ON PSEUDO-EINSTEIN RULED REAL HYPERSURFACES IN COMPLEX SPACE FORMS

YOUNG JIN SUH

Abstract. In this paper we define the new notion of pseudo-Einstein ruled real hypersurfaces, which are foliated by the leaves of pseudo-Einstein complex hypersurfaces in complex space forms $M_n(c)$, $c \neq 0$. Also we want to give a new characterization of this kind of pseudo-Einstein ruled real hypersurfaces in terms of the Ricci tensor and the certain integrability condition defined on the orthogonal distribution $T_0$ in $M_n(c)$


Key words: Einstein, Pseudo-Einstein ruled real hypersurface, Complex space form, Ricci tensor, Totally geodesic, Distribution, Weingarten map.

1 Introduction

A complex $n(\geq 2)$-dimensional Kaehler manifold of constant holomorphic sectional curvature $c$ is called a complex space form, which is denoted by $M_n(c)$. A complete and simply connected complex space form is a complex projective space $P_n(\mathbb{C})$, a complex Euclidean space $\mathbb{C}^n$ or a complex hyperbolic space $H_n(\mathbb{C})$, according as $c > 0$, $c = 0$ or $c < 0$. The induced almost contact metric structure of a real hypersurface $M$ of $M_n(c)$ is denoted by $(\phi, \xi, \eta, g)$.

Until now several kinds of real hypersurfaces have been investigated by many differential geometers from different view points ([2],[3],[4],[7],[12]and [14]). Among them in a complex projective space $P_n(\mathbb{C})$ [3] Cecil-Ryan and [7] Kimura proved that they are realized as the tubes of constant radius over Kaehler submanifolds if the structure vector field $\xi$ is principal. Also Berndt [2] showed recently that all real hypersurfaces with constant principal curvatures of a complex hyperbolic space $H_n(\mathbb{C})$ are realized as horospheres or the tubes of constant radius over certain submanifolds when the structure vector field $\xi$ is principal. Nowadays in $H_n(\mathbb{C})$ they are said to be of type $A_0, A_1, A_2,$ and $B$.

When the structure vector field $\xi$ is not principal, Kimura [8] and Ahn, Lee and the present author [1] have constructed an example of ruled real hypersurfaces foliated by totally geodesic leaves, which are integrable submanifolds of the distribution $T_0$ defined by the subspace $T_0(x) = \{X \in T_xM : X \perp \xi \}$, $x \in M$, along the direction of $\xi$ and Einstein complex hypersurfaces in $P_n(\mathbb{C})$ and $H_n(\mathbb{C})$ respectively. The expression of the Weingarten map is given by

$$A\xi = \alpha \xi + \beta U, \quad AU = \beta \xi \quad \text{and} \quad AX = 0, \quad (1.1)$$

\footnote{This paper was supported by the grant from BSRI, 1998-015-D0030, Korea Research Foundation, Korea and partly by TGRC-KOSEF.}
where we have defined a unit vector $U$ orthogonal to $\xi$ in such a way that $\beta U = A\xi - \alpha\xi$ and $\beta$ denotes the length of a vector field $A\xi - \alpha\xi$ and $\beta(x) \neq 0$ for any point $x$ in $M$, and for any $X$ in the distribution $T_0$ and orthogonal to $\xi$. Recently, several characterizations of such kind of ruled real hypersurfaces have been studied by the papers ([1],[8],[9],[10] and [15]). Moreover, among them there are so many ruled real hypersurfaces, which are foliated in parallel by the leaves of the distribution $T_0 = \{X \in T_xM : X \perp \xi\}$ along the integral curve of the structure vector $\xi$. Then in such a situation the vector field $U$ defined in above is always parallel along the direction of $\xi$.

Now as a general extension of this fact we introduce a new kind of ruled real hypersurfaces in $M_n(c)$ foliated by pseudo-Einstein leaves, which are integrable submanifolds of the distribution $T_0$ defined by the subspace $\{X \in T_xM : X \perp \xi\}$, along the direction of $\xi$ and pseudo-Einstein complex hypersurfaces in $M_n(c)$. Then such kind of ruled real hypersurfaces are said to be pseudo-Einstein, because its Ricci tensor of the integral submanifold $M(t)$ is given by

$$S' = \left(\frac{n}{2}c - \mu\right)I + (\mu - \lambda)(U \otimes U^* + \phi U \otimes (\phi U)^*) \cdot$$

Moreover, its expression of the Weingarten map is given by

$$AU = \beta\xi + \gamma U + \delta \phi U \quad \text{and} \quad A\phi U = \delta U - \gamma \phi U.$$

In Lemma 3.1 we know that the function $\lambda$ in above is given by $\lambda = 2(\gamma^2 + \delta^2)$. When $\lambda = \mu$, ruled real hypersurfaces foliated by such kind of leaves are said to be Einstein. In particular, $\lambda = \mu = 0$, this kind of Einstein ruled real hypersurfaces are congruent to ruled real hypersurfaces in $M_n(c)$ foliated by totally geodesic Einstein leaves $M_{n-1}(c)$, which are said to be totally geodesic ruled real hypersurfaces in the sense of Kimura [8] for $c > 0$ and Ahn, Lee and the present author [1] for $c < 0$. In such a situation the function $\gamma$ and $\delta$ both vanish identically.

