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THE MAXIMALITY OF THE GROUP OF EUCLIDEAN SIMILARITIES WITHIN THE AFFINE GROUP

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Abstract. *The purpose of this note is to show by elementary means that over the field of real numbers, or more generally over any Euclidean field K with Archimedean order, the group of n-dimensional Euclidean similarities is maximal within the group of all affine mappings having a determinant of the form* $\pm\lambda^n \neq 0$. *As a corollary it turns out that the orthogonal group* $O_n(K)$ *is maximal within the group* $SL_n(K)^{\pm}$ *of all matrices of determinant* ± 1.

1 Introduction

The maximality of certain classical groups within certain other ones is a subject that has been dealt with by many authors (cf. Dynkin [3] or the survey of Seitz [7]). In this note we shall consider a relatively easy particular instance of this type of question in an entirely elementary setting: the maximality of the group of Euclidean similarities within the group of affine transformations.

Let K be a field and $n \geq 2$. let us denote by $GL_n(K)$ the group of all invertible $n \times n$ matrices, by $SL_n(K)$ the group of all matrices having determinant one, by $O_n(K)$ the group of all orthogonal matrices, i.e. M such that $MM^{\perp} = I_n$ (where I_n denotes the $n \times n$ identity matrix), by $O_n^+(K)$ the group of all orthogonal matrices of determinant one, and finally by Λ_n the group of all matrices which have an element $\lambda \neq 0$ along the main diagonal and zeroes everywhere else.

In a vector space V of dimension n over K an arbitrary affine transformation may then be written as $X \to XM + B$, where $X, B \in V$, and $M \in GL_n(K)$. A mapping $X \to XM + B$, where $M \in \Lambda_n O_n(K)$ on the other hand, by analogy to the classical case may be called a Euclidean similarity. The maximality of the group of Euclidean similarities within the group of affine transformations is immediately seen to be equivalent to the maximality of $\Lambda_n O_n(K)$ within $GL_n(K)$. In general it is not reasonable to expect that $\Lambda_n O_n(K)$ should be maximal in $GL_n(K)$ since the first group contains only matrices with determinants of the form $\pm\lambda^n$. In an arbitrary field however, the numbers of this form need not exhaust the multiplicative group. We are therefore lead to consider the subgroup $GL_n(K)^*$ of $GL_n(K)$ which consists only of matrices with determinant ± 1 times a non-zero n-th power.

The purpose of this note is to show by an elementary argument using essentially only matrix multiplication that under suitable restrictions on K the group $\Lambda_n O_n(K)$ is maximal within the group $GL_n(K)^*$. As to the restrictions, we require that K is a Euclidean field which means that K is ordered and every positive element is a square. Moreover it is required that the order in K be Archimedean. In most of our arguments however, it is enough to assume that K is Pythagorean (cf. Bachmann [1], page 216 or Weyl [8], page 13) which amounts to the following two properties: i) the element -1 is a non-square and ii) the sum of two squares is a square. It is well-known that a Pythagorean field K can be ordered, but generally in more than one way. That is, if we set $a \leq b$ if and only if $b-a$ is a square we may get only a partial ordering.

Theorem 1 *For any Euclidean field K with Archimedean order the group $\Lambda_n O_n(K)$ is a maximal subgroup of $GL_n(K)^*$.*

In the field R of real numbers the set of n-th powers contains all positive numbers. Therefore $GL_n(R)^* = GL_n(R)$ and the group of Euclidean similarities is maximal within the affine group. It should be noted however, that for the field of real numbers theorem 1 can be deduced from two well-known facts on Lie groups. Namely, that $O_n^+(R)$ is a maximal compact subgroup of $SL_n(R)$ (cfr. [6], page 335) and that a maximal compact subgroup of a simple Lie group is maximal in the group theoretic sense (cf. [5], Chap. VI, Ex. A3 (iv) page 276).

