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ON THE GAUSS MAP OF EMBEDDED MINIMAL TUBES[1]

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Abstract. *The Gaussian image of the minimal tubes of arbitrary dimension is studied. If the angle between the flow-vector of such a surface $\mathcal{M}$ and its axe is equal to $\alpha(\mathcal{M}) > 0$ then the diameter of the Gauss image of $\mathcal{M}$ is at least $2\alpha(\mathcal{M})$. As a consequence we show that the length of a two-dimensional minimal tube $\mathcal{M}$ can be estimated by the angle $\alpha(\mathcal{M})$ and the total Gaussian curvature of $\mathcal{M}$.*

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1 Introduction

Let M be a Riemannian orientable manifold of dimension $(n-1)$ with $n \geq 3$ and $x = x(m)$: $M \to \mathbf{R}^n$ be an embedded surface $\mathcal{M}$. Further we identify $\mathcal{M}$ with $x(M)$. The following definition is due [9].

Definition. Let $t(\mathcal{M}) \subset \mathbf{R}^1$ be an open interval. A surface $\mathcal{M}$ is called a *tube* with the axe Ox_n (or the tube in e_n-direction) and the projection interval $t(\mathcal{M})$ if

1) $\forall t \in t(\mathcal{M})$ the sections $\Sigma_t = \mathcal{M} \cap \Pi_t$, $\Pi_t = \{x \in \mathbf{R}^n : x_n = t\}$, are nonempty compact sets lying in the interior of $\mathcal{M}$ (other words, the preimage $x^{-1}(\Sigma_t)$ is a compact subset of M);

2) $\forall t_1, t_2 \in t(\mathcal{M})$ any portion of $\mathcal{M}$ situated between Π_{t_1} and Π_{t_2} is compact.

The length $|t(\mathcal{M})|$ of the projection interval $t(\mathcal{M})$ is called the *life-time* of $\mathcal{M}$.

In the present paper we concern the minimal tubes (in the sense that $\mathcal{M}$ has zero mean curvature). We notice that the previous definition does not impose a priori restrictions on the topological structure of $\mathcal{M}$. Important results on twodimensional minimal tubes have been obtained by J.C.C. Nitsche in [13],[14]. We should mention also the substantial papers of W.H. Meeks and B. White [7],[8] and Y. Fang [3] concerning the minimal tubes with convex sections Σ_t.

The simplest exapmles of minimal tubes in higher dimensions give the rotationally symmetric minimal surfaces (so-called $(n-1)$-dimensional catenoids). In [11] V.M. Miklyukov and A.D. Vedenyapin obtained finiteness of the life-time of *every* minimal $(n-1)$-dimensional tube in $\mathbf{R}^n$ in the case $n \geq 4$.

The situation changes in the twodimensional case. The well-known example of a tube with infinite life-time gives the standard catenoid. On the other hand, there are properly embedded singly periodic minimal surfaces $\mathcal{M}$ constructed by B. Riemann in [15] and their

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generalizations given in [2] which produce by quotient the minimal tube $\mathcal{M}_1 = \mathcal{M}/\mathbf{Z}$ of finite life-time. The last means that $\mathcal{M}_1$ can not be a proper part of any minimal tube.

In the recent paper [17] the second author proposed a new approach to the problem of deciding whether given a minimal tube be of finite or infinite life-time. The main tool is the notion of the flow-vector of a minimal tube $\mathcal{M}$ which is defined below. In the present paper we show (Theorem 2 below) that the length of the Gaussian image of section Σ_t can be described in terms of slope of the flow-vector to the axe of $\mathcal{M}$.

Let t be a regular value of the coordinate function x_n. Then Σ_t splits in finite union of compact $(n-2)$-dimensional connected submanifolds of $\mathcal{M}$. Let ν be the unit exterior normal to Σ_t as a boundary of $\mathcal{M} \cap \{x_n < t\}$. In particularly, we have $\langle \nu, e_n \rangle > 0$ everywhere in Σ_t.