On the other hand, Okumura [13] and Montiel and Romero [12] respectively have considered real hypersurfaces in $P_n(\mathbb{C})$ and in $H_n(\mathbb{C})$ satisfying the condition that the structure tensor $\phi$ and the shape operator $A$ commute with each other, that is $\phi A = A\phi$, and have shown respectively that they are congruent to real hypersurfaces of type $A_1, A_2$ in $P_n(\mathbb{C})$ and of type $A_0, A_1$ and $A_2$ in $H_n(\mathbb{C})$. That is, we have the following

**Theorem A.** (Okumura [13], Montiel and Romero [12]) Let $M$ be a real hypersurface of $M_n(c)$, $c \neq 0$, and $n \geq 3$. If it satisfies the condition

$$A\phi - \phi A = 0,$$

then $M$ is locally congruent to one of the following spaces:

1. In case $M_n(c) = P_n(\mathbb{C})$

   (A1) a tube of radius $r$ over a hyperplane $P_{n-1}(\mathbb{C})$, where $0 < r < \frac{n}{2}$,

   (A2) a tube of radius $r$ over a totally geodesic $P_k(\mathbb{C})$ 

   (1 \leq k \leq n - 2), where $0 < r < \frac{n}{2}$.

2. In case $M_n(c) = H_n(\mathbb{C})$
(A₀) a horosphere in $H_\ell(C)$, i.e., a Montiel tube,
(A₁) a tube of radius $r > 0$ over a totally geodesic hyperplane $H_k(C)$ ($k = 0$ or $n - 1$),
(A₂) a tube of radius $r > 0$ over a totally geodesic $H_k(C)$ ($1 \leq k \leq n - 2$).

Let us consider a distribution $T_0$ defined by a subspace $T_0(x)$ of the tangent space $T_xM$ of $M$ at any point $x$ in $M$ such that $T_0(x) = \{ u \in T_xM : g(u, \xi(x)) = 0 \}$. Then such a distribution $T_0$ is said to be holomorphic in $M$, because it is invariant by the Kaehler structure $J$. Now we consider another condition on the distribution $T_0$ defined by

$$g((A\phi - \phi A)X, Y) = 0 \quad (I)$$

for any $X$ and $Y$ in $T_0$, which is much more weaker than (1.2), that is, the structure tensor $\phi$ and the second fundamental tensor $A$ commute with each other. Of course in the paper [1] and [8] we have shown that totally geodesic ruled real hypersurfaces in $M_n(c)$ satisfy the condition (1.1). So naturally they satisfy the formula (I).

On the other hand, the holomorphic distribution $T_0$ is said to be integrable when it satisfies

$$g((A\phi + \phi A)X, Y) = 0, \quad X, Y \in T_0. \quad (II)$$

Now let us consider the restricted Ricci tensor defined on the distribution $T_0$ in such a way that

$$g((S\phi - \phi S)X, Y) = f g(AX, Y), \quad X, Y \in T_0, \quad (III)$$

where $f$ is a smooth function defined on $M$. When the function $f$ vanishes on $M$ identically and its structure vector $\xi$ is principal, the formula (I) implies the formula (III). So naturally in such a situation real hypersurfaces of type $A$ in Theorem A satisfy the formula (III). But its distribution $T_0$ can not be integrable.

On the other hand, in section 3 it will be shown that pseudo-Einstein ruled real hypersurfaces also satisfy the formula (III). Moreover, its distribution $T_0$ is integrable. Then as a characterization of this kind of ruled real hypersurfaces in $M_n(c)$ we assert the following:

**Theorem B.** Let $M$ be a real hypersurface in $M_n(c)$, $c \neq 0, n \geq 2$. If it satisfies the condition (III) provided with $f \neq 0$ and the holomorphic distribution $T_0$ is integrable, then $M$ is locally congruent to a pseudo-Einstein ruled real hypersurface in $M_n(c)$.

2 Preliminaries

First of all, we recall fundamental properties of real hypersurfaces of a complex space form. Let $M$ be a real hypersurface of a complex $n$-dimensional complex space form $M_n(c)$ of constant holomorphic sectional curvature $c(\neq 0)$ and let $C$ be a unit normal field on the neighborhood of a point $x$ in $M$. We denote by $J$ an almost complex structure of $M_n(c)$. For a local vector field $X$ on a neighborhood of $x$ in $M$, the transformation of $X$ and $C$ under $J$ can be represented as

$$JX = \phi X + \eta(X)C, \quad JC = -\xi,$$
where $\phi$ defines a skew-symmetric transformation on the tangent bundle $TM$ of $M$, while $\eta$ and $\xi$ denote a 1-form and a vector field on a neighborhood of $x$ in $M$, respectively. Moreover, it is seen that $g(\xi, X) = \eta(X), where $g$ denotes the induced Riemannian metric on $M$. By properties of the almost complex structure $J$, the set $(\phi, \xi, \eta, g)$ of tensors satisfies

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi \xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1,$$

where $I$ denotes the identity transformation. Accordingly, the set is so called an almost contact metric structure. Furthermore the covariant derivative of the structure tensors are given by

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X \xi = \phi AX,$$  \hspace{1cm} (2.1)

where $V$ is the Riemannian connection of $g$ and $A$ denotes the shape operator with respect to the unit normal $C$ on $M$.

Since the ambient space is of constant holomorphic sectional curvature $c$, the equations of Gauss and Codazzi are respectively given as follows

$$R(Y, Z)U = \frac{1}{4} \left\{ g(Z, U)Y - g(Y, U)Z + g(\phi Z, U)\phi Y - g(\phi Y, U)\phi Z 
- 2g(\phi Y, Z)\phi U \right\} + g(AZ, U)AY - g(AY, U)AZ,$$  \hspace{1cm} (2.2)

and

$$(\nabla_X A)Y = (\nabla_Y A)X - c \left\{ \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \right\},$$  \hspace{1cm} (2.3)

where $R$ denotes the Riemannian curvature tensor of $M$ and $\nabla_X A$ denotes the covariant derivative of the shape operator $A$ with respect to $X$.