The proof of theorem 1 will be given in section 2 below. As an easy corollary we shall show that $O_n(K)$ is maximal within the group $SL_n(K)^{\pm}$ of matrices with determinant ± 1. In section 3 we shall give a counterexample showing that for the field of rational numbers the assertion of theorem 1 is not true. Of course, between this counterexample for rational numbers and the Euclidean fields in the hypothesis of theorem 1 there is still a wide gap.

2 The Maximality of $\Lambda_n O_n(K)$ in $GL_n(K)^*$

In this section we assume throughout that K is a Pythagorean field. When it is necessary to assume that K is Euclidean we shall state it explicitely. We shall proceed by induction on the dimension n and therefore we look at the case $n = 2$ first. We have to show that $\Lambda_2 O_2(K)$ is maximal in $GL_2(K)^*$. We shall proceed by several small steps in which we assume that $H_2(K)$ is a group such that $\Lambda_2 O_2(K) < H_2(K) \leq GL_2(K)^*$.

2.1 *In $H_2(K)$ there exists a matrix of the form $\begin{pmatrix} a & 0 \\ q & a^{-1}\varepsilon \end{pmatrix}$ where $\varepsilon^2 = 1$ and at least one of the inequalities $q \neq 0$ or $a^2 \neq 1$ is satisfied.*

Proof. Let U be a matrix in $H_2(K)$ which is not contained in $\Lambda_2 O_2(K)$. By multiplying with a suitable matrix X of $O_2(K)$ we can achieve that UX maps the vector $(1,0)$ to a scalar multiple of itself and thus λUX must have the form above for a suitable λ. The inequalities simply express the fact thet λUX does not belong to $O_2(K)$.

To find the matrix X assume $(1,0)U = (u_{11}, u_{12})$. The matrices in $O_2^+(K)$ have the general form $X = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$ where $c^2 + s^2 = 1$. We wish to solve $-u_{11}s + u_{12}c = 0$ for c and s satisfying $c^2 + s^2 = 1$. if $u_{11} = 0$ take $s = 1$, $c = 0$. If $u_{11} \neq 0$ we have $s = u_{11}^{-1} u_{12} c$. Now $c^2 + s^2 = 1$ gives the condition $(1 + u_{11}^{-2} u_{12}^2)c^2 = 1$, i.e. $c = \frac{1}{\sqrt{1 + u_{11}^{-2} u_{12}^2}}$. □

In the following let us assume that $\begin{pmatrix} a & 0 \\ q & a^{-1}\varepsilon \end{pmatrix}$ is a fixed matrix as in (2.1) contained in $H_2(K)$. Of course it is not contained in $\Lambda_2 O_2(K)$.

2.2 *The following matrices* $\begin{pmatrix} a & 0 \\ q & a^{-1} \end{pmatrix}$, $\begin{pmatrix} a & 0 \\ q & -a^{-1} \end{pmatrix}$, $\begin{pmatrix} a & 0 \\ -q & a^{-1} \end{pmatrix}$, *and* $\begin{pmatrix} 1 & 0 \\ 2a^{-1}q & 1 \end{pmatrix}$ *are also contained in* $H_2(K)$.

Proof. $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ belongs to $O_2(K)$ and hence $\begin{pmatrix} a & 0 \\ q & a^{-1}\varepsilon \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} =$ $\begin{pmatrix} a & 0 \\ q & -a^{-1}\varepsilon \end{pmatrix}$ belongs to $H_2(K)$ which proves that the first two of the matrices in lemma (2.2) are in $H_2(K)$.

Now $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a & 0 \\ q & a^{-1} \end{pmatrix} = \begin{pmatrix} a & 0 \\ -q & -a^{-1} \end{pmatrix}$ is in $H_2(K)$ and hence so is the matrix $\begin{pmatrix} a & 0 \\ -q & a^{-1} \end{pmatrix}$.

Finally the inverse of $\begin{pmatrix} a & 0 \\ q & a^{-1} \end{pmatrix}$ is $\begin{pmatrix} a^{-1} & 0 \\ -q & a \end{pmatrix}$ and hence $\begin{pmatrix} a^{-1} & 0 \\ q & a \end{pmatrix}$ is in $H_2(K)$.