Definition. A union $\Sigma = \bigcup_{i=1}^{k} \Sigma^i$ of components of Σ_t with the induced orientation is called a *cycle*. The linear functional $F(e)$ defined by

$$F_\Sigma(e) = \int_\Sigma \langle e, \nu \rangle : \mathbf{R}^n \to \mathbf{R}^1$$

generates the dual element $J(\Sigma) \in \mathbf{R}^n$ such that

$$F_\Sigma(e) = \langle e, J(\Sigma) \rangle. \tag{1}$$

Then

$$J_n(\Sigma) \equiv \langle J(\Sigma), e_n \rangle = \int_\Sigma \langle e_n, \nu \rangle = \int_\Sigma |e_n^\top| > 0, \tag{2}$$

and it follows that $J(\Sigma) \neq 0$. Here and subsequently we use the notation a^V for the orthogonal projection of $a \in \mathbf{R}^n$ onto a subspace $V \subset \mathbf{R}^n$ and by $T = T_m\mathcal{M}$ we denote the tangent space of the surface $\mathcal{M}$ at $m \in \mathcal{M}$.

Definition. Two oriented cycles Σ' and Σ'' are *equivalent* in $\mathcal{M}$, or $\Sigma' \overset{\mathcal{M}}{\sim} \Sigma''$, in there exists an open subset $D \subset \mathcal{M}$ such that $\partial D = (-\Sigma') \cup \Sigma''$ (here $-\Sigma'$ is the opposite oriented cycle to Σ). This notation is actually the oriential bordism equivalence (see [5, §7]). A connected cycle Σ is called *simple* if it is equivalent to zero cycle in the hyperplane Π_t. Other words, it is a boundary of an open subset of Π_t.

Proposition 1 *We have for* $\Sigma = \bigcup_{i=1}^{k} \Sigma^i$

$$J(\Sigma) = J(\Sigma^1) + \ldots + J(\Sigma^k).$$

Moreover, if $\Sigma' \overset{\mathcal{M}}{\sim} \Sigma''$ *then* $J(\Sigma') = J(\Sigma'')$.

Proof. The first property is direct consequence of the above definitions. To prove the second one we recall that all coordinate functions of minimal immersion are harmonic [6]. Let $D \subset \mathcal{M}$ be an open set in the definition such that $\partial D = (-\Sigma') \cup \Sigma''$. Then for arbitrary coordinate vector $e_k \in \mathbf{R}^n$

$$\langle J(\Sigma''), e_k \rangle - \langle J(\Sigma'), e_k \rangle = \int_{\partial D} \langle e_k, \nu \rangle = \int_{\partial D} \langle \nabla f_k, \nu \rangle = \int_D \Delta f_k = 0,$$

where $\nabla f_k = e_k^\top$ is the gradient of $f_k = \langle e_k, x(m)\rangle$. □

Definition. We call $J(\mathcal{M}) = J(\Sigma_t)$ to be the *flow-vector* of the tube $\mathcal{M}$.

Remark. It follows from Proposition 1 that the flow-vector $J(\Sigma_t)$ does not depend on a choice of $t \in t(\mathcal{M})$. One easy to see also that both the angle $\alpha(\mathcal{M})$ between $J(\mathcal{M})$ and e_n and the norm $\|J(\mathcal{M})\|$ are invariants under the action of the orthogonal subgroup of $\mathbf{R}^n$ preserving the axe Ox_n. Moreover, we emphasize that the flow-vector of $\mathcal{M}$ is a local characteristic of $\mathcal{M}$ in the sense that it can be computed if we consider only a portion of $\mathcal{M}$ situated between Π_{t_1} and Π_{t_2} for t_1 and t_2 arbitrarily close.

Let S^{n-1} be the unit sphere in the Euclidean space $\mathbf{R}^n$ and $d(E)$ be the spherical diameter of a set $E \subset S^{n-1}$. By $\gamma : \mathcal{M} \to S^{n-1}$ we denote the Gaussian map of $\mathcal{M}$, where $\gamma(m)$ is the unit normal at $m \in \mathcal{M}$; by $\gamma(E)$ we denote the Gaussian image of a set $E \subset \mathcal{M}$.

Our main results is the following lower estimate of the diameter of the Gaussian image.