Now let us suppose that the structure vector $\xi$ is a principal vector with principal curvatures $\alpha$, that is, $A\xi = \alpha \xi$. Then, differentiating this, we have

$$(\nabla_X A)\xi = (X\alpha)\xi + \alpha \phi AX - A\phi AX,$$  \hspace{1cm} (2.4)

where we have used (2.1). Then it follows

$$g(\langle \nabla_X A, \xi \rangle, \xi) = (X\alpha)\eta(Y) + \alpha g(Y, \phi AX)$$

$$- g(Y, A\phi AX)$$  \hspace{1cm} (2.5)

for any tangent vector fields $X$ and $Y$ on $M$. By the equation of Codazzi (2.3), we have

$$2A\phi AX - \frac{c}{2} \phi X = \alpha(\phi A + A\phi)X.$$  \hspace{1cm} (2.6)

Therefore if a vector field $X$ orthogonal to $\xi$ is a principal vector with a principal curvature $\lambda$, then $\phi X$ is also principal with principal curvature $\mu = \frac{(2\alpha + c)}{2(2\lambda - \alpha)}$, namely we have

$$A\phi X = \mu \phi X, \quad \mu = \frac{2\alpha + c}{2(2\lambda - \alpha)}.$$  \hspace{1cm} (2.7)

Accordingly, the Ricci tensor $S$ is given by

$$S = \frac{1}{4} \left\{ (2n + 1)I - 3\eta \otimes \xi \right\} + hA - A^2$$  \hspace{1cm} (2.8)

where $h$ is the trace of the second fundamental tensor $A$ of $M$.

Now in order to get our results, we introduce a lemma due to Ki and the present author [5] as follows:
Lemma 2.1 Let $M$ be a real hypersurface in a complex space form $M_n(c)$, $n \geq 2$. If it satisfies
\[ A\phi + \phi A = 0, \]  
then we have $c = 0$.

3 Pseudo-Einstein ruled real hypersurfaces

This section is concerned with the necessary properties about \textit{pseudo-Einstein ruled} real hypersurfaces. Before going to give the notion of pseudo-Einstein ruled ones, we recall a ruled real hypersurface $M$ of $M_n(c), c \neq 0$ which is defined in Kimura [7]. Let us denote by $\mathcal{D}$ a $J$-invariant integrable $(2n - 2)$-dimensional distribution defined on $M_n(c)$ whose integral manifolds are holomorphic planes normal to the plane spanned by unit normals $C$ and $JC$ and let $\gamma : I \to M_n(c)$ be an integral curve for the vector $\xi = -JC$.

For any $t \in I$ let $M^{(t)}_{n-1}(c)$ be a totally geodesic complex hypersurface through the point $\gamma(t)$ of $M_n(c)$ which is orthogonal to a holomorphic plane spanned by $\gamma'(t)$ and $J\gamma'(t)$. Set $M = \{ x \in M_n^{(t)}_{n-1}(c) : t \in I \}$. Then the construction of $M$ asserts that $M$ is a real hypersurface of $M_n(c)$, which is called a ruled real hypersurface. This means that there exists a ruled real hypersurfaces of $M_n(c)$ with the given distribution $\mathcal{D}$. This kind of ruled real hypersurface is foliated by leaves, which are totally geodesic complex hypersurfaces $M^{(t)}_{n-1}(c)$. Then from its construction it can be easily seen that the expression of the Weingarten map is given by
\[ A\xi = \alpha \xi + \beta U, \quad AU = \beta \xi \text{ and } AX = 0, \]  
where $U$ is a unit vector orthogonal to $\xi$ and $\alpha$ and $\beta$ $(\beta \neq 0)$ denote certain differentiable function defined on $M$ and for any $X$ in $\mathcal{D}$ orthogonal to $U$. Moreover, it can be easily seen that the Ricci tensor $S'$ of the complex hypersurface $M(t)$ in $M_n(c)$ is propotional to its Riemannian metric such that $S' = \frac{\beta \epsilon}{2} g$. That is, all of its leaves are Einstein complex hypersurfaces in $M_n(c)$. So such a ruled real hypersurface is naturally said to be \textit{Einstein ruled}.

Now let us consider more generalized notion than the above ones. We want to consider a generalized ruled real hypersurface $M$, which is foliated by \textit{pseudo-Einstein} leaves. Here, the meaning of \textit{pseudo-Einstein} leaves are integrable submanifolds of the distribution $\mathcal{D}$ which are \textit{pseudo-Einstein} complex hypersurfaces in $M_n(c)$. Then in this case, this kind of generalized ruled real hypersurface is said to be \textit{pseudo-Einstein ruled} real hypersurfaces.

For the construction of this, let us consider two shape operators $A_C$ and $A_\xi$ of any integral submanifold $M(t) = M^{(t)}_{n-1}(c)$ of $\mathcal{D}$ in $M_n(c)$ in the direction of $C$ and $\xi$. For any unit vector field $V$ along $\mathcal{D}$, let $V^*$ be the corresponding 1-form defined by $V^*(V) = g(V, V) = 1$. If the Ricci tensor of $M(t)$ is given by
\[ S' = \left( \frac{n}{2} c - \mu \right) I + (\mu - \lambda) \{ V \otimes V^* + \phi V \otimes (\phi V)^* \} \]  
for a certain vector field $V$, where $\lambda$ and $\mu$ are smooth functions on $M$, then the real hypersurface $M$ with the given distribution $\mathcal{D}$ of $M_n(c)$ is said to be \textit{pseudo-Einstein ruled}. In
particular, if $\lambda = \mu$, then it is said to be *Einstein ruled* and if $\lambda = \mu = 0$, then it is said to be *totally geodesic and Einstein ruled*, and is the ruled real hypersurface as discussed in above. Accordingly, we say that the real hypersurface $M$ is *pseudo-Einstein ruled*, *Einstein ruled* or *totally geodesic ruled*, then it is easily seen that any integral submanifold of $\mathcal{D}$, which is a submanifold of real codimension 2 in $M_n(c)$, is *pseudo-Einstein, Einstein or totally geodesic*, respectively.

