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Art galleries with k-guarded guards

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Abstract. The k-guarded art gallery problem asks for the minimum number of k-guarded point guards that can collectively monitor a simple polygon with n vertices. A guard is *k-guarded* if it can see k other guards. For $k = 0$, this problem is equivalent to the classical art gallery problem of Klee. For $k = 1$, a tight bound of $\lfloor \frac{3n-1}{7} \rfloor$ was shown recently by Michael and Pinciu and, independently, by Żyliński. In this paper, we settle the problem for every $k \geq 2$ and show that $k\lfloor \frac{n}{5} \rfloor + \lfloor \frac{n+2}{5} \rfloor$ k-guarded guards are always sufficient and sometimes necessary to guard a simple polygon with n vertices.

Keywords: The art gallery problem, guarded guards

MSC 2000 classification: primary 52C99, secondary 05C90

Introduction

The *art gallery problem* asks how many guards are sufficient to see every point of the interior of an n-vertex simple polygon. The guard is a stationary point who can *see* any point that can be connected to it with a line segment within the polygon. The art gallery problem was first raised by Klee in 1973. In 1975, Chvátal [2] proved that $\lfloor \frac{n}{3} \rfloor$ guards are occasionally necessary and always sufficient to cover a polygon with n vertices. Since then many different variations of this problem have arisen; see [11], [13] for more details.

Herein we analyze the concept of k-guarded guards that was raised by Michael and Pinciu [9]. A set of points S in a polygon P is a *k-guarded guard* set for P provided that (i) for every point x in P there is a point g in S such that g sees x; and (ii) every point of S is visible from at least k other points in S. For a polygon P, we define

$$gg(P,k) = \min\{|S| : S \text{ is a } k\text{-guarded guard set for } P\}.$$

Liaw, Huang, and Lee [6], [7] referred to a 1-guarded guard set as a *weakly cooperative* guard set and showed that the computation of $gg(P, 1)$ is an NP-hard problem. Let

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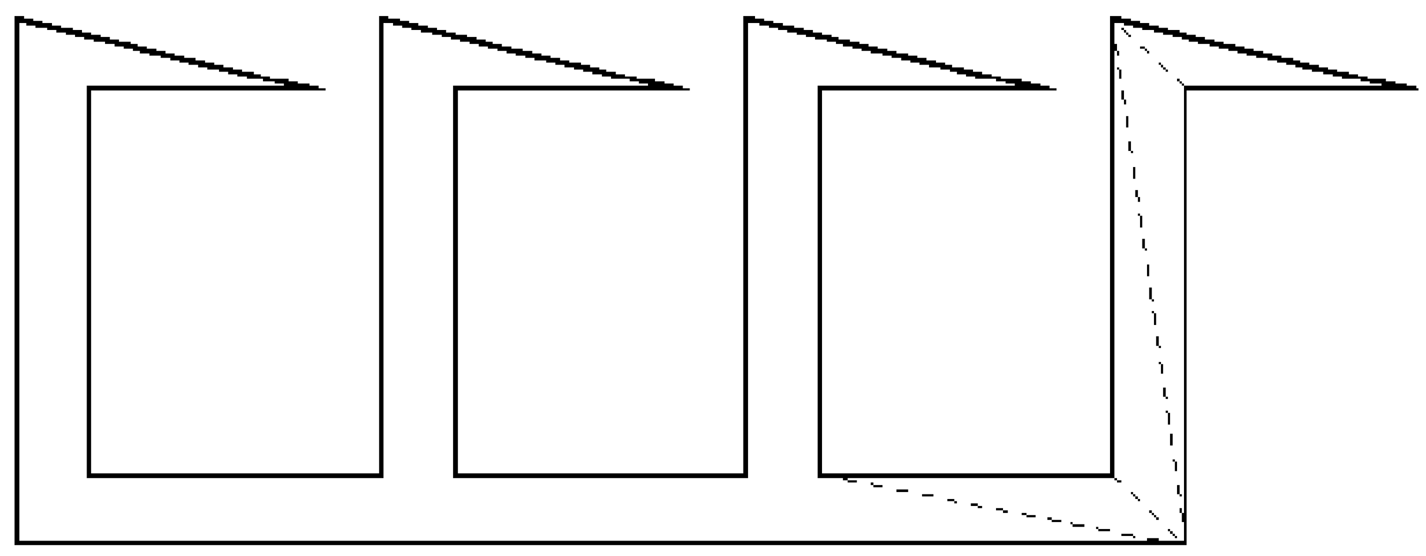

