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Elementary axiomatizations of projective space and of its associated Grassmann space

Victor Pambuccian
Department of Integrative Studies, Arizona State University,
West Phoenix AZ 85069-7100, U.S.A.
pamb@math.west.asu.edu

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Abstract. It is pointed out that the axiomatizations of projective geometry by means of point-line incidence, or by means of line-intersection, and that of the Grassmann space of the lines in a projective space are mutually translatable, so that one can obtain an axiomatization of any of these theories from the other, which in particular allows us to elementarily axiomatize the Grassmann space representing the lines of a projective space, a subject which has been extensively studied since 1981, when Tallini provided a non-elementary characterization for it.

Keywords: projective geometry, Grassmann space, first-order axiomatization, mutual interpretability.

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1 Introduction

To axiomatize a geometry one needs a language in which to write the axioms, and a logic by means of which to deduce consequences from those axioms. Based on the work of Skolem, Hilbert and Ackermann, Gödel, and Tarski, a consensus had been reached by the end of the first half of the 20th century that, as Skolem had emphasized since 1923, "*if* we are interested in producing an axiomatic system, we can *only* use first-order logic" ([3, p. 472]).

This had not been the case in 1899, as evidenced by Hilbert's axiomatization of Euclidean geometry in [6], in which the underlying logic is left unspecified, but where the strength of the theory axiomatized, Euclidean geometry over the field of real numbers, renders the use of a higher-order logic unavoidable. For, by the Löwenheim-Skolem theorem, no axiom system in first-order logic can admit a *unique* infinite model.

The language of first-order logic (also referred to as *predicate logic*) consists of the logical symbols $\wedge$ (and), $\vee$ (or), $\rightarrow$ (implies), $\neg$ (not), $\leftrightarrow$ (if and only if), a denumerable list of symbols called *individual variables*, as well as denumerable lists of n-ary *predicate (relation)* and *function (operation)* symbols for all natural numbers n, as well as *individual constants* (which may be thought of as 0-ary function symbols), together with two quantifiers, $\forall$ and $\exists$ which can bind *only* individual variables, but *not* sets of individual variables nor predicate or function symbols. Its axioms and rules of deduction are those of classical logic. A *theory* $\mathcal{T}$ is a set of sentences closed under deduction, i. e. containing with any set of sentences their consequences.

First-order logic with several *sorts* of individual variables, is called *multi-sorted.* In this paper we shall deal with one-sorted and two-sorted languages only. At the cost of cluttering the sentences of a two-sorted theory, we can express them in a one-sorted language by adding two new unary predicates P_1 and P_2 to its language, and by changing any sentence of the form $(\exists X^i)\,\varphi(X^i)$ into $(\exists x)\,P_i(x) \wedge \varphi(x)$ and $(\forall X^i)\,\varphi(X^i)$ into $(\forall x)\,P_i(x) \to \varphi(x)$ (here the X^i stand for variables of sort i. For more on multi-sorted logic see [11]. We shall refer to an axiomatization as *non-elementary* if it is not expressed in first-order logic.

Dimension-free projective and affine geometry (of unspecified dimension ≥ 3) were first axiomatized by H. Lenz [9], the former in a first-order (elementary) language with two sorts of individuals, standing for *points* and *lines*, and a binary relation I of incidence whose first argument is a point- and whose second argument is a line-variable, the latter in a language containing, beside the notions listed above, a binary relation $\|$ of parallelism between lines.

Three-dimensional projective geometry had been axiomatized much earlier in a first-order language with lines as the only individual variables and the binary relation of line-intersection as single non-logical notion first by M. Pieri [13], next by E. R. Hedrick and L. Ingold [5], who simplified Pieri's system, then, more than fifty years later by S. Trott [18], to be followed twenty years later by E. Koźniewski [7], both of whom were apparently unaware of their predecessors, and finally by H.-J. Stoß [15, Chap. 7]. That this is possible for all finite dimensions ≥ 3 is implicit in W.-L. Chow [2]. An axiomatization for dimension-free projective geometry for dimensions ≥ 4 (i. e. there is no upper-dimension axiom, so any projective space of dimension ≥ 4 is a model of that axiom system) has been provided by E. Koźniewski [8]. We shall show that such a line-intersection based axiomatization is possible for dimension-free projective geometry by means of a simple translation procedure.

In 1981 G. Tallini [17] gave a non-elementary characterization of the Grassmann space representing the lines of a projective space, i. e. the incidence structure consisting of lines and pencils of lines passing through a given point (all lines in a given plane passing through a given point), the incidence being the inclusion of a line in a pencil. The characterization is non-elementary as it uses families of maximal subspaces and cannot be reformulated without the use of set-theoretical (higher-order) notions. It was followed by another non-elementary characterization of the same structure by N. Melone and D. Olanda [10] and more recently by a first elementary axiomatization by E. Ferrara Dentice and N. Melone [4]. A non-elementary axiomatization of the Grassmann space representing the lines of an affine space had been provided right after Tallini's original paper by A. Bichara and F. Mazzocca [1].

The purpose of this note is to show that all of these theories are equivalent, in a sense we shall make precise, and that any axiom system for one of them, say the point-line projective geometry, can be translated in a straightforward — if somewhat cumbersome — way into an axiom system for another one, say for the Grassmann space representing the lines of a projective space.