Thus $\begin{pmatrix} a & 0 \\ q & a^{-1} \end{pmatrix}\begin{pmatrix} a^{-1} & 0 \\ q & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2a^{-1}q & 1 \end{pmatrix}$ also is an element of $H_2(K)$. □

2.3 *The matrix* $\begin{pmatrix} a\cdot c^2 - q\cdot s\cdot c + a^{-1}\varepsilon\cdot s^2 & (a-a^{-1}\varepsilon)\cdot s\cdot c - q\cdot s^2 \\ (a-a^{-1}\varepsilon)\cdot s\cdot c + q\cdot c^2 & a\cdot s^2 + q\cdot c\cdot s + a^{-1}\varepsilon\cdot c^2 \end{pmatrix}$ *is contained in* $H_2(K)$ *for any admissible pair* $c^2+s^2=1$. *Here the value of* ε *may be freely chosen to be* 1 *or* -1.

Proof. The matrix $\begin{pmatrix} c & -s \\ s & c \end{pmatrix}$ and its inverse $\begin{pmatrix} c & s \\ -s & c \end{pmatrix}$ both belong to $O_2(K)$ and the matrix in (2.3) is equal to $\begin{pmatrix} c & -s \\ s & c \end{pmatrix}\begin{pmatrix} a & 0 \\ q & a^{-1}\varepsilon \end{pmatrix}\begin{pmatrix} c & s \\ -s & c \end{pmatrix}$. □

2.4 *When* $q=0$ *in (2.1) then*

$$M = \begin{pmatrix} ((a^2+a^{-2})/2^{\frac{1}{2}} & 0 \\ (a^2-a^{-2}/(2(a^2+a^{-2}))^{\frac{1}{2}} & -(2/(a^2+a^{-2})^{\frac{1}{2}} \end{pmatrix} \in H_2(K).$$

Proof. From (2.3) with $c=s=\frac{1}{2}\sqrt{2}$ we obtain that $\frac{1}{2}\begin{pmatrix} a+a^{-1} & a-a^{-1} \\ a-a^{-1} & a+a^{-1} \end{pmatrix} = M_1$ is in $H_2(K)$. Now $M_2 = \lambda\begin{pmatrix} a+a^{-1} & a-a^{-1} \\ a-a^{-1} & -(a+a^{-1}) \end{pmatrix}$ where $\lambda = \frac{1}{\sqrt{2(a^2+a^{-2}}}$ belong to $\Lambda_2 O_2(K)$ and M_1M_2 belong to $H_2(K)$ and is equal to M. □

By (2.4) it is possible to assume $q\neq 0$ in (2.1) since we have $a^2-a^{-2}\neq 0$ when $a^2\neq 1$.

2.5 *For some* $q\neq 0$ *and for* $c^2+s^2=1$ *the matrix* $\begin{pmatrix} 1 & 0 \\ 2q(q\cdot c\cdot s - c^2+s^2) & 1 \end{pmatrix}$ *is contained in* $H_2(K)$.

Proof. From (2.2) we know that $\begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix}$ belongs to $H_2(K)$ for some $q\neq 0$. Now

$\begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} = \begin{pmatrix} c-q\cdot s & -s \\ s+q\cdot c & c \end{pmatrix} = M_1$ belongs to $H_2(K)$ and $M_2 = p$ $\begin{pmatrix} c-q\cdot s & s \\ -s & c-q\cdot s \end{pmatrix}$ belongs to $\Lambda_2 O_2(K)$ where $p = 1/(1-2q\cdot c\cdot s+q^2\cdot s^2)^{\frac{1}{2}}$. Note that the expression $1-2q\cdot c\cdot s+q^2\cdot s^2$ is positive and has a square root for all admissible pairs c,s since it is the sum of the two squares $(c-q\cdot s)^2$ and s^2 which cannot vanish simultaneously. Thus $M_1M_2 = \begin{pmatrix} p^{-1} & 0 \\ -q(q\cdot c\cdot s-c^2+s^2)p & p \end{pmatrix}$. The inverse of M_1M_2 is equal to $\begin{pmatrix} p & 0 \\ q(q\cdot c\cdot s-c^2+s^2)p & p^{-1} \end{pmatrix}$. From (2.2) we obtain that $\begin{pmatrix} 1 & 0 \\ 2q(q\cdot c\cdot s-c^2+s^2) & 1 \end{pmatrix} \in H_2(K)$. □