On the other hand, the distribution $T_0(= \mathcal{D})$ is integrable, we see

$$g((A\phi + \phi A)X, Y) = 0 \quad (II)$$

for any vector fields $X$ and $Y$ in $T_0$.

Now from the notion of pseudo-Einstein ruled real hypersurfaces $M$ in $M_n(c)$ we are going to give an expression of $A_\xi^2 + A_C^2$ of two shape operators $A_\xi$ and $A_C$ of the integral submanifold $M(t)$ of the distribution $\mathcal{D}$, which is a pseudo-Einstein submanifold of real codimension 2 in $M_n(c)$. Of course this expression will be useful to get a complete expression of the shape operator $A$ of $M$ (See Lemma 3.1). Since $M(t)$ is a submanifold of codimension 2, $\xi$ and $C$ are orthonormal vector fields on its leaf in $M_n(c)$. So we have the equation of Gauss

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)C$$

$$= \nabla'_X Y + g(A_\xi X, Y)\xi + g(A_C X, Y)C,$$

where $\tilde{\nabla}$ and $\nabla'$ are the covariant derivatives in the ambient space $M_n(c)$ and in the submanifold $M(t)$, respectively and moreover $A_C$ and $A_\xi$ are the shape operators in the direction of $C$ and $\xi$, respectively. Then we have

$$g(\tilde{\nabla}_X Y, \xi) = g(\nabla'_X Y, \xi) = -g(\nabla_X \xi, Y) = g(A_\xi X, Y),$$

for any $X, Y \in T_0$, from which it implies that

$$A_\xi X = -\phi AX, \; X \in T_0. \quad (3.2)$$

On the other hand, by the equation of Gauss we have

$$g(AX, Y) = g(A_C X, Y), \; X, Y \in T_0$$

and therefore

$$A_C X = AX - \beta g(X, U)\xi, \; X \in T_0. \quad (3.3)$$

By (II) we have

$$A\phi X = -\phi AX - \beta g(X, \phi U)\xi, \; X \in T_0. \quad (3.4)$$

From this it can be easily seen that the traces of these two shape operators $A_\xi$ and $A_C$ are both equal to zero. Now the curvature tensor of the integral submanifold $M(t)$ is given by

$$g(R^I(X, Y)Z, W) = \frac{\xi}{4} \{g(Y, Z)g(X, W) - g(X, Z)g(Y, W) + g(\phi Y, Z)g(\phi X, W)$$

$$- g(\phi X, Z)g(\phi Y, W) - 2g(\phi X, Y)g(\phi Z, W)\}$$

$$+ g(A_\xi Y, Z)g(A_\xi X, W) + g(A_C Y, Z)g(A_C X, W)$$

$$- g(A_\xi X, Z)g(A_\xi Y, W) - g(A_C X, Z)g(A_C Y, W)$$
for any vector fields $X, Y, Z$ and $W$ in $\mathcal{D}$. Since the traces of the above two shape operators $A_\xi$ and $A_C$ are both equal to zero, its Ricci tensor $S'$ of $M(t)$ in $M_n(c)$ is given by

$$
\begin{align*}
g(S'Y, Z) &= \sum_{i=1}^{2n-2} g(R'(e_i, Y)Z, e_i) \\
&= \frac{n}{2} c g(Y, Z) - g((A_\xi + A_C^2)Y, Z)
\end{align*}
$$

(3.5)

for any $Y, Z$ in $\mathcal{D}$. In such a situation we can define the Ricci tensor $S'$ of the pseudo-Einstein submanifold $M(t)$ in such a way that

$$
S' = (\frac{n}{2} c - \mu)I + (\mu - \lambda)\{U \otimes U^* + \phi U \otimes (\phi U)^*\}.
$$

Then by (3.5) it can be easily checked that the expression of the Ricci tensor $S'$ is equivalent to the expression of the tensor $A_\xi^2 + A_C^2$ of $M(t)$ given by

$$
\begin{align*}
\begin{cases}
(A_\xi^2 + A_C^2)U &= \lambda U, \\
(A_\xi^2 + A_C^2)\phi U &= \lambda \phi U, \\
(A_\xi^2 + A_C^2)X &= \mu X, \quad X \in \mathcal{D} \perp U, \phi U,
\end{cases}
\end{align*}
$$

(3.6)

where $\lambda$ and $\mu$ are smooth functions on $M(t)$.

Now we give some examples of pseudo-Einstein ruled real hypersurfaces in complex projective space $P_n(C)$.

**Example 1** Let $M$ be a ruled real hypersurface in $P_n(C)$ foliated by complex hyperplane $P_{n-1}(C)$. Then the expression (3.1) implies that

$$
A_\xi X = 0 \text{ and } A_C X = 0
$$

for any $X \in \mathcal{D}$, where $\mathcal{D}$ denotes the distribution of $P_{n-1}(C)$. This implies $A_\xi^2 + A_C^2 = 0$ on the distribution $\mathcal{D}$. Then its Ricci tensor is given by $S' = \frac{n}{2} c g$. So we know that $M$ is a totally geodesic Einstein ruled real hypersurface in $P_n(C)$.