2 Equivalence of theories with variables to be interpreted differently

When we say that two theories $\mathcal{T}_1$ and $\mathcal{T}_2$, in languages L_1 and L_2, which we shall, for simplicity's sake, assume to be at most two-sorted (they each have at most two sorts of individual variables, which we may differentiate with superscripts, such as X^1 and X^2, X^1 denoting a variable of the first sort (say a *point*), and X^2 a variable of the second sort (say a *line*) with possibly different intended interpretations in the two theories, e. g. *points and lines* and *lines and pencils*), axiomatize the 'same geometry', what we mean is the following:
There are natural numbers k_i^j for $i, j = 1, 2$ such that (addition in the indices is mod 2):
(i) with $k = k_i^j$, one can identify the individuals X^j of L_i with any k-tuple $(x_1, \dots, x_k)$ of individuals from L_{i+1} which satisfies a certain formula with k free variables $\varphi_i^j(x_1, \dots, x_k)$ of L_{i+1};
(ii) there is a definition for the equivalence of two k-tuples, in terms of an L_{i+1} formula ψ_{i+1} with $2k$ free variables, such that $(x_1, \dots, x_k) \equiv (y_1, \dots, y_k)$ if and only if $\psi_{i-1}(x_1, \dots, x_k, y_1, \dots y_k)$ holds;
(iii) for every n-ary relation symbol π of L_i, there is an L_{i+1}-formula δ_π with kn free variables, such that $\pi(X_1^{j_1}, \dots, X_n^{j_n})$ holds if and only if $\delta_\pi(x_{1,1}, \dots x_{1,k}, \dots, x_{n,1}, \dots, x_{n,k})$ holds, where $(x_{i,1}, \dots, x_{i,k})$ is a k-tuple associated via (i) to $X_i^{j_i}$;
(iv) For every formula ϑ, if $\mathcal{T}_i \vdash \vartheta$ then $\mathcal{T}_{i+1} \vdash \overline{\vartheta}$, where $\overline{\vartheta}$ is the L_{i+1}-formula obtained from ϑ by replacing all of its individual variables with k-tuples satisfying φ_i^j, all equality symbols with the $\equiv$-relation, and all occurring relation symbols with the L_{i+1}-formulas that correspond to them by (iii).

The definition provided here for the equivalence of theories is similar to those given by Previale [14] and Szczerba [16]. The notion of equivalence of theories is also called *mutual interpretability* in [16]. It should be noted however that there is no purely logical definition of a *faithful mutual interpretability*, which would be the fruit of a translation that not only can translate to and fro in an automatic way and turn true statements from one language into true statements in the other language, but would also preserve the *intention* of the statements. For example, as emphasized in [16], Euclidean and hyperbolic geometry are equivalent or mutually interpretable under our definition.

To make this paper self-contained we shall first present the axiom systems of Lenz [9] for dimension-free projective spaces, the axiom systems from [5] and [18] for the three-dimensional projective geometry of line-intersection and make some observations regarding the complexity of their axioms, as well as the axiom system for [4, Th. 3] both in the language of line-pencil incidence and its translation in the language of line-intersection, by which we obtain our first axiom system expressed in terms of line-intersection for projective spaces for dimensions ≥ 4. We then translate Lenz's axiom system both into an axiom system for line-intersection projective geometry and line-pencil incidence Grassmann space, thus obtaining an elementary axiom system for Grassmann spaces associated with projective spaces of dimension ≥ 4. By translating the line-intersection axiom system for 3-dimensional projective geometry from [5] into the language of line-pencil incidence, we eventually obtain an alternate axiom system for the theory axiomatized in [4, Th. 3].

3 Lenz's axiom system for higher-dimensional projective geometry

Lenz's axiom system is expressed in the two sorted language L_{I}, with variables for *points* and *lines* to be denoted by upper- and lowercase letters, and a binary relation I between points and lines, with $A\,\mathrm{I}\,l$ to be read as 'A is incident with l'. We shall use the following convenient abbreviations: $(A_1, \ldots, A_n\,\mathrm{I}\,l)$ for $A_1\,\mathrm{I}\,l \wedge \ldots \wedge A_n\,\mathrm{I}\,l$, $A\,\mathrm{I}\,l_1, \ldots, l_n$ for $A\,\mathrm{I}\,l_1 \wedge \ldots \wedge A\,\mathrm{I}\,l_n$, and $\neq (A_1 \ldots A_n)$ for $\bigwedge_{i\neq j} A_i \neq A_j$.[1] Its axioms are:

1 L. $(\forall AB)(\exists l)(\forall l')\, A \neq B \rightarrow (A, B\,\mathrm{I}\,l) \wedge [(A, B\,\mathrm{I}\,l' \rightarrow l' = l)]$

2 L. $(\forall ABCDElmnp)(\exists P) \neq (ABCD) \wedge (A, B, E\,\mathrm{I}\,l) \wedge (C, D, E\,\mathrm{I}\,m)$
$\wedge (A, C\,\mathrm{I}\,n) \wedge (B, D\,\mathrm{I}\,p) \rightarrow (P\,\mathrm{I}\,n,p)$

3 L. $(\forall l)(\exists ABC) \neq (ABC) \wedge (A, B, C\,\mathrm{I}\,l)$

4 L. $(\exists lm)(\forall P)\, \neg(P\,\mathrm{I}\,l, m)$

Here L1 states that there is a unique line incident with two distinct points, L2 is Veblen's axiom (cf. also [12]), L3 states that on every line there are three

[1] Identical abbreviations will be used in $\mathsf{L}_{\in}$ without further mention.

points, and L4 that there are two skew lines. We shall refer to the theory axiomatized by this axiom system as $\mathcal{L}$.