2.6 *If K is a Euclidean and Archimedean field it follows from (2.5) that all matrices of the form $\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$ belong to $H_2(K)$.*

Proof. Consider the function $f(c) = q\cdot c\cdot s-c^2+s^2$. Since $\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ x+y & 1 \end{pmatrix}$ it suffices to show that the set of values $y = f(c)$ assumed by the function f contains some interval, e.g. $f(-1) = -1 \leq y \leq 1 = f(0)$. Thus for given y in the interval $-1 \leq 0$ we have to solve the equation $q\cdot c\cdot s-c^2+s^2 = y$ for c and s. Note that $s = \sqrt{1-c^2}$ so that we get the biquadratic equation

$$(4+q^2)c^4-(4+q^2-4y)c^2+1-2y+y^2 = 0.$$

This equation is of the form $Ac^4-Bc^2+C = 0$ where $A,B > 0$, $C \geq 0$. We see at once that a solution exists in K if, and only if, the discriminant B^2-4AC is non-negative. Evaluating the discriminant we find $B^2-4AC = q^4+q^2-4q^2y^2 > 0$ in view of $-1 \leq y \leq 0$. The solution for c^2 is

$$c^2 = \frac{1}{2A}(B \pm \sqrt{B^2-4AC}).$$

We must be sure that $0 \leq c^2 \leq 1$ since otherwise $\sqrt{1-c^2} = s$ would not exist in K. As long as we take the positive root in the formula above there is no problem with the lower bound. So we need only look at the upper bound. Now

$$\frac{B}{2A} = \frac{4+q^2-4y}{2(4+q^2} \leq \frac{1}{2}$$

and

$$\frac{1}{2A}\sqrt{B^2-4AC} = \frac{1}{2(4+q^2)}\sqrt{q^4+4q^2-4q^2y^2} \leq \frac{1}{2}.$$

Hence we may take the positive root in the formula for c^2 and all our requirements to find a solution for c and s are fulfilled. □

From (2.6) it may be concluded that if K is Euclidean and Archimedean then $H_2(K)$ contains all unimodular matrices and hence $H_2(K) = GL_2(K)^*$.

Let us now assume by induction that if $H_{n-1}(K)$ for $n-1 \geq 2$ is a subgroup of $GL_{n-1}(K)^*$ which properly contains $\Lambda_{n-1}O_{n-1}(K)$ then $H_{n-1}(K) = GL_{n-1}(K)^*$. We wish to show that the analogous statement with $n-1$ replaced by n is also true.

Let K^n denote the vector space of all n-tuples in K. Consider a sequence of subspaces $0 = U_0 \subset \ldots \subset U_{n-1} \subset U_n = K^n$ where $dim U_i = i$, $i = 0, 1, \ldots, n$. Such a sequence is usually called a flag. We can choose an orthonormal basis $b_1, b_2, \ldots, b_n$ such that $b_i \in U_i \backslash U_{i-1}$ and this implies that the group $O_n(K)$ acts transitively on flags. Denote by F the special flag $0 \subset U_1 = K(1,0,\ldots,0) \subset U_2 = U_1 + K(0,1,0,\ldots,0) \subset \ldots \subset U_n = U_{n-1} + K(0,\ldots,0,1)$. Its stabilizer $GL_n(K)_F$ within the full matrix group $GL_n(K)$ consists of all triangular matrices

$$T = \begin{pmatrix} a_{11} & 0 & \ldots & 0 \\ a_{21} & a_{22} & \ldots & 0 \\ & & \ddots & \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} \text{ where } \det T = a_{11}a_{22}\ldots a_{mm} \neq 0.$$