**Example 2** Let $M$ be a real hypersurface in $P_n(C)$ foliated by complex quadric $Q^{n-1}$. Then it is known that in Kimura [10] the shape operator $A_C$ defined on the distribution of the complex quadric $Q^{n-1}$ satisfies

$$
A_C^2 = \lambda^2 I.
$$

Moreover, we know that $A_\xi X = -\phi AX$ for $X \in \mathcal{D}$. Then we know

$$
\begin{align*}
A_\xi^2 X &= \phi A\phi AX \\
&= \phi A\phi A_C X \\
&= -\phi^2 A A_C X \\
&= -\phi^2 \{A_\xi^2 X + \beta g(A_0 X, U)\xi\} \\
&= -\phi^2 \{\lambda^2 X\} \\
&= \lambda^2 X,
\end{align*}
$$

where in the third equality we have used the integrability of the distribution $\mathcal{D}$. So it follows that $(A_\xi^2 + A_C^2)X = 2\lambda^2 X$ for any $X \in \mathcal{D}$. Then the Ricci tensor $S'$ is given by $S' = \{\frac{n}{2} c - 2\lambda\} g$. From this we conclude that $M$ is not totally geodesic Einstein ruled real hypersurface.
Example 3  Let $\Gamma$ be a complex curve in $P_n(\mathbb{C})$. Now let us consider

$$\phi_\frac{\pi}{2}(\Gamma) = \bigcup_{x \in \Gamma} \{ \exp_{\frac{x}{2}} v \mid v \text{ is a unit normal vector of } \Gamma \text{ at } x \}.$$ 

Then $\phi_\frac{\pi}{2}(\Gamma)$ is an $(n-1)$-dimensional complex hypersurface in $P_n(\mathbb{C})$ (See [8],[9]), which is a submanifold of real codimension 2 in $P_n(\mathbb{C})$. Moreover, it is a pseudo-Einstein complex hypersurface in $P_n(\mathbb{C})$. Then we construct a real hypersurface $M$ in $P_n(\mathbb{C})$ foliated by such kind of leaves along the integral curve of the normal vector field $\xi = -JC$.

For this, we consider a regular curve $\gamma : I \to M_n(c)$. Then we can construct a ruled real hypersurface $M$ foliated by pseudo-Einstein complex hypersurfaces in such a way that

$$M = \bigcup_t \gamma(t) \times \phi_\frac{\pi}{2}(\Gamma) = \bigcup_t \phi_\frac{\pi}{2}^{(t)}(\Gamma).$$

Moreover, let us take a structure vector $\xi$ such that $\xi(\gamma(t)) = \gamma'(t)$ orthogonal to the tangent space of $\phi_\frac{\pi}{2}(\Gamma)$ at $\gamma(t)$. The vector $\xi(\gamma(t))$ can be smoothly extended to any point in $\phi_\frac{\pi}{2}^{(t)}(\Gamma)$ by parallel displacement $P$ in such a way that $P\xi(\gamma(t)) \perp T_x \phi_\frac{\pi}{2}^{(t)}(\Gamma)$ for any $x$ in $\phi_\frac{\pi}{2}^{(t)}(\Gamma)$. Then in this case we call such a real hypersurface in $P_n(\mathbb{C})$ pseudo-Einstein ruled real hypersurface. Now let us show that its leaves are pseudo-Einstein complex hypersurfaces in $P_n(\mathbb{C})$.

In fact, if we consider the principal curvatures of the shape operator $A_C$ defined on the distribution of $\phi_\frac{\pi}{2}(\Gamma)$, it is given by

- $\cot \left( \frac{\pi}{2} + \theta \right)$ with multiplicity 1,
- $\cot \left( \frac{\pi}{2} - \theta \right)$ with multiplicity 1,
- 0 with multiplicity $2n - 4$.

Then from this expression of the shape operator $A_C$ we can put

$$A_C U = \cot \left( \frac{\pi}{2} + \theta \right) U, \quad A_C \phi U = \cot \left( \frac{\pi}{2} - \theta \right) \phi U,$$

and $A_C X = 0$ for a certain vector field $U \in \mathcal{D}$ and any vector field $X \in \mathcal{D}$ orthogonal to $U$ and $\phi U$, where $\mathcal{D}$ denotes the distribution of $\phi_\frac{\pi}{2}(\Gamma)$ orthogonal to the structure vector $\xi$. Then it can be easily seen that

$$A_C^2 U = \cot^2 \left( \frac{\pi}{2} + \theta \right) U = \frac{\lambda}{2} U,$$

$$A_C^2 \phi U = \cot^2 \left( \frac{\pi}{2} - \theta \right) \phi U = \frac{\lambda}{2} \phi U,$$

$$A_C X = 0$$

for any $X$ orthogonal to $U, \phi U$. Also if we apply the same method as in Example 2, the shape operator $A_\xi$ can be calculated. So naturally it follows that

$$\left( A_\xi^2 + A_C^2 \right) U = \lambda U,$$

$$\left( A_\xi^2 + A_C^2 \right) \phi U = \lambda \phi U,$$

$$\left( A_\xi^2 + A_C^2 \right) X = 0$$

for any $X$ orthogonal to $U$ and $\phi U$. Accordingly, we have our assertion.
Now from the formula (3.6) it follows

**Lemma 3.1** Let $M$ be a proper pseudo-Einstein ruled real hypersurfaces in $M_n(c)$, $c \neq 0, n \geq 3$. Then we have

\[
\begin{align*}
AU &= \beta \xi + \gamma U + \delta \phi U, \\
A \phi U &= \delta U - \gamma \phi U, \\
\lambda &= 2(\gamma^2 + \delta^2). 
\end{align*}
\]  
(3.7)

In particular, if it is totally geodesic, we have $\gamma = \delta = 0$.