Theorem 2 *Let $\mathcal{M}$ be an embedded minimal tube in $\mathbf{R}^n$; $\Sigma \subset \Sigma_t$ be a simple cycle with the flow-vector $J(\Sigma)$. Then the diameter of $\gamma(\Sigma)$ satisfies*

$$d(\gamma(\Sigma)) \geq 2\alpha(\Sigma), \tag{3}$$

where $\alpha(\Sigma)$ is the angle between $J(\Sigma)$ and e_n.

As a consequence in Section 4 we obtain the upper estimate on the life-time of minimal tubes of finite total Gaussian curvature.

Theorem 3 *Let $\mathcal{M}$ be a twodimensional minimal tube in $\mathbf{R}^3$ of finite total Gaussian curvature $-G(\mathcal{M})$. If $\alpha(\mathcal{M}) > 0$ then $\mathcal{M}$ has finite life-time and*

$$|t(\mathcal{M})| \leq \|J(\mathcal{M})\| G(\mathcal{M}) \frac{\cos\alpha(\mathcal{M})}{16\alpha^2(\mathcal{M})}. \tag{4}$$

Corollary 4 *Let $\mathcal{M}$ be a twodimensional minimal tube in $\mathbf{R}^3$ with univalent Gaussian map. Then $\mathcal{M}$ has finite life-time provided that $\alpha(\mathcal{M}) > 0$.*

Now we indicate the main idea of the proof of Theorem 3. In this case $\dim \Sigma = 1$ and it follows that all one-dimensional cycles are simple (see [5, §7]). Moreover, (3) implies that the Gaussian image of every section Σ_t is uniformally "large" provided the angle between $J(\mathcal{M})$ and e_n is strictly positive. On the other hand, in the two-dimensional case $\dim \mathcal{M} = 2$ the Gaussian map is conformal and (3) yields that $\mathcal{M}$ must be a surface of hyperbolic conformal type. The final step is to use the connection between the conformal module of minimal tube and its life-time value.

We notice that Theorem 3 fails if we drop the requirements of finiteness of the total Gaussian curvature. Really, in the previous paper [18] we have constructed the corresponding examples by using the suitable Weierstrass representation for minimal tubes. Namely, given arbitrary $\alpha(\mathcal{M}) > 0$ there exists a properly embedded minimal tube of infinite life-time.

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2 Preliminary facts

By $\Lambda(\mathbf{R}^n)$ and $\Lambda^k(\mathbf{R}^n)$ we denote the exterior algebra of $\mathbf{R}^n$ and the subspace of all $k-$form respectively. We specify an orthonormal basis $\{e_k\}_{k=1}^n$ in $\mathbf{R}^n$ and by $\omega = e_1 \wedge \ldots \wedge e_n$ we denote the volume-form of $\mathbf{R}^n$. We write $a \simeq b$ if $a = \pm b$.

Given $u \in \Lambda(\mathbf{R}^n)$ we define the inner product $u \dashv \cdot$ on $\Lambda(\mathbf{R}^n)$ by

$$\langle x, u \dashv y\rangle \equiv \langle u \wedge x, y\rangle, \quad \forall x, y \in \Lambda(\mathbf{R}^n). \tag{5}$$

Then the Hodge $*$-operator: $\Lambda^k(\mathbf{R}^n) \to \Lambda^{n-k}(\mathbf{R}^n)$, can be written for every k-form x by

$$*x = x \dashv \omega. \tag{6}$$

The following facts are elementary and can be found in [16].

(i) $**x = (-1)^{k(n-k)}x \simeq x, \quad \forall x \in \Lambda^k(\mathbf{R}^n)$;

(ii) $x \wedge *y = \langle x, y\rangle\omega, \quad \forall x \in \Lambda^k(\mathbf{R}^n), \forall y \in \Lambda^{n-k}(\mathbf{R}^n)$;

(iii) $\langle *x, *y\rangle = \langle x, y\rangle, \quad \forall x, y \in \Lambda^k(\mathbf{R}^n)$.