As $O_n(K)$ and hence a fortiori $\Lambda_n O_n(K)$ act transitively on flags it follows from the assumption $\Lambda_n O_n(K) \subset H_n(K)$ that there exists a triangular matrix $M \in H_n(K)$ which is not contained in $\Lambda_n O_n(K)$. A triangular matrix is contained in $\Lambda_n O_n(K)$ if, and only if, all elements off the diagonal are zero and the elements in the diagonal differ at most by a factor $\varepsilon = \pm 1$. Hence the matrix M above has non-zero elements off the diagonal or distinct elements on the diagonal which are not equal up to a factor ± 1.

Let us now consider the stabilizer of the subspace $U_{n-1} = \{(x_1, \ldots, x_{n-1}, 0) | x_i \in K\}$. It consists of matrices of the form

$$\left(\begin{array}{ccc|c} a_{1,1} & \ldots & a_{1,n-1} & 0 \\ \vdots & & \vdots & \vdots \\ a_{n-1,1} & \ldots & a_{n-1,n-1} & 0 \\ \hline p_1 & \ldots & p_{n-1} & \alpha \end{array}\right)$$

which we shall symbolically denote by (A, P, α).
Multiplication of two such matrices (A, P, α) and (B, Q, β) follows the rule $(A, P, \alpha)\ (B, Q, \beta) = (AB, PB + \alpha Q, \alpha\beta)$. We wish to prove:

2.7 *For each* $(n-1) \times (n-1)$ *matrix* A *such that* $\det A = \pm\lambda^{n-1} \neq 0$ *the group* $H_n(K)$ *contains at least one matrix of the form* (A, P, α).

Proof. Note first that $H_n(K)$ contains all matrices of the form $(B, 0, \pm\lambda)$ where $B \in \Lambda_{n-1} O_{n-1}(K)$ and $\det B = \pm\lambda^{n-1}$. Therefore, if in the matrix M above a deviation that causes it not to belong to $\Lambda_n O_n(K)$ takes place above the n-th row we can use the induction hypothesis. It follows that $H_n(K)$ contains matrices (A, P, α) with arbitrarily prescribed A such that $\det A = \pm\lambda^{n-1} \neq 0$.

Otherwise the matrix M has the form

$$\left(\begin{array}{cccc|c} a\varepsilon_1 & 0 & \ldots & & 0 \\ 0 & a\varepsilon_2 & \ldots & & 0 \\ & & \ddots & & \\ 0 & & \ldots & a\varepsilon_{n-1} & 0 \\ \hline p_1 & p_2 & \ldots & p_{n-1} & \alpha \end{array}\right).$$

Here $\varepsilon_i = \pm 1$ and at least one of the p_i is not zero or $\alpha \neq \pm a$. We can get rid of any factors $\varepsilon_i = -1$ in the main diagonal by multiplying from the left with the matrix $\begin{pmatrix} \varepsilon_1 & 0 & \dots & & 0 \\ 0 & \varepsilon_2 & & & \\ \vdots & & \ddots & & \vdots \\ 0 & & & \varepsilon_{n-1} & 0 \\ 0 & & & 0 & 1 \end{pmatrix}$.

If all the p_i are zero we conjugate by $\begin{pmatrix} 1 & 0 & \dots & & 0 & 0 \\ 0 & 1 & & & & \\ \vdots & & \ddots & & \vdots & \vdots \\ & & & 1 & & \\ 0 & & \dots & & 0 & 1 \\ 0 & & \dots & & 1 & 0 \end{pmatrix}$ and obtain

$$\begin{pmatrix} a & 0 & \dots & & 0 & 0 \\ 0 & a & & & & \\ \vdots & & \ddots & & \vdots & \vdots \\ 0 & & & a & & \\ 0 & & \dots & & \alpha & 0 \\ 0 & & \dots & & 0 & a \end{pmatrix}.$$

If on the other hand the vector $(p_1, \dots, p_{n-1})$ is not zero we may assume that $M = (aI_{n-1}, P, \alpha)$ and (2.7) follows by the same reasoning from the assertion below.