4 Axiom systems for three-dimensional projective geometry

The axiom systems in [5] and [18] for three-dimensional projective geometry can be expressed in a one-sorted language $\mathsf{L}_\sim$, with individual variables to be interpreted as *lines*, with a single binary relation $\sim$, with $a \sim b$ to be read as 'line a intersects line b' (a and b being different lines; in [18] equality of 'intersecting' lines is allowed, but we reformulate that axiom system here in terms of this strict intersection predicate).

We use the abbreviations $(a_1, \ldots, a_n \sim b_1, \ldots, b_m)$ for $\bigwedge_{1\leq i\leq n, 1\leq j\leq m} a_i \sim b_j$, and $a \simeq b$ for $a \sim b \vee a = b$ (so that $a \not\simeq b$ stands for $a \not\sim b \wedge a \neq b$).

The axiom system in [5] consists of the following 7 axioms (we omit the universal quantifiers in universal axioms throughout this paper):

1 HI. $a \not\sim a$

2 HI. $a \sim b \rightarrow b \sim a$

3 HI. $(\exists ab)\, a \sim b$

4 HI. $(\forall ab)(\exists cd)\, a \sim b \rightarrow (c, d \sim a, b) \wedge c \not\simeq d$

5 HI. $a \sim b \wedge (l, l', m, n \sim a, b) \wedge l \not\simeq l' \wedge m \not\sim n \wedge \neq (lmn)$
$\rightarrow ((l \sim m \wedge l \not\sim n) \vee (l \sim n \wedge l \not\sim m))$

6 HI. $(\forall ab)(\exists c)\, a \sim b \rightarrow a \not\sim c \wedge b \not\sim c$

7 HI. $(\forall abc)(\exists l)(\forall m)\, a \sim b \rightarrow ((l \sim a, b) \vee l = a \vee l = b) \wedge l \sim c$
$\wedge ((m \sim a, b) \rightarrow l \simeq m)$

The one in [18] consists of HI1 and HI2, as well as of the following 6 axioms (the addition in the indices is (here and throughout the paper), whenever the sum exceeds the upper bound of the index-range, mod 3)

1 T. $(\exists ab)\, a \neq b$

2 T. $(\forall ab)(\exists mn)\, m \not\simeq n \wedge (m, n \sim a, b)$

3 T. $(\forall abcc')(\exists d)\, c \not\simeq c' \wedge a \sim b \wedge (a, b \sim c, c') \rightarrow (d \sim a, b, c, c')$

4 T. $(a_1 \sim a_2) \wedge \bigwedge_{1\leq i\leq 2, 1\leq j\leq 3} (a_i \sim b_j) \rightarrow \bigvee_{j=1}^{3} b_j \simeq b_{j+1}$

5 T. $(\forall abcd)(\exists ef)\, (b \sim a, c) \wedge c \sim d \wedge a \not\simeq c \wedge b \not\simeq d$
$\rightarrow (e, f \sim a, b, c, d) \wedge e \not\simeq f$

6 T. $(b \sim a, c) \wedge c \sim d \wedge a \not\simeq c \wedge b \not\simeq d \wedge \bigwedge_{i=1}^{3} (e_i \sim a, b, c, d)$
$\rightarrow \bigvee_{i=1}^{3} e_i = e_{i+1}$

The first axiom system has the remarkable property of having each axiom formulated as a prenex statement by means of no more than 6 variables (although we don't know whether this is the minimum number required to axiomatize three dimensional projective geometry in $\mathsf{L}_\sim$), whereas in the second axiom system each axiom is a $\forall\exists$-statement, i. e. all universal quantifiers (if any) precede all existential quantifiers (if any). Its last axiom requires 7 variables, so in this sense it is less simple than the first one. The axiom systems in [7] and [15] were not presented here as their axioms can neither be all written as $\forall\exists$-statements, nor all be expressed by means of at most 6 variables.

The $\mathsf{L}_\sim$-theory of dimension-free projective spaces will be denoted by $\mathcal{P}$.

5 An axiomatization of the Grassmann space

The axiom system in [4, Th. 3] for the Grassmann space representing the lines of a projective space of an arbitrary dimension ≥ 3 can be expressed in the two-sorted language $\mathsf{L}_\in$, with individuals for *lines* and *pencils*, denoted by lowercase Latin and lowercase Greek letters, and a binary relation $\in$ of line-pencil incidence (pencils may be thought of as collections of lines in a fixed plane passing through a fixed point, and a line is said to be incident with a pencil, if it belongs to it). When expressed in this language without the use of the set-theoretical definitions which make the axiom system look so simple in [4], we end up with the following rather complex-looking axiom system:

1 FDM. $(p, q \in \lambda, \lambda') \to \lambda = \lambda' \vee p = q$

2 FDM. $(\forall\lambda)(\exists pq)\, p \neq q \wedge (p, q \in \lambda)$

3 FDM. $(\forall p)(\exists\lambda)\, (p \in \lambda)$

4 FDM. $(\forall\lambda)(\exists pq)(\forall\mu a)(\exists\alpha\beta)\, \neg(p, q \in \mu) \wedge (a \in \lambda \to (p, a \in \alpha) \wedge (q, a \in \beta))$

5 FDM. $(\forall\lambda p_1 p_2 p_3)(\exists a\pi)(\forall\alpha_1\alpha_2\alpha_3)\, (a \in \lambda \wedge (\bigvee_{i=1}^3 \neg(p_i, a \in \alpha_i)))$
$\vee \bigvee_{i=1}^3 (p_i, p_{i+1} \in \pi)$

6 FDM. $(\forall\lambda\mu l m)(\exists\pi p\xi)(\forall a b p_1 p_2 x)(\exists u a_1 b_1 \alpha\beta\alpha'\beta')(\forall\alpha_1\alpha_2\beta_1\beta_2\gamma)$
$(p \in \lambda, \mu) \vee (l \in \lambda \wedge m \in \mu \to (l, m \in \pi)) \vee \{(a \in \lambda \wedge b \in \mu$
$\to (a, p \in \alpha) \wedge (b, p \in \beta)) \wedge ((a_1 \in \lambda \wedge b_1 \in \mu \wedge \neg(\bigwedge_{i=1}^2((a_1, p_i \in \alpha_i)$
$\wedge(b_1, p_i \in \beta_i)))) \vee \bigvee_{i=1}^2 p_i = p \vee p_1 = p_2) \wedge ((p \notin \lambda \wedge p \notin \mu)$
$\vee((p \in \lambda \vee p \in \mu) \wedge ((u \in \lambda \wedge \neg(x, u \in \gamma)) \vee (u \in \mu \wedge \neg(x, u \in \gamma))$
$\vee x \in \xi) \wedge (x \in \xi \wedge a \in \lambda \wedge b \in \mu \to (x, a \in \alpha') \wedge (x, b \in \beta'))))\}$

We shall denote the theory axiomatized by these axioms by $\mathcal{G}$.

6 The translations

We now show how to translate an axiom system for $\mathcal{L}'$, $\mathcal{P}'$, or $\mathcal{G}'$ — which are the theories $\mathcal{L}$, $\mathcal{P}$, and $\mathcal{G}$ to which an axiom stating that the dimension is ≥ 4 has been added — expressed in L_{I}, $\mathsf{L}_{\sim}$, or $\mathsf{L}_{\in}$ into one for any of the other two theories expressed in the language corresponding to it. In all these translations, the two languages will share one individual variable with the same intended interpretation, namely *lines*, which will get translated identically into the lines of the other language. We first define in $\mathsf{L}_{\sim}$ the ternary co-punctuality predicate S, with $S(abc)$ standing for 'a, b, c are three different lines passing through the same point' and the closely related ternary predicate $\overline{S}$, where $\overline{S}(abc)$ stands for 'c passes through the intersection point of a and b', and then E with $E(abx)$ standing for 'line x lies in the same plane as the intersecting lines a and b and goes through their intersection point' by:

$$\begin{aligned}
S(a_1a_2a_3) \quad &:\Leftrightarrow \quad (\forall g)(\exists h)\, g \sim h \wedge \bigwedge_{i=1}^{3}(a_i \sim a_{i+1}, h),\\
\overline{S}(abc) \quad &:\Leftrightarrow \quad S(abc) \vee (a \sim b \wedge (c = a \vee c = b)),\\
E(a_1a_2a_3) \quad &:\Leftrightarrow \quad a_1 \sim a_2 \wedge \bigwedge_{i=1}^{2} a_3 \simeq a_i \wedge [(\exists b_1b_2)\, b_1 \not\simeq b_2 \bigwedge_{1\leq i\leq 2, 1\leq j\leq 3} b_i \sim a_j].
\end{aligned}$$

In $\mathsf{L}_{\in}$, the predicates Σ, $\overline{\Sigma}$, $E_{\in}$ (the latter of which which we defined the way we did, and not by means of just $(\exists\alpha)\,(a_1, a_2, a_3 \in \alpha) \wedge a_1 \neq a_2$ for reasons which will become apparent in the function the axiom GHI5 plays), having the same intuitive interpretations as S, $\overline{S}$, and E, are defined by (here and in the sequel $P(ab\alpha)$ stands for $a \neq b \wedge (a, b \in \alpha)$):

$$\begin{aligned}
\Sigma(a_1a_2a_3) \quad &:\Leftrightarrow \quad (\forall g)(\exists h\gamma\pi_1\pi_2\pi_3\alpha_1\alpha_2\alpha_3)\, P(gh\gamma)\\
&\qquad \wedge \bigwedge_{i=1}^{3}(P(a_ia_{i+1}\pi_i) \wedge P(a_ih\alpha_i)),\\
\overline{\Sigma}(abx) \quad &:\Leftrightarrow \quad \Sigma(abx) \vee ((\exists\alpha)\, P(ab\alpha) \wedge (x = a \vee x = b)),\\
E_{\in}(a_1a_2a_3) \quad &:\Leftrightarrow \quad (\exists b_1b_2\alpha\alpha_{11}\alpha_{12}\alpha_{13}\alpha_{21}\alpha_{22}\alpha_{23})(\forall\mu)\,(a_1, a_2, a_3 \in \alpha)\\
&\qquad \wedge a_1 \neq a_2 \wedge \bigwedge_{1\leq i\leq 2, 1\leq j\leq 3} P(b_ia_j\alpha_{ij}) \wedge \neg(b_1, b_2 \in \mu).
\end{aligned}$$