Let $V \subset \mathbf{R}^n$ be an oriented k-dimensional subspace and $v_1, \ldots, v_k$ be an orthonormal basis of V. By

$$\sigma(V) \equiv v_1 \wedge \ldots \wedge v_k$$

we denote the volume form of V. Further we use the operator

$$\pi_V(\xi) = *(\sigma(V) \wedge \xi).$$

Let $V \subset \mathbf{R}^n$ be an oriented hyperspace, $\dim V = n-1$. If ξ is a unit vector orthogonal to V (i.e. $\xi \in V^\perp$) then it follows from (ii)

$$\xi \simeq *\sigma(V). \tag{7}$$

Lemma 5 *For all $a \in \Lambda^r(\mathbf{R})$, $b \in \Lambda^k(\mathbf{R}^n)$ one holds*

$$a \dashv (*b) = *(b \wedge a).$$

Proof. Let $\xi \in \Lambda^{n-k-r}(\mathbf{R}^n)$ be choosen arbitrary. Then

$$\langle a \dashv (*b), \xi\rangle = \langle *b, a \wedge \xi\rangle = \langle b \dashv \omega, a \wedge \xi\rangle =$$

$$= \langle \omega, b \wedge a \wedge \xi\rangle = \langle (b \wedge a) \dashv \omega, \xi\rangle = \langle *(b \wedge a), \xi\rangle.$$

Then by duality we have $*(b \wedge a) = a \dashv (*b)$ and the lemma is proved. □

Lemma 6 *Let $V \subset \mathbf{R}^n$ be a subspace, $\dim V = k$, and $V^\perp$ be its orthogonal complement. Then for all $x \in \mathbf{R}^n$*

$$x^{V^\perp} = \sigma(V) \dashv (\sigma(V) \wedge x). \tag{8}$$

Proof. We choose $v_1,\dots,v_k$ to be an orthonormal basis of V and consider its complement in $\mathbf{R}^n$: $w_1,\dots,w_{n-k}$. Given arbitrary $x\in\mathbf{R}^n$ we have $x=x_1v_1+\dots+x_kv_k+y_1w_1+\dots+y_{n-k}w_{n-k}$. Then for every v_i

$$\langle\sigma(v)\dashv(\sigma(v)\wedge x),v_i\rangle=\langle\sigma(v)\wedge x,\sigma(v)\wedge v_i\rangle=0.$$

On the other hand,

$$\langle\sigma(v)\dashv(\sigma(v)\wedge x),w_\alpha\rangle=\langle\sigma(v)\wedge x,\sigma(v)\wedge w_\alpha\rangle=$$

$$=\sum_{j=1}^{n-k}\langle\sigma(v)\wedge w_j,\sigma(v)\wedge w_\alpha\rangle=\sum_{j=1}^{n-k}y_j\delta_{j\alpha}=y_\alpha.$$

Hence, by the definition we have the identity

$$\sigma(v)\dashv(\sigma(v)\wedge x)=y_1w_1+\dots+y_{n-k}w_{n-k}=x^{V^\perp},$$

which proves the lemma. □

3 Proof of Theorem 1

In this section by $\Pi=\Pi_0$ we denote a hyperspace $x_n=0$ in $\mathbf{R}^n$ and by $T\equiv T_m\Sigma$ — the tangent space to the section Σ being considered as submanifold of Π. Let $\gamma=\gamma(m)$ be the unit normal to $\mathcal{M}$ at m. We specify an orthonormal basis $\tau_1,\dots,\tau_{n-2}$ in T and by $\tau\equiv\sigma(T)$ denote the volume form of $T_m\Sigma$.

We need the following auxilliary assertion

Lemma 7 *Let ξ and η be two unit vectors such that $\xi,\eta,\tau_1,\dots,\tau_{n-2}$ form oriented orthonormal basis of $\mathbf{R}^n$. Then for every $q\in\mathbf{R}^n$*

$$\langle q,\xi\rangle\simeq\langle\eta,\pi_T(q)\rangle.$$

Proof. We have from (7) that

$$\xi\simeq*(\eta\wedge\tau),$$

and by virtue of (iii) and Lemma 5 we obtain

$$\langle q,\xi\rangle\simeq\langle q,*(\eta\wedge\tau)\rangle\simeq\langle q,*(\tau\wedge\eta)\rangle\simeq\langle q,\eta\dashv*\tau\rangle\simeq$$

$$\simeq\langle\eta\wedge q,*\tau\rangle\simeq\langle\eta,q\dashv*\tau\rangle\simeq\langle\eta,*(\tau\wedge q)\rangle\simeq\langle\xi,\pi_T(q)\rangle$$