2.8 *Let* $P = (p_1, p_2, \dots, p_{n-1})$ *and* $d^2 = p_1^2 + p_2^2 + \ldots + p_{n-1}^2$. *Then any matrices of the form* (aI_{n-1}, P, α) *and* $\begin{pmatrix} a & 0 & \dots & & & 0 \\ 0 & a & & & & \\ \vdots & & \ddots & & & \vdots \\ & & & a & & \\ 0 & & \dots & a & 0 & \\ 0 & & \dots & d & \alpha & \\ 0 & & & & & a \end{pmatrix}$ *are conjugate within* $H_n(K)$.

Proof. We choose vectors $(q_{1,1}, \dots, q_{1,n-1})$, …, $(q_{n-1,1}, \dots, q_{n-1,n-1})$ such that $(q_{n-1,1}, \dots, q_{n-1,n-1}) = (p_1, \dots, p_{n-1})$ and $(q_{i,1}, \dots, q_{i,n-1})\ (q_{j,1}, \dots, q_{j,n-1}) = \delta_{ij} d^2$ where $d^2 = p_1^2 + p_2^2 + \ldots + p_{n-1}^2$.

The matrix $Q = \begin{pmatrix} q_{1,1} & \dots & q_{1,n-1} \\ \vdots & & \vdots \\ q_{n-1,1} & \dots & q_{n-1,n-1} \end{pmatrix}$ belongs to $\Lambda_{n-1} O_{n-1}(K)$ and so $(Q, 0, d) \in \Lambda_n O_n(K)$.

Also $(Q, 0, d)^{-1} = (d^{-2} Q^\top, 0, d^{-1}) \in \Lambda_n O_n(K)$ and by an easy computation it follows that $(Q, 0, d)\ (aI_{n-1}, P, \alpha)\ (d^{-2} Q^\top, 0, d^{-1}) = (aI_{n-1}, (0, \dots, 0, d), \alpha)$.

Now conjugationg successively by $\begin{pmatrix} 1 & 0 & \dots & & & 0 \\ 0 & 1 & & & & \\ \vdots & & \ddots & & & \vdots \\ & & & 1 & & \\ 0 & & \dots & 0 & 1 & \\ 0 & & \dots & 1 & 0 & \\ 0 & & \dots & & & 1 \end{pmatrix}$ and by

$$\begin{pmatrix} 1 & 0 & \dots & & & 0 & 0 \\ 0 & 1 & & & & & \\ \vdots & & \ddots & & & & \vdots \\ & & & 1 & & & \\ 0 & & \dots & & 1 & & \\ 0 & & \dots & & & 0 & 1 \\ 0 & & \dots & & & 1 & 0 \end{pmatrix} \text{ yields } \begin{pmatrix} a & 0 & \dots & & & 0 \\ 0 & a & & & & \\ \vdots & & \ddots & & & \vdots \\ & & & a & & \\ 0 & & \dots & & a & 0 \\ 0 & & \dots & & d & \alpha \\ 0 & & & & & & a \end{pmatrix}.$$

□

It is now possible to go a little beyond (2.7). If as above we find a matrix $M \in H_n(K) \setminus \Lambda_n O_n(K)$ that has the shape $M = (M', 0, \mu)$ and M' does not belong to $\Lambda_{n-1}\, O_{n-1}(K)$ then for each $A \in GL_{n-1}(K)^*$ we can find a matrix $(A, 0, \alpha)$ in $H_n(K)$. This follows because we can use the induction hypothesis on the subgroup of $H_n(K)$ formed by its matrices of the form $(B, 0, \beta)$.