In L_{I}, the predicates S_{I}, E_{I}, K, the first two having the same intuitive interpretation as S, E, and $K(abc)$ standing for 'the three lines a, b, c are concurrent and lie in the same plane', are defined by:

$$
\begin{aligned}
S_{\mathrm{I}}(a_1a_2a_3) \quad &:\Leftrightarrow \quad (\exists A) \bigwedge_{i=1}^{3} (A \,\mathrm{I}\, a_i \wedge a_i \neq a_{i+1}),\\
K(a_1a_2a_3) \quad &:\Leftrightarrow \quad (\exists g A_1A_2A_3)\, S_{\mathrm{I}}(a_1a_2a_3) \wedge \bigwedge_{i=1}^{3} (A_i \neq A_{i+1} \wedge (A_i \,\mathrm{I}\, a_i, g)),\\
E_{\mathrm{I}}(abx) \quad &:\Leftrightarrow \quad K(abx) \vee [(\exists P)\,(P \,\mathrm{I}\, a, b) \wedge a \neq b \wedge (x = a \vee x = b).]
\end{aligned}
$$

Notice that the definitions for S, $\overline{S}$, Σ, $\overline{\Sigma}$, are valid *only* for dimensions ≥ 4, whereas the definition for E, $E_{\in}$, S_{I}, K, E_{I} are valid for dimensions ≥ 3. We now indicate how to translate back and forth between $\mathcal{L}$, $\mathcal{P}$, $\mathcal{G}$ (the translations from $\mathcal{L}$ to $\mathcal{P}$, and from $\mathcal{L}$ to $\mathcal{G}$ work only if the dimension is known to be ≥ 4, so they are actually translations from $\mathcal{L}'$ to $\mathcal{P}'$, and from $\mathcal{L}'$ to $\mathcal{G}'$).

To translate from $\mathcal{L}'$ into $\mathcal{P}'$, a point A is identified with a couple (a_1, a_2) of lines with $a_1 \sim a_2$. We have[2]

$$(a_1, a_2) \equiv (b_1, b_2) :\Leftrightarrow \bigwedge_{i=1}^{2} \overline{S}(a_1a_2b_i), \tag{6.1}$$

$A \,\mathrm{I}\, l$ if and only if $\overline{S}(a_1a_2l)$,

where (a_1, a_2) is the couple associated with A.

To translate from $\mathcal{P}$ to $\mathcal{L}$, all we need is $l \sim m \Leftrightarrow (\exists A)\,(A \,\mathrm{I}\, l, m)$.

To translate from $\mathcal{L}'$ to $\mathcal{G}'$, a point A is identified with a triple (a_1, a_2, α) with $P(a_1a_2\alpha)$, such that

$$(a_1, a_2, \alpha) \equiv (b_1, b_2, \beta) :\Leftrightarrow \bigwedge_{i=1}^{2} \overline{\Sigma}(a_1a_2b_i), \tag{6.2}$$

$A \,\mathrm{I}\, l$ if and only if $\overline{\Sigma}(a_1a_2l)$,

where a_1 and a_2 are the two lines associated with A.

To translate from $\mathcal{G}$ into $\mathcal{L}$, a pencil α is identified with a triple (A, a_1, a_2),

[2]In the definiens of each $\equiv$ and in those of I and $\in$ we have omitted, for simplicity's sake, the conditions on the couples or triples which are required so they actually stand for points (in (1) and (2)) or pencils (in (3) and (4)).

with $a_1 \neq a_2 \wedge (A \,\mathrm{I}\, a_1, a_2)$, and

$$(A, a_1, a_2) \equiv (B, b_1, b_2) :\Leftrightarrow \bigwedge_{i=1}^{2} E_{\mathrm{I}}(a_1 a_2 b_i), \tag{6.3}$$

$$a_3 \in \alpha \text{ if and only if } (\exists A_1 A_2 A_3 g)\, A \,\mathrm{I}\, a_3 \wedge \bigwedge_{i=1}^{3} (A_i \neq A \wedge (A_i \,\mathrm{I}\, a_i, g)),$$

where (A, a_1, a_2) is the triple associated with α.

To translate from $\mathcal{P}$ to $\mathcal{G}$, we note that $l \sim m$ if and only if $(\exists \alpha)\, P(lm\alpha)$, and to translate from $\mathcal{G}$ to $\mathcal{P}$ we identify a pencil α with a pair (a_1, a_2) of lines with $a_1 \sim a_2$, such that

$$(a_1, a_2) \equiv (b_1, b_2) :\Leftrightarrow \bigwedge_{i=1}^{2} E(a_1 a_2 b_i), \tag{6.4}$$

$l \in \alpha$ if and only if $E(a_1 a_2 l)$,

where (a_1, a_2) is the pair associated with α.

We are now ready to find the axiom systems we were after.