and the lemma is proved. □

By the Sard's theorem and regularity of t we conclude that Σ is a smooth submanifold of Π_t. Assume that $\eta=\eta(m)$ is the unit normal vector field to Σ in Π_t oriented such that the pair $(T;\eta)$ is an oriented basis of Π. Then by Lemma 6 we have for every $q\in\mathbf{R}^n$

$$\int_\Sigma\langle\pi_T(q),e_n\rangle=\int_\Sigma\langle*(\tau\wedge q),e_n\rangle\simeq$$

$$\simeq \int_{\Sigma} \langle \tau \wedge q, \tau \wedge \eta \rangle \simeq \int_{\Sigma} \langle q, \tau \dashv (\tau \wedge \eta) \rangle \simeq \int_{\Sigma} \langle q, \eta \rangle \tag{9}$$

To show that in fact the last integral vanishes we observe that by the definition, the simple cycle Σ is the boundary of some open subset $\Omega \subset \Pi_t$. The by the Stokes' formula we obtain

$$\int_{\partial\Omega} \langle q, \eta \rangle = \int_{\Omega} \operatorname{div} q = 0. \tag{10}$$

Thus (9) yields the following identity

$$\int_{\Sigma} \langle \pi_T(q), e_n \rangle = 0. \tag{11}$$

Choose $q \neq 0$ arbitrarily such that the equality $\langle q, J(\Sigma) \rangle = 0$ holds. Then taking into account the mutual orthogonality of γ, ν and the tangent space $T_m\Sigma$, we obtain from (1) and Lemma 7

$$0 = \int_{\Sigma} \langle q, \nu \rangle = \int_{\Sigma} \langle \pi_T(q), \gamma \rangle.$$

Hence, we conclude from (11) that

$$\int_{\Sigma} \langle \pi_T(q), \gamma \pm e_n \rangle = 0. \tag{12}$$

By virtue of regularity of t, the expressions $\gamma \pm e_n$ does not vanish everywhere in Σ. It follows that along Σ the vector fields

$$v \pm (m) = \frac{e_n \pm \gamma(m)}{||e_n \pm \gamma(m)||}$$

are well-defined.

Using the mean value theorem we deduce from (12) that there exist two points m_- and m_+ in Σ such that

$$\langle \pi_{T_\pm}(q), v_\pm \rangle = 0, \tag{13}$$

where $T_\pm = T_{m_\pm}\mathcal{M}$ and $v_\pm = v_\pm(m_\pm)$.

Noe we observe that the set $(v_-, v_+, \tau_1, \ldots, \tau_{n-2})$ forms an orthonormal basis $\mathbf{R}^n$. Thus, applying Lemma 7 to (13) we obtain

$$0 = \langle \pi_{T_+}(q), v_+ \rangle \simeq \langle v_-, q \rangle$$

at m_- and similarily at m_+:

$$0 = \langle \pi_{T_-}(q), v_- \rangle \simeq \langle v_+, q \rangle.$$

By getting rid of the denominator in the definition of the vectors $v_\pm$ we arrive at

$$\langle q, e_n \pm \gamma(m_\pm) \rangle = 0.$$

It follows that

$$\langle \gamma(m_+) - \gamma(m_-), q \rangle = 2q_n,$$

where $q_n = \langle q, e_n \rangle$. Finally, applying the Cauchy's integral inequality in the last identity yields

$$||\gamma(m_+) - \gamma(m_-)|| \geq \frac{2q_n}{||q||}. \tag{14}$$

Taking into account that $\gamma(m_\pm)$ are points on the unit sphere lying in the Gaussian image of Σ, we obtain $||\gamma(m_+) - \gamma(m_-)|| = 2\sin\frac{\beta}{2}$, where β is the angle between $\gamma(m_+)$ and $\gamma(m_-)$. It follows by the definition of the spherical diameter of $\gamma(\Sigma) \subset S^{n-1}$ that $d(\gamma(\Sigma)) \geq \beta$. Thus, by virtue of (14) we conclude that

$$d(\gamma(\Sigma)) \geq \arcsin\frac{q_n}{||q||}.$$

To find the maximum of the right part of the last expression we assume $\alpha(\Sigma)$ to be equal the angle between $J(\Sigma)$ and e_n. Then, by orthogonality of q to the flow-vector $J(\Sigma)$ one easily sees that

$$\max_{q \perp J(\Sigma)} \frac{q_n}{||q||} = \sin\alpha(\Sigma).$$

Hence,

$$d(\gamma(\Sigma)) \geq 2\alpha(\Sigma),$$

and Theorem 2 is proved completely.