Figure 1. An n-vertex polygon P with $gg(P) = k\lfloor \frac{n}{5} \rfloor + \lfloor \frac{n+2}{5} \rfloor$;
here $n = 20$, and $gg(P) = 4k + 4$.

$$gg(n,k) = \max\{gg(P,k) : P \text{ is a polygon with } n \text{ vertices}\},$$
$$gg_{\perp}(n,k) = \max\{gg(P,k) : P \text{ is an orthogonal polygon with } n \text{ vertices}\}.$$

The 1-guarded guards problem for orthogonal polygons was solved by Hernández-Peñalver [5] and, independently, by Michael and Pinciu [9], who proved that $gg_{\perp}(n,1) = \lfloor \frac{n}{3} \rfloor$. The 1-guarded guards problem for general simple polygons has been completely settled by Michael and Pinciu [8], and, independently, by Żyliński [14], who proved that $gg(n,1) = \lfloor \frac{3n-1}{7} \rfloor$. If $k \geq 2$, then Michael and Pinciu [9] established that $gg_{\perp}(n,k) = k\lfloor \frac{n}{6} \rfloor + \lfloor \frac{n+2}{6} \rfloor$. The case of general bound for arbitrary polygons has been remained open, and in this paper our main result is the following theorem.

1 Theorem. *For $n \geq 5$, $gg(n,k) = k\lfloor \frac{n}{5} \rfloor + \lfloor \frac{n+2}{5} \rfloor$.*

The necessity is established by the gallery P shown in Fig. 1. Each wave requires $k+1$ k-guarded guards, and it is clear that for $n \equiv 0 \bmod 5$, $gg(P,k) = k\lfloor \frac{n}{5} \rfloor + \lfloor \frac{n+2}{5} \rfloor$; the case of $n \equiv 1, 2, 3, 4 \bmod 5$ is indicated with dashed lines. As far as the sufficiency is considered, then the proof is based upon the reduction to a combinatorial problem on a fixed triangulation of the polygon (Section 1). The upper bound follows from an induction argument for $n = 13, 14, 15, 16$, and $n \geq 18$ (Section 2); for the remaining cases, we prove Theorem 1 by a thorough case analysis (Section 1).

Before we proceed further, let us make an interesting comment. Consider the 12-vertex polygon P that is shown in Fig. 2. It is easy to see that it requires five 1-guarded guards: each prong requires a guard, but these guards will form a hidden set, as they do not see each other. As any additional guard see at most two of these three guards, the fifth guard is needed [8, 14]. However, if we consider the k-guarded guards problem with $k \geq 2$, then $2k+2$ k-guards are enough: we have to place $k-1$ guards at vertex x_1, $k-1$ guards at vertex x_2, and one guard per each of vertices v_1, v_2, v_3 and x_3. This shows the discrepancy between $\lfloor \frac{3n-1}{7} \rfloor$- and $k\lfloor \frac{n}{5} \rfloor + \lfloor \frac{n+2}{5} \rfloor$-bound, whereas in the case of orthogonal