7 Axiom systems for $\mathcal{P}$ and $\mathcal{G}$

Consider the following $\mathsf{L}_{\sim}$-axioms:

1 P. $(\forall a_1 b_1 a_2 b_2)(\exists p)(\forall q) \bigwedge_{i=1}^{2} a_i \sim b_i \wedge \neg \overline{S}(a_1 b_1 a_2) \rightarrow \bigwedge_{i=1}^{2} \overline{S}(a_i b_i p)$
$\wedge (\bigwedge_{i=1}^{2} \overline{S}(a_i b_i q) \rightarrow p = q)$

2 P. $(l \sim n, m, p) \wedge (m \sim p, n) \wedge \neg(\overline{S}(nlm) \vee \overline{S}(nlp) \vee \overline{S}(mlp) \vee \overline{S}(mnp))$
$\rightarrow n \sim p$

3 P. $(\forall l)(\exists a_1 a_2 a_3) \not\sim (a_1 a_2 a_3) \wedge \bigwedge_{i=1}^{3} (l \sim a_i \wedge \neg S(l a_i a_{i+1}))$

4 P. $\overline{S}(abl) \wedge \overline{S}(abm) \wedge l \neq m \rightarrow l \sim m$

5 P. $(\exists abcd)(\forall ef)\, (b \sim a, c) \wedge c \sim d \wedge a \not\sim c \wedge b \not\sim d$
$\wedge ((e, f \sim a, b, c, d) \rightarrow e = f)$

{HI1, P1-P5} axiomatizes $\mathcal{P}'$, and to see this, we shall think of a couple of intersecting lines (a, b) as a *point* (and we shall write all such virtual concepts in italics), and interpret *point equality* and *point-line incidence* as in (6.1). P1-P3 thus become L1-L3, and P5 implies $(\exists ac)\, a \not\sim c$, which becomes L4. so we know that our lines and our *points*, together with *point-line incidence* are those of dimension-free projective geometry. What we don't yet know is the meaning of $\sim$, and it is the function of axioms HI1 and P4 to "define" it. P4 tells us that if two different lines l and m have a *point* in common, then $l \sim m$, so all lines that *intersect* are in the relation $\sim$. Conversely, if two lines l and m are in the relation $\sim$, then there is a *point*, namely (l, m), which is incident with both l and m. HI1 ensures that $\sim$ applies only to different lines that have a point in common (the symmetry of $\sim$, i. e. HI2, follows from P4 with $a = m$ and $b = l$). Given P5, which is a slight strengthening of the negation of T5, the dimension of this space is ≥ 4, so the definitions in (6.1) have the intended interpretation, thus all axioms in which S and $\overline{S}$ occur are sound (are valid in $\mathcal{P}'$).

Let ϑ denote the conjunction of the axioms HI1, HI2, T1-T6, and let π denote the conjunction of the axioms HI1, P1-P5. Then $\vartheta \vee \pi$ is a sentence of quantifier complexity $\forall\exists\forall\exists$ which axiomatizes $\mathcal{P}$.

We are now going to provide an alternate axiomatization of the same quantifier complexity for $\mathcal{P}$ by way of translating Ferrara Dentice and Melone's [4, Th. 3] axiom system into $\mathsf{L}_{\sim}$.

The axiom system obtained in this manner consists of HI1 and

1 N. $(\forall p)(\exists q)\, p \sim q$

2 N. $(\forall l_1 l_2)(\exists pq)(\forall a)\, p \not\simeq q \wedge (E(l_1 l_2 a) \rightarrow p \sim a \wedge q \sim a)$

3 N. $(\forall l_1 l_2 p_1 p_1 p_2 p_3)(\exists a)\, l_1 \sim l_2 \wedge \bigwedge_{i=1}^{3} p_i \not\simeq p_{i+1} \rightarrow E(l_1 l_2 a) \wedge (\bigvee_{i=1}^{3} p_i \not\sim a)$

4 N. $(\forall l_1 l_2 m_1 m_2 l m)(\exists p x_1 x_2)(\forall a b p_1 p_2 x)(\exists u a_1 b_1)\, l_1 \sim l_2 \wedge m_1 \sim m_2$
$\rightarrow \{(E(l_1 l_2 p) \wedge E(m_1 m_2 p)) \vee (E(l_1 l_2 l) \wedge E(m_1 m_2 m) \rightarrow l \simeq m)$
$\vee[(E(l_1 l_2 a) \wedge E(m_1 m_2 b) \rightarrow (p \simeq a, b)) \wedge ((E(l_1 l_2 a_1) \wedge E(m_1 m_2 b_1)$
$\wedge\neg(\bigwedge_{i=1}^{2}(p_i \sim a_1, b_1))) \vee \bigvee_{i=1}^{2} p_i = p \vee p_1 = p_2)$
$\wedge((\neg E(l_1 l_2 p) \wedge \neg E(m_1 m_2 p)) \vee ((E(l_1 l_2 p) \vee E(m_1 m_2 p))$
$\wedge((E(l_1 l_2 u) \wedge x \not\simeq u) \vee (E(m_1 m_2 u) \wedge x \not\simeq u) \vee E(x_1 x_2 x))$
$\wedge(E(x_1 x_2 x) \wedge E(l_1 l_2 a) \wedge E(m_1 m_2 b) \rightarrow (x \simeq a, b)))))]\}$

5 N. $p \neq q \wedge E(abp) \wedge E(abq) \wedge E(a'b'p) \wedge E(a'b'q) \rightarrow E(aba') \wedge E(abb')$

6 N. $E(abp) \wedge E(abq) \rightarrow p \simeq q$

7 N. $(\forall a_1 a_2)(\exists b_1 b_2)\, a_1 \sim a_2 \rightarrow b_1 \not\simeq b_2 \bigwedge_{1\leq i\leq 2, 1\leq j\leq 2} b_i \sim a_j$.