Remark. We notice that in the twodimensional case the assertion of Theorem 2 is still true provided $\mathcal{M}$ is properly immersed minimal tube. To check this fact we observe that the unique place in the proof of the theorem where we essentially needed the embeddeness hypothesis is formula (10). The valideness of this formula in the twodimensional immersed case is a direct consequence of Green integration formula.

4 Applications to twodimensional minimal tubes

To prove Theorem 3 we need some terminology from potential theory.

Let us consider an embedded minimal hypersurface $\mathcal{M}$ in $\mathbf{R}^n$ which is a tube in e_n-direction. Given t_1, t_2 from the interval $t(\mathcal{M})$ we notice by $\mathcal{M}(t_1;t_2)$ the portion of $\mathcal{M}$ situated in the slab $t_1 < x_n < t_2$. Then the quantity

$$\operatorname{cap} \mathcal{M}(t_1;t_2) = \inf \int\int_{\mathcal{M}(t_1;t_2)} |\nabla\varphi|^2,$$

where the infimum is taken over all Lipschitzian functions $\varphi(m)$ on $\mathcal{M}(t_1;t_2)$ such that $\varphi(m) = 0$ on Σ_{t_1} and $\varphi(m) = 1$ on Σ_{t_2} is called the *capacity* of $\mathcal{M}(t_1;t_2)$.

Let Γ be a family of locally rectifiable curves $\gamma \subset \mathcal{M}$ and $\rho(m) \geq 0$ be a Baire function with the property

$$\int_\gamma \rho(z)ds \geq 1,$$

for every $\gamma \in \Gamma$. The infimum

$$\operatorname{mod}\Gamma = \inf \int\int_{\mathcal{M}(t_1;t_2)} \rho^2(m)$$

over all such $\rho(m)$ is called the *module* of the family Γ.

If $\dim \mathcal{M} = 2$ the following connection between the capacity of $\mathcal{M}(t_1;t_2)$ and the module of the family $\Gamma(t_1;t_2)$ of all curves which connect two boundary components of $\mathcal{M}(t_1;t_2)$ is well-known

$$\mathrm{mod}\,\Gamma(t_1;t_2) = \cap\mathcal{M}(t_1;t_2)$$

(see for the Euclidean case [4] and for the Riemannian case [10] respectively).

In his paper [9] V.M. Miklyukov has studied the higherdimensional minimal tubes in $\mathbf{R}^n$ and has established the following connection between the capacity of $\mathcal{M}(t_1;t_2)$ and its life-time

$$\mathrm{cap}\mathcal{M}(t_1;t_2) = \frac{t_2 - t_1}{\langle J(\mathcal{M}, e_3\rangle}. \tag{15}$$

STEP 1. Let us first assume that $\mathcal{M}$ be a twodimensional embedded minimal tube to be homeomorphic to an annulus. Then from (15) we obtain

$$\mathrm{mod}\,\Gamma(t_1;t_2) = \frac{t_2 - t_1}{\langle J(\mathcal{M}), e_3\rangle}. \tag{16}$$

By virtue of conformality of the Gaussian map of a minimal surface, we have for every tangent vector $X \in T_m M$

$$||d\gamma_m(X)|| = \lambda(m)||X||,$$

and the Gaussian curvature is $K(m) = -\lambda^2(m)$.

Given an arbitrary $t \in t(\mathcal{M})$ we write by change coordinates formula

$$\int_{\Sigma_t} \lambda ds \geq \int_{\gamma(\Sigma_t)} ds_1 \geq 2d(\gamma(\Sigma_t)),$$

where ds_1 is the metric element on the unit sphere. By virtue of Theorem 2 we obtain

$$\int_{\Sigma_t} \lambda ds \geq 4\alpha(\mathcal{M}), \tag{17}$$

where $\alpha(\mathcal{M})$ is the angle between the flow-vector $J(\mathcal{M})$ and e_3.