We are now going to show that the stronger assertion with $P = 0$ in (2.7) must be true anyway:

2.11 *For each $(n-1) \times (n-1)$ matrix A such that $detA = \pm\lambda^{n-1} \neq 0$ the group $H_n(K)$ contains at least one matrix of the form $(A, 0, \alpha)$.*

Proof. If for some matrix A the group $H_n(K)$ contains two matrices (A, P, α) and (A, P', α) with $P \neq P'$ we may form the product $(A, P, \alpha)\ (A, P', \alpha)^{-1} = (I_{n-1}, Q, q)$. Since Q cannot be zero by (2.8) we are in a situation which implies the assertion.

We may therefore assume that for given A and α there exists at most one matrix (A, P, α) in $H_n(K)$. In other words, the vector $P = P(A, \alpha)$ is a function of A and α. Now multiplying $(A, P(A, \alpha), \alpha)$ from the left by $(I_{n-1}, 0, -1)$ we obtain $(A, -P(A, \alpha), -\alpha)$ hence $P(A, -\alpha) = -P(A, \alpha)$. Multiplying from the right by the same matrix we obtain $(A, P(A, \alpha), -\alpha)$ and hence $P(A, -\alpha) = P(A, \alpha)$. It follows that if P is a function of A and α it must be identically zero. □

A commutator of two arbitrary matrices $(A, 0, \alpha)$ and $(B, 0, \beta)$ takes the form $(ABA^{-1}B^{-1}, 0, 1)$. Hence it follows that $H_n(K)$ contains all matrices of the form $(C, 0, 1)$ where $\det C = 1$. By (2.8) we may conclude that all matrices $(I_{n-1}, P, 1)$ are contained in $H_n(K)$. These are matrices of transvections with respect to the hyperplane U_{n-1} introduced above. Since $H_n(K)$ is transitive on hyperplanes it follows that $H_n(K)$ contains all matrices of transvections. It is well-known that the matrices of transvections generate the subgroup $SL_n(K)$ of matrices of determinant 1 (see e.g. Dieudonné [2], page 37).

Thus $SL_n(K) \subseteq H_n(K)$ and hence $H_n(K) = GL_n(K)^*$. Theorem 1 is proved.

It is now easy to prove the following corollary where $SL_n(K)^{\pm}$ denotes the group of all matrices of determinant ± 1.

Corollary *If K is a Euclidean and Archimedean field then $O_n(K)$ is maximal in $SL_n(K)^{\pm}$.*

Proof. Assume that $O_n(K) < X \leq SL_n(K)^{\pm}$. Then $\Lambda_n O_n(K) < \Lambda_n X \leq GL_n(K)^*$. By theorem 1 it follows that $\Lambda_n X = GL_n(K)^*$ and this implies that $X = SL_n(K)^{\pm}$. □

For odd n it follows easily from the corollary that $O_n^+(K)$ is also maximal in $SL_n(K)$. For even n this is still an open question.

In the proof of theorem 1 there was only one step, namely 2.6, where we actually did require the field K to be Euclidean and Archimedean. Therefore it may be concluded that if K is Pythagorean then theorem 1 remains true for K provided it is true for $n = 2$.

If K is Pythagorean but non-Archimedean, the corollary and hence theorem 1 are not true. For in such a field the set S of all numbers u which have absolute value less than some natural number n form a proper subring which contains all solutions c, s of $c^2 + s^2 = 1$. Therefore the 2×2 matrices with entries in S and determinant ± 1 form a subgroup X such that $O_n(K) < X < SL_n(K)^{\pm}$.

3 Real Number Fields

In this section we give a simple counterexample showing that the results of section 2 do not remain true for fields which are not Pythagorean. Let K denote a subfield of R the field of real numbers. Let A_K denote the set of angles α such that $\cos\alpha$ and $\sin\alpha$ are both in K. It is an immediate consequence of the addition theorems of cos and sin that A_K is a subgroup of all real angles under addition of angles. Therefore the group $O_2^+(K)$ may be parameterized in the usual form

$$O_2^+(K) = \left\{ \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \mid \ \alpha \in A_K \right\}.$$

The only difference to the usual parameterization for the field of real numbers is that the angles now are restricted to the subgroup A_K.