To see that it actually does axiomatize $\mathcal{P}$, notice that if we think that two lines a and b with $a \sim b$ determine a *pencil* (a, b) (H2, which is needed so that the notion of pencil depends on the set $\{a, b\}$ and not on the pair (a, b), follows from N6 and N7), and that (4) holds for the identity of two pencils and for the line-pencil incidence relation, then axiom FDM3 follows from N1, FDM4 from N2, FDM5 from N3, FDM6 from N4, and FDM1 and FDM2 from our definition of a pencil and N5, HI1. Thus all the axioms of the axiom system in [4, Th. 3] hold for the structure of lines and defined pencils, so lines, pencils, and $\in$ do have the desired interpretation. To prove that $\sim$ has the desired interpretation as well, notice that, by N7, we know that, if $p \sim q$, then p and q belong to the pencil (p, q), so that p and q must *intersect*. Conversely, suppose p and q are two lines which *intersect*. Then there is a pencil (a, b), to which they belong, and so, by N6, $p \sim q$, so $\sim$ does indeed stand for line-intersection.

We can also axiomatize the Grassmann space representing the lines of a three-dimensional projective space by translating the axioms of [5] and adding a few axioms to make sure that pencils and $\in$ get interpreted in the intended manner. The axiom system consists of FDM1, FDM2, as well as

1 GHI. $(\forall a_1a_2\alpha)(\exists m_1m_2\mu_{11}\mu_{12}\mu_{21}\mu_{22})(\forall\mu)\, P(a_1a_2\alpha)$
$\to \bigwedge_{1\leq i,j\leq 2} P(m_ia_j\mu_{ij}) \wedge \neg(m_1, m_2 \in \mu)$

2 GHI. $(\forall a_1a_2b_1b_2b_3b_4\alpha\alpha_{11}\alpha_{12}\alpha_{13}\alpha_{14}\alpha_{21}\alpha_{22}\alpha_{23}\alpha_{24})(\exists\beta)(\forall\beta')$
$P(a_1a_2\alpha) \wedge \bigwedge_{1\leq i\leq 2, 1\leq j\leq 4} P(a_ib_j\alpha_{ij}) \to [(P(b_1b_3\beta) \wedge \neg(b_1, b_4 \in \beta'))$
$\vee(P(b_1b_4\beta) \wedge \neg(b_1, b_3, \beta')) \vee (b_1, b_2 \in \beta) \vee (b_3, b_4 \in \beta) \vee b_1 = b_3 \vee b_1 = b_4]$

3 GHI. $(\forall ab\alpha)(\exists c)(\forall\beta)\, P(ab\alpha) \to \neg(a, c \in \beta) \wedge \neg(b, c \in \beta)$

4 GHI. $(\forall abc\alpha)(\exists l\alpha'\beta'\lambda)(\forall m\mu\nu)(\exists\beta)\, P(ab\alpha)$
$\to (l, a \subset \alpha') \wedge (l, b \subset \beta') \wedge P(cl\lambda) \wedge (P(ma\mu) \wedge P(mb\nu) \to (l, m \subset \beta))$

5 GHI. $(\forall a_1a_2a_3)(\exists b_1b_2\alpha\alpha_{11}\alpha_{12}\alpha_{13}\alpha_{21}\alpha_{22}\alpha_{23})(\forall\mu)$
$(a_1, a_2, a_3 \in \alpha) \wedge a_1 \neq a_2 \to \bigwedge_{1\leq i\leq 2, 1\leq j\leq 3} P(b_ia_j\alpha_{ij}) \wedge \neg(b_1, b_2 \in \mu)$

From FDM1, FDM2, GHI1-GHI4, we get, by interpreting the existence of α with $P(ab\alpha)$ as 'lines a and b intersect', i. e. as $a \sim b$, that the axioms HI1-HI7 are satisfied, so that we may think of lines as the lines of a 3-dimensional projective geometry, and deduce from $P(ab\alpha)$ that the lines a and b intersect. Thus $E_\in$ has the desired interpretation, so that GHI5 (which states that $(a, b, l \in \alpha) \wedge a \neq b \to E_\in(abl)$, the converse implication being an immdeiate consequence of the definition of $E_\in$) ensures that pencils are what they should be, and $\in$ has the desired interpretation. Thus our axiom system does indeed axiomatize the Grassmann space representing the lines of a three-dimensional projective space. Let δ denote the conjunction of the axioms FDM1, FDM2, and GHI5, and let γ denote the conjunction of the axioms GHI1-GHI4.