Substituting $\rho(m) = \lambda(m)$ in the definition of the module, we get from (17)

$$\mathrm{mod}\,\Gamma(t_1;t_2) \leq \frac{1}{16\alpha^2(\mathcal{M})} \int\int_{\mathcal{M}(t_1;t_2)} (-K),$$

and by (16) we arrive at

$$t_2 - t_1 \leq \frac{J_3(\mathcal{M})G(\mathcal{M}(t_1;t_2))}{16\alpha^2(\mathcal{M})}.$$

Then the required estimate in the annulus case yields from the arbitrariness of t_1 and t_2.

To prove the general case we need the following elementary fact

Lemma 8 *Let $v_1, \dots v_l$ be a system of nonzero vectors from $\mathbf{R}^n$ such that for some $e \in \mathbf{R}^n$ one holds $\alpha_i \leq \pi/2$, where α_i is the angle between v_i and e. Let $v = \sum_{i=1}^{l} v_i$. Then we have for the angle α between v and e:*

$$\alpha \leq \max\{\alpha_1, \dots, \alpha_l\}.$$

Proof. By virtue of $\langle v_i, e\rangle \geq 0$ we notice that $a_i \leq \pi/2$ and by the triangle inequality obtain

$$\cos\alpha = \frac{\langle v, e\rangle}{||v||} \geq \frac{\langle v, e\rangle}{\sum_{i=1}^{l}||v_i||} = \frac{\sum_{i=1}^{l}\langle v_i, e\rangle}{\sum_{i=1}^{l}||v_i||} =$$

$$= \frac{\sum_{i=1}^{l}||v_i||\cos\alpha_i}{\sum_{i=1}^{l}||v_i||} \geq \min_{1\leq j\leq l}\cos\alpha_i = \cos(\max_{1\leq j\leq l}\alpha_j),$$

as required.

STEP 2. Let $\mathcal{M}$ be a properly embedded minimal tube in $\mathbf{R}^3$ of general topological structure. First we notice that for arbitrary closed subinterval $[\alpha;\beta] \subset t(\mathcal{M})$ there exist at most finitely many points $m \in \mathcal{M}\ (\alpha;\beta)$ such that $\gamma(m) = \pm e_3$. Really, the coordinate function $f_3(m) = \langle e_3, x(m)\rangle$ is harmonic on $\mathcal{M}$ and it follows that the critical set $H = \{m \in \mathcal{M} : \nabla f_3 \equiv e_3^{\top} = 0\}$ has no accumulation points inside of $\mathcal{M}$. Other words, for any compact part $\mathcal{M}$ $(\alpha;\beta)$ the set H is finite.

Let $c_1 < c_2 < \ldots c_{k-1}$ are all the values of $f_3(m)$ when m runs H and $\alpha = c_0$, $\beta = c_k$. Then by the Morse theory, every part of $\mathcal{M}_i = \mathcal{M}\ (c_{i-1}, c_i)$, $i = 1, \ldots, k$, is a union of annuli $\mathcal{M}_i^1, \ldots, \mathcal{M}_i^{l_i}$. From positivity of the third coordinate of the flow vector and Proposition 1 we have $J_3(\mathcal{M}_i^j) \leq J_3(\mathcal{M})$. Applying Step 1 we obtain

$$c_i - c_{i-1} \leq \frac{J_3(\mathcal{M})G(\mathcal{M}_i^j)}{16\alpha^2(\mathcal{M}_i^j)}, \qquad 1 \leq j \leq \ell_i. \tag{18}$$

On the other hand, by virtue of Proposition 1

$$J(\mathcal{M}) = J(\mathcal{M}_i) = J(\mathcal{M}_i^1) + \ldots + J(\mathcal{M}_i^{\ell_i}).$$

Let the index j_0 corresponds to the maximum angle $\alpha(\mathcal{M}_i^j)$ when i is fixed. Then applying Lemma 8 to $e = e_3$ and $v_j = J(\mathcal{M}_i^j)$ we obtain

$$\alpha(\mathcal{M}_i^{j_0}) \geq \alpha(\mathcal{M}_i) = \alpha(\mathcal{M}).$$

