M_0 and the distribution $L(\xi, U, \phi U)$ is A-invariant.

Proof. We consider only the non-empty open subset M_0. Taking an inner product (4.6) with $X = U$ and V respectively, we have

$$f\gamma + 2h\delta g(U, \phi V) = 0,\ f\delta + (\beta^2 - 2h\gamma)g(U, \phi V) = 0. \tag{4.8}$$

Next, let us take an inner product (4.6) with ϕU and ϕV, respectively. Then we have

$$f\delta g(V, \phi U) + (2h\gamma - \beta^2) = 0,\ f\gamma g(U, \phi V) + 2h\delta = 0. \tag{4.9}$$

By using (4.9) to eliminate the second terms of (4.8) respectively, we have

$$f\gamma\{g(U, \phi V)^2 - 1\} = 0,\ f\delta\{g(U, \phi V)^2 - 1\} = 0. \tag{4.10}$$

Suppose that $g(U, \phi V) \neq \pm 1$. Then we see that $\gamma = \delta = 0$, because of the assumption $f \neq 0$. So by (4.9) we know $\beta = 0$ on M_0, a contradiction. Hence we have $g(U, \phi V) = \pm 1$. Since U and ϕV are unit, $\phi V = \pm U$. Without loss of generality, we may assume that $V = \phi U$.

The mutual relation among the coefficients is given by (4.2),(4.3),(4.4) and (4.5). It completes the proof. □

Lemma 4.2 *If it satisfies (II) and (III) and $f \neq 0$, then we have*

$$AX = 0, \quad X \in T'. \tag{4.11}$$

Proof. By Lemma 4.1 the distribution T' is also A-invariant, because T' is an orthogonal complement of $L(\xi, U, \phi U)$ in the tangent bundle TM and the shape operator A is symmetric. For a principal vector X in T' with principal curvature λ, by (4.2) we have $A\phi X = -\lambda\phi X$. Accordingly, we have by (4.3)

$$2h\lambda g(\phi X, Y) + f\lambda g(X, Y) = 0$$

for any vector field Y in T', which yields that $\lambda = 0$, because of the assumption. It completes the proof. □

Now we are in a position to prove the main theorem

Proof of the Theorem. Lemma 4.2 and the condition of the Theorem we have

$$AX = 0 \tag{4.12}$$

for any vector field X in T' on M_0. By the continuity of principal curvatures we see that the shape operator satisfies the conditions (4.7) and (4.12) on the whole M.

In fact if we consider the set $int(M - M_0)$, then ξ is principal. From this together with the condition (II) we assert $A\phi + \phi A = 0$. Thus Lemma 2.1 implies $c = 0$, which makes a contradiction. Accordingly the set M_0 should be dense in M. So we have the above assertion.

Since the distribution T_0 is integrable on M by the definition, the integral manifold of T_0 can be regarded as the submanifold of codimension 2 in $M_n(c)$ whose normal vectors are ξ and $J\xi = C$.

By the definition of the second fundmental form, we see

$$g(\bar{\nabla}_X Y, C) = -g(\bar{\nabla}_X C, Y) = g(A_C X, Y) = g(AX, Y), \tag{4.13}$$

$$g(\bar{\nabla}_X Y, \xi) = g(\nabla_X Y, \xi) = -g(\nabla_X \xi, Y) = g(A_\xi X, Y), \tag{4.14}$$

for any vector fields X and Y in T_0, where $\bar{\nabla}$ denotes the Riemannian connection of $M_n(c)$ and A_ξ or A_C denotes the shape operator of the integral submanifold $M(t)$ of the disribution T_0 in $M_n(c)$ in the direction of the normal ξ or C, respectively. For any point x in the integral submanifold $M(t)$ we denote by $\{e_i, \phi e_i\}$, $i = 1, \dots, n-1$, an orthonormal basis of the tangent space $T_x M(t)$. Then by (3.2) and (4.2) we have

$$g_x(A_\xi \phi e_i, \phi e_i) = -g_x(A_\xi e_i, e_i).$$

On the other hand, by (3.4) and (4.2) we have

$$g_x(A_C \phi e_i, \phi e_i) = -g_x(\phi A_C e_i, \phi e_i) = -g_x(A_C e_i, e_i).$$

These mean that the integral submanifold $M(t)$ is *minimal* in the ambient space $M_n(c)$. Since T_0 is also J-invariant, its integral manifold is a complex hypersurface and moreover, it is seen that these shape operators satisfy

$$\begin{aligned}\bar{\nabla}_X Y &= \nabla_X Y + g(AX, Y)C \\ &= \nabla'_X Y + g(A_\xi X, Y)\xi + g(A_C X, Y)C\end{aligned}$$

where ∇^t denotes the Riemannian connection of the integral submanifold of T_0. Thus we see

$$\begin{array}{rcl} A_C X &=& AX + g(A_C X - AX, \xi)\xi = AX - \beta g(X,U)\xi, \quad X \in T_0 \\ A_\xi X &=& -\phi AX, \; X \in T_0, \end{array}$$

on M_0, because we have

$$g(\nabla_X Y, \xi) = -g(\nabla_X \xi, Y) = -g(\phi AX, Y) \quad X, Y \in T_0,$$

by (2.1). Since it is discussed in above that the open subset M_0 is dense in M, by means of the continuity of principal curvatures we have

$$\begin{array}{rcl} AU &=& \beta\xi + \gamma U + \delta\phi U, \; A\phi U = \delta U - \gamma\phi U, \\ AX &=& 0, \quad X \in T' \end{array} \tag{4.15}$$

on M and therefore it is seen that another shape operator A_ξ of the integral submanifold of T_0 satisfies

$$A_\xi X = \begin{cases} \delta U - \gamma\phi U, & X = U; \\ -\gamma U - \delta\phi U, & X = \phi U; \\ 0, & X \in T' \end{cases} \tag{4.16}$$

on M, where $X \in T'$ is principal, and it is also seen that another shape operator A_C of the integral submanifold of T_0 satisfies

$$A_C X = \begin{cases} \gamma U + \delta\phi U, & X = U; \\ \delta U - \gamma\phi U, & X = \phi U; \\ 0, & X \in T'. \end{cases} \tag{4.17}$$

on M, where $X \in T'$ is principal. By combining (4.16) with (4.17) and by the direct calculation, it is trivial that we have

$$(A_\xi^2 + A_C^2)X = 2(\gamma^2 + \delta^2)X, \quad X = U \text{ and } \phi U.$$

In the case where X is in T', we see

$$(A_\xi^2 + A_C^2)X = 0, \quad X \in T'.$$

This shows that an integral submanifold is pseudo-Einstein. Thus M is a pseudo- Einstein ruled real hypersurface, because the Ricci tensor S^t of any integral manifold $M(t)$ of the distribution T_0 in a complex space form $M_n(c)$ is given by

$$S^t = \frac{n}{2}cI - 2(\gamma^2 + \delta^2)\{U \otimes U^* + \phi U \otimes \phi U^*\}.$$

Conversely, let M be a pseudo-Einstein ruled real hypersurface in $M_n(c)$. Then we have shown in section 3 that M satisfies (3.14), which is equivalent to the condition (III). Also by its construction we know that it satisfies the condition (II). So it completes the proof. □

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