**Proof.** Naturally let us put

\[
\begin{align*}
A \xi &= \alpha \xi + \beta U, \\
AU &= \beta \xi + \gamma U + \delta \phi U + \epsilon X, \\
A \phi U &= -\gamma \phi U + \delta U - \epsilon \phi X, 
\end{align*}
\]  
(3.8)

for some vector field $X$ orthogonal to $\xi, U$ and $\phi U$ where in the third equation we have used the condition (II), because the distribution $\mathcal{D}$ is integrable. Since $M$ is supposed to be proper pseudo-Einstein, we may put $\lambda \neq \mu$. In order to prove $\epsilon = 0$, firstly let us prove the following

\[
A^2 U = (\alpha + \gamma) \beta \xi + (\beta^2 + \frac{\lambda}{2}) U. 
\]  
(3.9)

Indeed, (3.2), (3.3) and the first formula of (3.6) imply

\[
\begin{align*}
\lambda U &= -A \xi \phi AU + A_C(AU - \beta \xi) \\
&= \phi A \phi AU + A(AU - \beta \xi) - \beta g(AU - \beta \xi, U) \xi \\
&= 2 \{A^2 U - \beta A \xi - \beta g(AU, U) \xi\},
\end{align*}
\]

where in the third equality we also have used the condition (II).

Secondly, we calculate the following

\[
A^2 \phi U = \beta \delta \xi + \frac{\lambda}{2} \phi U. 
\]  
(3.10)

In fact, (3.2), (3.3) and the second formula of (3.6) give

\[
\begin{align*}
\lambda \phi U &= (A^2 + A^2 \phi) \phi U \\
&= \phi A^2 U + A^2 \phi U - \beta^2 \phi U - \beta g(A \phi U, U) \xi.
\end{align*}
\]

So by (3.8) we get the above (3.10).

Finally we give the following for any $X$ orthogonal to $\xi, U$ and $\phi U$.

\[
A^2 X = \beta e \xi + \frac{\mu}{2} X, 
\]  
(3.11)

because the third formula of (3.6) and the condition (II) imply that

\[
\begin{align*}
\mu X &= -A \xi \phi AX + A_C\{AX - \beta g(X, U) \xi\} \\
&= 2(A^2 X - \beta g(AX, U) \xi).
\end{align*}
\]
Now let us apply the shape operator $A$ to the second formula of (3.8) and use also (3.8) and (3.9). Then
\[\varepsilon AX = \left(\frac{1}{2} - \gamma^2 - \delta^2\right)U - \gamma eX + \delta e\phi X\]
\[= \varepsilon^2 U - \gamma eX + \delta e\phi X,\]
where we have used
\[||A\phi U||^2 = \gamma^2 + \delta^2 + \varepsilon^2 = \frac{1}{2},\]
which can be obtained from (3.8) and (3.10). So let us assume $\varepsilon \neq 0$, then $AX = \varepsilon U - \gamma X + \delta \phi X$. This implies
\[A^2X = \varepsilon AU - \gamma AX + \delta A\phi X\]
\[= (\beta \xi + \gamma U + \delta \phi U + eX) - \gamma (\varepsilon U - \gamma X + \delta \phi X)\]
\[-(e\phi U - \gamma \phi X - \delta X)\]
\[= \varepsilon\beta \xi + (\varepsilon^2 + \gamma^2 + \delta^2)X.\]

From this together with (3.11) it follows
\[\mu = 2(\gamma^2 + \delta^2 + \varepsilon^2).\]

Then by (3.12) we have $\lambda = \mu$, which makes a contradiction. So we should have $\varepsilon = 0$. It completes the proof of Lemma 3.1.

Now let us suppose that the coefficients $\alpha, \beta, \gamma$ and $\delta$ of the vector $A\xi$ and $AU$ satisfy $\beta^2 \gamma = 2 \alpha (\gamma^2 + \delta^2)$. A smooth function $f$ is defined by $f\gamma = 2\alpha \delta$, and then it satisfies
\[f\delta = \beta^2 - 2\alpha \gamma.\]

Moreover, if we put $AX = \lambda X$ for any $X \in T'$, where $T'$ denotes the orthogonal complement of $L(\xi, U, \phi U)$, then (3.4) gives $A\phi X = -\lambda \phi X$. From this and (3.7) it follows $h = TrA = \alpha$. Thus (3.7) together with these formulas imply
\[2hA\phi U + \beta^2 \phi U - fA U \equiv 0 \pmod{\xi}, \text{ and}\]
\[-2hAU + \beta^2 U - fA\phi U \equiv 0 \pmod{\xi}.\]

where in the second equation we have used the condition (II). When the function $\mu$ in (3.6) vanishes, then (3.11) implies $||AX||^2 = 0$ for any $X \in T'$, where $T'$ denotes the orthogonal complement of $L(\xi, U, \phi U)$. So it follows
\[2hA\phi X - fAX \equiv 0 \pmod{\xi}, \ X \in T'.\]

Consequently, if the coefficients $\alpha$, $\beta$ and $\gamma$ satisfy $\beta^2 \gamma = 2 \alpha (\gamma^2 + \delta^2)$, then they satisfy
\[2hA\phi X - fAX + \beta^2 \{g(X, \phi U)U + g(X, U)\phi U\} \equiv 0 \pmod{\xi},\]
for a smooth function $f$ by $\beta \alpha \delta / \gamma$. Then (3.13) is equivalent to
\[g((S\phi - \phi S)X, Y) = fg(AX, Y), \ X, Y \in T_0.\]
4 Proof of the Theorem

Let $M$ be a real hypersurface of $M_n(c), \ c \neq 0, \ n \geq 2$. Assume that it satisfies
\[
g((A\phi + \phi A)X, Y) = 0, \ X, Y \in T_0, \quad (I)\]
\[
g((S\phi - \phi S)X, Y) = fg(AX, Y), \ X, Y \in T_0, \quad (III)\]
where $f$ is a smooth function on $M$. The condition (III) is equivalent to
\[
g([h(A\phi - \phi A) - (A^2\phi - \phi A^2) - fA]X, Y) = 0 \quad (4.1)\]
for any vector fields $X$ and $Y$ in $T_0$.