Hence, by (18) and positiveness of the absolute total Gaussian curvature G

$$c_i - c_{i-1} \leq \frac{J_3(\mathcal{M})G(\mathcal{M}_i^{j_0})}{16\alpha^2(\mathcal{M})} \leq \frac{J_3(\mathcal{M})G(\mathcal{M}_i)}{16\alpha^2(\mathcal{M})}.$$

Summing of the last inequalities over all $i = 1, \ldots, k$ we arrive at

$$\beta - \alpha \leq \frac{J_3(\mathcal{M})G(\mathcal{M}(c_0;c_k))}{16\alpha^2(\mathcal{M})} \leq \frac{J_3(\mathcal{M})G(\mathcal{M})}{16\alpha^2(\mathcal{M})} = ||J(\mathcal{M})||G(\mathcal{M})\frac{\cos\alpha(\mathcal{M})}{16\alpha^2(\mathcal{M})}.$$

By arbitrariness of subinterval $[\alpha;\beta] \subset t(\mathcal{M})$ we obtain the assertion of Theorem 3.

References

[1] L. Ahlfors, *Lectures on quasiconformal mappings*, New York, Van Nostrand Math. Studies, 1966.

[2] M. Callahan, D. Hoffman and W.H. Meeks, "Embedded minimal surfaces with an infinite number of ends", *Invent. math.* **96** (1989), 459-505.

[3] Y. Fang, "Minimal annuli in $\mathbf{R}^3$ bounded by non-compact complete convex curves in parallel planes", *J. Austral. Math. Soc. (Series A)* **60** (1996), 369-388.

[4] B. Fuglede, "Extremal length and functional completetion", *Acta Math.*, **98** (1957), N 3-4, 171-219.

[5] M.W. Hirsch, *Differential topology*, Springer-Verlag, New York Heidelberg Berlin, 1976.

[6] Sh. Kobayashi and K. Nomizu, *Foundations of differential geometry*. Vol. 2, Interscience, 1969.

[7] W.H. Meeks and B. White, "The Minimal surfaces bounded by convex curves in parallel planes", *Comment. Math. Helv.*. **66** (1991) 263-278.

[8] W.H Meeks, B. White, "The space of minimal annuli bounded by an extremal pair of planar curves", *Comment. Anal. and Geom.* **1** (1993), No 3, 415-437.

[9] V.M. Miklyukov, "On some properties of tubular minimal surfaces in $\mathbf{R}^n$", *Soviet Math. Dokl.* **247** (1997), N 3, 549-552.

[10] V.M. Miklyukov, "Some parabolicity and hyperbolicity criteria for boundariy sets of surfaces", *Izv. Ross. Acad. Nauk Ser. Mat.* **60** (1996), *no.* 4, 111-158.

[11] V.M. Miklyukov and V.D. Vedenyapin, "Extrinsic dimension of tubular minimal hypersurfaces", *Math. USSR-Sb.*, **131** (1986), 240-250.

[12] V.M. Miklyukov and V.G. Tkachev, "Denjoy-Ahlfors Theorem for Harmonic Functions on Riemannian Manifolds and External Structure of Minimal Surfaces", *Commun. in Anal. and Geometry*, **4** (1996), N4, 547-587.

[13] J.C.C. Nitsche, "A characterization of the catenoid", *J. Mathem. Mech.*, **11** (1962), 293-302.

[14] J.C.C. Nitsche, *Lectures on minimal surfaces. Vol. 1*, Cambridge Univ. Press, Cambridge New York Melbourne Sydney, 1989.

[15] B. Riemann, *Oeuvres Mathematiques de Riemann*, Paris, Gautheriers-Villars, 1898.

[16] S. Sternberg, *Lectures on differential geometry*, Prentice Hall, Inc. Englewood Cliffs, New York, 1964.

[17] V.G. Tkachev, "Minimal tubes and coefficients of holomorphic functions in annulus", *Bull. de la Soc. Sci. de Lódź., Recherches sur deform.*, Paris, 1995, V. XX, 19-26.

[18] V.G. Tkachev, "Minimal tubes of finite total curvature", *Siberian Math. J.*, **39** (1998), N 1, 181-190.

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