Let $[\cos A_K]$ denote the subring of K generated by all values $\cos\alpha$ where α runs through A_K. Note that since $\sin\alpha = \cos\left(\frac{\pi}{2} - \alpha\right)$ and $\frac{\pi}{2} \in A_K$ the subring $[\cos A_K]$ contains also all values $\sin\alpha$, where $\alpha \in A_K$. It contains 1 since $0 \in A_K$.

Proposition 2 *If $[\cos A_K]$ is a proper subring of K then there exist subgroups properly contained between each of the pairs of groups $O_2^+(K)$ and $SL_2(K)$, $O_2(K)$ and $SL_2(K)^{\pm}$, and $\Lambda_2 O_2(K)$ and $GL_2(K)^*$.*

Proof. Consider the set U of all 2×2 matrices with entries in $[\cos A_K]$ and determinant 1. Then because of the above parametrization of $O_2^+(K)$ we see that U is a subgroup containing $O_2^+(K)$. Moreover U lies properly in between $O_2^+(K)$ and $SL_2(K)$ since a matrix of the form $\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$ is in U if, and only if, $x \in [\cos A_K]$. In a similar way let U_1 be the set of all 2×2 matrices with entries in $[\cos A_K]$ and determinant ± 1. Then U_1 is a subgroup between $O_2(K)$

and $SL_2(K)^{\pm}$. This follows since the elements of $O_2(K)$ which are not in $O_2^+(K)$ can be written as $\begin{pmatrix} -\cos\alpha & \sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$.
Finally it follows that $\Lambda_2 U_1 = U_2$ is a subgroup properly contained between $\Lambda_2 O_2(K)$ and $GL_2(K)^*$.

If K is a Pythagorean field then it is easy to see that every entry a_{ik} of an orthogonal matrix is also an element of the set $\cos A_K$ since it satisfies an equation $a_{i1}^2 + \ldots + a_{ik}^2 + \ldots a_{in}^2 = 1$. Hence for Pythagorean subfields of R the above construction of subgroups between $O_n^+(K)$ and $SL_n(K)$ etc. works for all n provided $[\cos A_K]$ is a proper subring. Whether such Pythagorean fields exist must remain an open question since the following is only a much weaker example.

Proposition 3 $[\cos A_Q]$ *is a proper subring of the field Q of rational numbers.*

Proof. To find all values $\cos\alpha$ where $\alpha \in A_Q$ it is necessary to look at pairs of rational numbers c and s such that $c^2 + s^2 = 1$. Let $c = n/m$ and $s = n_1/m_1$ then we get a Pythagorean triple of integers $x = n \cdot m_1$, $y = n_1 \cdot m$, $z = m \cdot m_1$, and $x^2 + y^2 = z^2$. Conversely each Pythagorean triple of integers gives us a pair of rationals x/z, y/z such that $(x/z)^2 + (y/z)^2 = 1$. We may assume that $(x,y) = 1$ in such a triple since multiplying with a common factor n will not change the pair of rationals obtained in the way above. But the assumption $(x,y) = 1$ implies that x and y do not have the same parity and hence, z is odd (cf. Hardy and Wright [4], page 190).

To determine the subring $[\cos A_K]$ it would be necessary to use the precise knowledge on the set of solutions of $x^2 + y^2 = z^2$ (cf. [4], loc. cit.). Here it suffices to know that z is odd. This means that each possible pair of values $c = x/z$, $s = y/z$ is a pair contained in the set $Odd(Q)$ of rationals that can be written with odd nominator. That is to say, we have proved that $\cos A_Q \subseteq Odd(Q)$. If p, q are in $Odd(Q)$ then so are $p+q$, $p-q$, and pq. It follows that the subring generated by any subset of $Odd(Q)$ is contained in $Odd(Q)$ which is itself a proper subring of Q since it does not contain e.g. $\frac{1}{2}$. Therefore $[\cos A_Q]$ is a proper subring of Q. □

4 Aknowledgements

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