Without loss of generality, we may suppose that $\xi$ is not principal. Then we can put $A\xi = \alpha \xi + \beta U$, where $\alpha$ and $\beta$ are smooth functions on $M$ and $\beta$ does not vanish identically on $M$. Let $M_0$ be an open subset in $M$ consisting of points $x$ at which $\beta(x) \neq 0$. Since $\xi$ is supposed to be not principal, $M_0$ is not empty.

From the condition (II) it follows that
\[
(A\phi + \phi A)X = \beta g(\phi X, U)\xi, \ X \in T_0 \quad (4.2)\]
from which together with (4.1) it follows that
\[
2hg(A\phi X, Y) - \beta^2 \{g(X, U)g(\phi Y, U) + g(Y, U)g(\phi X, U)\} = fg(AX, Y), \quad (4.3)\]
for any vector fields $X, Y$ in $T_0$.

In fact we have
\[
g(A\phi X, Y) = -g(\phi AX, Y)\]
for any vector fields $X$ and $Y$ in $T_0$ by (II). So it follows
\[
g(A^2\phi X, Y) = g(A\phi X, AX)
= g(A\phi X, (AY)_0) + \beta g(Y, U)g(A\phi X, \xi)
= -g(\phi AX, (AY)_0) + \beta^2 g(Y, U)g(\phi X, U)
= -g(A\phi AX, Y) + \beta^2 g(Y, U)g(\phi X, U)
= -g(A\phi AX)_0, Y) + \beta^2 g(Y, U)g(\phi X, U)
= g(\phi A^2X, Y) + \beta^2 \{g(X, U)g(\phi Y, U) + g(Y, U)g(\phi X, U)\}\]
for any vector fields $X$ and $Y$ in $T_0$. Then substituting these formulas into (4.1), we have the formula (4.3).

Now let us take $U$ in place of $X$ in (4.3), we have
\[
2hA\phi U - fAU + \beta^2 \phi U \equiv 0 \quad (\text{mod } \xi). \quad (4.4)\]
Assume that the holomorphic distribution $T_0$ is integrable. Namely, we assume
\[
g((A\phi + \phi A)X, Y) = 0, \ X, Y \in T_0. \quad (II)\]
Suppose that $\xi$ is principal. Then by the condition (II) we see
\[
A\phi + \phi A = 0.\]
By Lemma 2.1 due to Ki and the present author implies that $c = 0$, a contradiction. Hence we may suppose that $\xi$ is not principal. Then we can put $A\xi = \alpha \xi + \beta U$, where $\alpha$ and $\beta$ are smooth functions on $M$ and $\beta$ does not vanish identically on $M$. Let $M_0$ be an open subset in $M$ consisting of points $x$ at which $\beta(x) \neq 0$. Since $\xi$ is supposed to be not principal, $M_0$ is not empty. On the open subset $M_0$ we put $AU = \beta \xi + \gamma U + \delta V$, where $\xi, U$ and $V$ are orthonormal, where $\gamma$ and $\delta$ are smooth functions on $M_0$. We denote by $L(\xi, U)$ or $L(\xi, U, V)$ a distribution spanned by $\xi, U$ and $V$ in the tangent bundle $TM$, respectively.

Now let us put

$$
AU = \beta \xi + \gamma U + \delta V
$$
$$
A\phi U = -\phi AU = -\gamma \phi U - \delta \phi V
$$

Substituting these into (4.4), we have

$$
f \gamma U + f \delta V + (2h\gamma - \beta^2)\phi U + 2h \delta \phi V \equiv 0 \pmod{\xi}. \tag{4.6}
$$

Now firstly we assert that an information for the distribution $L(\xi, U, \phi U)$ is given.

**Lemma 4.1** If it satisfies (II) and (III) and $f \neq 0$, then we have

$$
\begin{align*}
A\xi &= \alpha \xi + \beta U; \\
AU &= \beta \xi + \gamma U + \delta \phi U; \\
A\phi U &= \delta U - \gamma \phi U;
\end{align*}
$$

on the open subset $M_0$ and the distribution $L(\xi, U, \phi U)$ is $A$-invariant.

**Proof.** We consider only the non-empty open subset $M_0$. Taking an inner product (4.6) with $X = U$ and $V$ respectively, we have

$$
f \gamma + 2h g(U, \phi V) = 0, \quad f \delta + (\beta^2 - 2h\gamma) g(U, \phi V) = 0. \tag{4.8}
$$

Next, let us take an inner product (4.6) with $\phi U$ and $\phi V$, respectively. Then we have

$$
f \delta g(V, \phi U) + (2h\gamma - \beta^2) = 0, \quad f \gamma g(U, \phi V) + 2h \delta = 0. \tag{4.9}
$$

By using (4.9) to eliminate the second terms of (4.8) respectively, we have

$$
f \gamma \{g(U, \phi V)^2 - 1\} = 0, \quad f \delta \{g(U, \phi V)^2 - 1\} = 0. \tag{4.10}
$$

Suppose that $g(U, \phi V) \neq \pm 1$. Then we see that $\gamma = \delta = 0$, because of the assumption $f \neq 0$. So by (4.9) we know $\beta = 0$ on $M_0$, a contradiction. Hence we have $g(U, \phi V) = \pm 1$. Since $U$ and $\phi V$ are unit, $\phi V = \pm U$. Without loss of generality, we may assume that $V = \phi U$.

The mutual relation among the coefficients is given by (4.2),(4.3),(4.4) and (4.5). It completes the proof.