Based on Lenz's axiom, we find that the axioms FDM1, FDM2, GHI5, as well as the following axioms, form an axiom system for $\mathcal{G}'$:

1 G. $(\forall a_1a_2b_1b_2\alpha\beta)(\exists l)(\forall l')\, P(a_1a_2\alpha) \wedge P(b_1b_2\beta) \wedge \neg\overline{\Sigma}(a_1a_2b_1)$
$\to \overline{\Sigma}(a_1a_2l) \wedge \overline{\Sigma}(b_1b_2l) \wedge (\overline{\Sigma}(a_1a_2l') \wedge \overline{\Sigma}(b_1b_2l') \to l = l')$

2 G. $(\forall\alpha\beta\gamma\delta\epsilon lmnp)(\exists\pi)\, (l \in \alpha, \beta, \epsilon) \wedge (m \in \gamma, \delta, \epsilon) \wedge (n \in \alpha, \gamma) \wedge (p \in \beta, \delta)$
$\wedge\neg(\overline{\Sigma}(lnp) \vee \overline{\Sigma}(lmn) \vee \overline{\Sigma}(lpm) \vee \overline{\Sigma}(mnp)) \to (n, p \in \pi)$

3 G. $(\forall l)(\exists a_1a_2a_3\alpha_1\alpha_2\alpha_3)\ \bigwedge_{i=1}^{3}(P(a_il\alpha_i) \wedge \neg\Sigma(a_ia_{i+1}l) \wedge a_i \neq a_{i+1})$

4 G. $(\exists lm)(\forall\alpha)\, \neg(l, m \in \alpha)$

5 G. $(\forall ablm)(\exists\alpha)\, \overline{\Sigma}(abl) \wedge \overline{\Sigma}(abm) \to (l, m \in \alpha)$

6 G. $(\exists a_1 a_2 a_3 a_4 \alpha_1 \alpha_2 \alpha_3)(\forall \mu_1 \mu_2 e_1 e_2 \epsilon_{11} \epsilon_{12} \epsilon_{13} \epsilon_{14} \epsilon_{21} \epsilon_{22} \epsilon_{23} \epsilon_{24})$
$\bigwedge_{i=1}^{3} P(a_i a_{i+1} \alpha_i) \wedge \bigwedge_{i=1}^{2} \neg(a_i, a_{i+2} \in \mu)$
$\wedge(\bigwedge_{1\leq i\leq 2, 1\leq j\leq 4} P(e_i a_j \epsilon_{ij}) \rightarrow e_1 = e_2)$

Again, we can think of triples (l, m, α) with $P(lm\alpha)$ as *points* with *point-equality* as well as *point-line incidence* defined by (6.2). It then turns out that G1-G4 can be read as L1-L4 (to get L2 from G2, let A, B, C, D, E be the names of (l, n, α), (l, p, β), (m, n, γ), (p, m, δ), (l, m, ϵ), and use G6, and to get from G4 to L4, we need to use G5 as well), so that the "*point*, *line*, point-line incidence theory" is that of $\mathcal{L}$. What we do not know by this observation alone is what pencils are to be interpreted as, nor what the meaning of the relation $\in$ is.

From G5 we conclude that, if two lines *contain* the same *point*, then they belong to some pencil. From FDM2 we conclude that every pencil λ contains two different lines, p and q, which must intersect as they both pass, by (6.2), through (p, q, λ). Thus every pencil contains two intersecting lines, and any two intersecting lines determine a pencil. GHI5 ensures that a line is contained in the pencil determined by the lines l and m if and only if it passes through the intersection point of l and m, and lies in the same plane as these. Thus all notions have the intended interpretation, so our axiom system does indeed axiomatize $\mathcal{G}'$, as G6, which is the $\mathsf{L}_{\in}$-counterpart of P5, ensures that the dimension is ≥ 4, so that $\overline{\Sigma}$ does indeed have the intended interpretation. Let γ' denote the conjunction of G1-G6. Then $\delta \wedge (\gamma \vee \gamma')$ is a sentence axiomatizing $\mathcal{G}$.

The quantifier complexity of this sentence is $\forall\exists\forall\exists$, a lesser complexity than that of the axiom system in [4], but one that most likely can be further simplified. With this we do not mean to suggest that our axiom system is in an informal sense simpler than the one in [4, Th. 3], as it is significantly longer, and it was not the purpose of this paper to find the simplest axiom system for $\mathcal{G}$, but rather to show that an axiom system for $\mathcal{G}$ may be obtained by means of a straightforward translation process.

The same can be done for dimension-free affine geometry, also first axiomatized in [9], and we leave it as an exercise for the reader to turn that axiom system, expressed in terms of *points*, *lines*, point-line incidence, and line-parallelism, into one expressed in $\mathsf{L}_{\sim}$ and into one expressed in $\mathsf{L}_{\in}$. The latter solves the problem of elementarily characterizing the Grassmann spaces representing the lines of an affine space, which received a non-elementary characterization in [1]. The procedure is entirely analogous to the one performed for projective spaces, given that the definitions of S, E, and their variants stays the same (and that for affine geometry all the definitions are valid for all dimensions ≥ 3, so there is no need to take the disjunction of the conjunction of two axiom systems as we had to do) and that line-parallelism can be defined in terms of $\sim$

(and thus in terms of $\in$ as well) by

$$a \parallel b :\Leftrightarrow (\exists cde)\, a = b \vee (a \not\sim b \wedge S(acd) \wedge S(bce) \wedge (d \sim b, e) \wedge (e \sim a)).$$

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