**Lemma 4.2** If it satisfies (II) and (III) and $f \neq 0$, then we have

$$
AX = 0, \quad X \in T'. \tag{4.11}
$$
**Proof.** By Lemma 4.1 the distribution $T'$ is also $A$-invariant, because $T'$ is an orthogonal complement of $L(\xi, U, \phi U)$ in the tangent bundle $TM$ and the shape operator $A$ is symmetric. For a principal vector $X$ in $T'$ with principal curvature $\lambda$, by (4.2) we have $A\phi X = -\lambda\phi X$. Accordingly, we have by (4.3)

$$2h\lambda g(\phi X, Y) + f\lambda g(X, Y) = 0$$

for any vector field $Y$ in $T'$, which yields that $\lambda = 0$, because of the assumption. It completes the proof. □

Now we are in a position to prove the main theorem

**Proof of the Theorem.** Lemma 4.2 and the condition of the Theorem we have

$$AX = 0 \quad (4.12)$$

for any vector field $X$ in $T'$ on $M_0$. By the continuity of principal curvatures we see that the shape operator satisfies the conditions (4.7) and (4.12) on the whole $M$.

In fact if we consider the set int$(M - M_0)$, then $\xi$ is principal. From this together with the condition (II) we assert $A\phi + \phi A = 0$. Thus Lemma 2.1 implies $c = 0$, which makes a contradiction. Accordingly the set $M_0$ should be dense in $M$. So we have the above assertion.

Since the distribution $T_0$ is integrable on $M$ by the definition, the integral manifold of $T_0$ can be regarded as the submanifold of codimension 2 in $M_n(c)$ whose normal vectors are $\xi$ and $J\xi = C$.

By the definition of the second fundamental form, we see

$$g(\check{\nabla}X, C) = -g(\check{\nabla}X, C) = g(A_C X, Y) = g(A_X, Y), \quad (4.13)$$

$$g(\check{\nabla}X, \xi) = g(\check{\nabla}X, \xi) = -g(\check{\nabla}X, \xi) = g(A_\xi X, Y), \quad (4.14)$$

for any vector fields $X$ and $Y$ in $T_0$, where $\check{\nabla}$ denotes the Riemannian connection of $M_n(c)$ and $A_\xi$ or $A_C$ denotes the shape operator of the integral submanifold $M(t)$ of the distribution $T_0$ in $\check{M}_n(c)$ in the direction of the normal $\xi$ or $C$, respectively. For any point $x$ in the integral submanifold $M(t)$ we denote by $\{e_i, \phi e_i\}$, $i = 1, \ldots, n - 1$, an orthonormal basis of the tangent space $T_x M(t)$. Then by (3.2) and (4.2) we have

$$g_x(A_\xi \phi e_i, \phi e_i) = -g_x(A_\xi e_i, e_i).$$

On the other hand, by (3.4) and (4.2) we have

$$g_x(A_C \phi e_i, \phi e_i) = -g_x(\phi A_C e_i, e_i) = -g_x(A_C e_i, e_i).$$

These mean that the integral submanifold $M(t)$ is minimal in the ambient space $M_n(c)$. Since $T_0$ is also $J$-invariant, its integral manifold is a complex hypersurface and moreover, it is seen that these shape operators satisfy

$$\check{\nabla}X = \nabla X + g(AX, Y)C$$

$$= \nabla' X + g(A_\xi X, Y)\xi + g(A_C X, Y)C$$
where $\nabla^I$ denotes the Riemannian connection of the integral submanifold of $T_0$. Thus we see

$$\begin{align*}
A_C X &= AX + g(A_C X - AX, \xi)\xi = AX - \beta g(X, U)\xi, \quad X \in T_0 \\
A_\xi X &= -\phi AX, \quad X \in T_0,
\end{align*}$$

on $M_0$, because we have

$$g(\nabla_X Y, \xi) = -g(\nabla_X \xi, Y) = -g(\phi AX, Y) \quad X, Y \in T_0,$$

by (2.1). Since it is discussed in above that the open subset $M_0$ is dense in $M$, by means of the continuity of principal curvatures we have

$$\begin{align*}
AU &= \beta \xi + \gamma U + \delta \phi U, \quad A\phi U = \delta U - \gamma \psi U, \\
AX &= 0, \quad X \in T',
\end{align*}$$

(4.15)

on $M$ and therefore it is seen that another shape operator $A_\xi$ of the integral submanifold of $T_0$ satisfies

$$A_\xi X = \begin{cases} 
\delta U - \gamma \phi U, & X = U; \\
-\gamma U - \delta \phi U, & X = \phi U; \\
0, & X \in T'
\end{cases}$$

(4.16)

on $M$, where $X \in T'$ is principal, and it is also seen that another shape operator $A_C$ of the integral submanifold of $T_0$ satisfies

$$A_C X = \begin{cases} 
\gamma U + \delta \phi U, & X = U; \\
\delta U - \gamma \phi U, & X = \phi U; \\
0, & X \in T'.
\end{cases}$$

(4.17)

on $M$, where $X \in T'$ is principal. By combining (4.16) with (4.17) and by the direct calculation, it is trivial that we have

$$(A^2_\xi + A^2_C)X = 2(\gamma^2 + \delta^2)X, \quad X = U \text{ and } \phi U.$$

In the case where $X$ is in $T'$, we see

$$(A^2_\xi + A^2_C)X = 0, \quad X \in T'.$$

This shows that an integral submanifold is pseudo-Einstein. Thus $M$ is a pseudo- Einstein ruled real hypersurface, because the Ricci tensor $S'$ of any integral manifold $M(t)$ of the distribution $T_0$ in a complex space form $M_n(c)$ is given by

$$S' = \frac{n}{2}cI - 2(\gamma^2 + \delta^2)\{U \otimes U^* + \phi U \otimes \phi U^*\}.$$

Conversely, let $M$ be a pseudo-Einstein ruled real hypersurface in $M_n(c)$. Then we have shown in section 3 that $M$ satisfies (3.14), which is equivalent to the condition (III). Also by its construction we know that it satisfies the condition (II). So it completes the proof.
References


Received December 9, 1998 and in revised form March 2, 1999
Department of Mathematics
Kyungpook National University
Taegu 702-701
KOREA
e-mail: yjsuh@bh.kyungpook.ac.kr