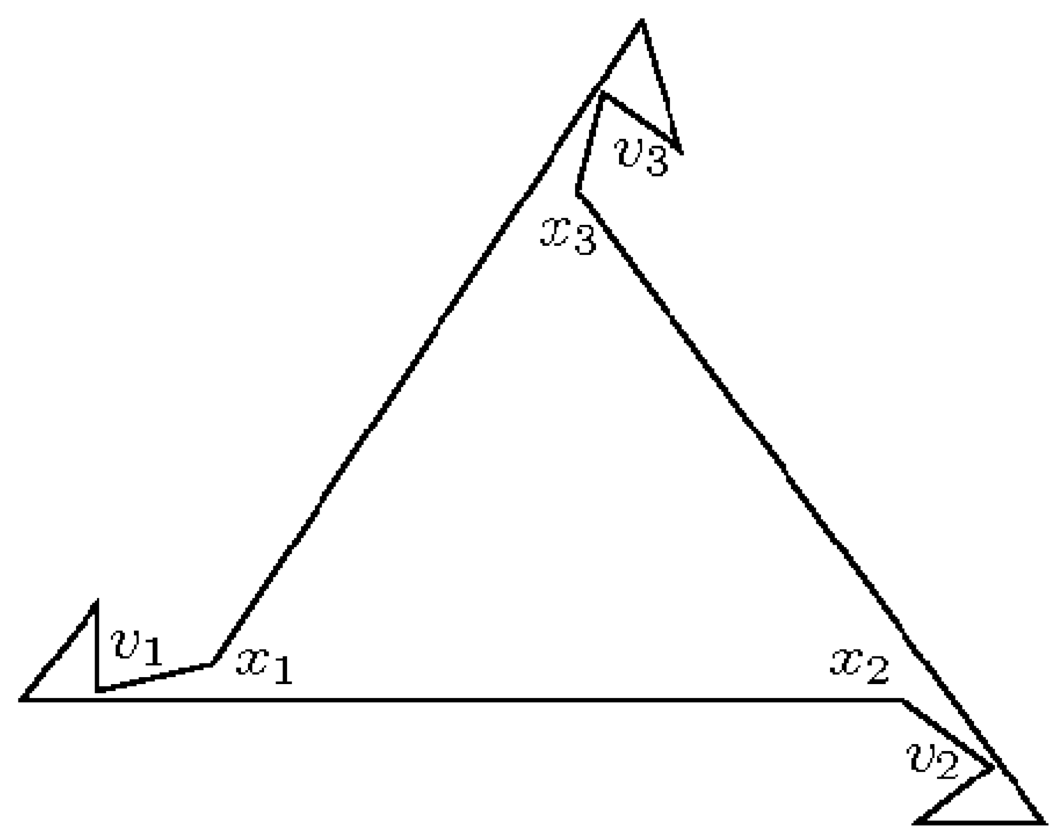

Figure 2. For $k \geq 2$, a polygon with 12 vertices can be guarded $2k+2$ k-guards.

polygons, the tight bound for $k \geq 2$ easily follows from that for $k = 1$, and one could expect that the same property would hold for general simple polygons.

1 Guard definitions and small triangulation graphs

An *art gallery* is a simple polygon P, i.e., a region bounded by a simple polyline $\bar{P}$ (together with $\bar{P}$). A *guard* g is any point of P. A point $x \in P$ is said to be *seen* by a guard g if the line segment with endpoints x and g is a subset of P. A collection of guards S is said to *cover* polygon P if every point $x \in P$ can be seen by some guard $g \in S$. A set of guards S is called *k-guarded* provided that each guard $g \in S$ sees at least k elements from S.

1.1 Reduction to combinatorial guards

A *triangulation* T of polygon P is a partitioning of P into a set of triangles with pairwise disjoint interiors in such a way that the edges of those triangles are either edges or internal diagonals of P joining pairs of vertices. It is easy to see that any polygon with n vertices can be partitioned into $n-2$ triangles by the addition of $n-3$ internal diagonals. A *triangulation graph* G_T of an n-vertex polygon P is a graph whose vertices correspond to n vertices of P and whose edges correspond to the n edges of the polygon P and $n-3$ internal diagonals of triangulation T.

The reason for introducing triangulation graphs is that a proof of the sufficiency of a certain number of combinatorial k-guarded guards in a triangulation graph of a polygon P establishes the sufficiency of the same number of geometric k-guarded guards in polygon P. Formally, a *vertex guard* in a triangulation graph G_T is a single vertex of G_T, and a set of guards S is said to *dominate* G_T if every triangular face of G_T has at least one of its vertices assigned as a guard. Next, the multiset of guards S is said to be *k-guarded* if every element of S is adjacent in G_T to at least k elements of S; note that we allow to choose a vertex many times for the location of a guard with the assumption that any two copies of the same vertex are adjacent in G_T. Then it easy to see that the following lemma holds.

2 Lemma. *Let P be a simple polygon and G_T be one of its triangulation graphs. If G_T can be dominated by m combinatorial k-guarded guards, then P can be covered by m geometric vertex k-guarded guards.* QED

1.2 Small triangulation graphs

As the proof of Theorem 1 is based upon the reduction to guarding triangulation graphs and it is an inductive proof, let us first establish the sufficiency of $k\lfloor\frac{n}{5}\rfloor + \lfloor\frac{n+2}{5}\rfloor$ k-guarded guards for small triangulation graphs. From now on, for simplicity, a k-guarded guard will be simply referred to as a k-guard.

In 1983, while considering the mobile guard problem in simple polygons, O'Rourke [10] introduced the concept of diagonal guards in triangulation graphs. Similarly to vertex guards, a *diagonal guard* in a triangulation graph G_T is a single edge of G_T. A set of guards (diagonals) S is said to *dominate* G_T if every triangular face of G_T has at least one of its vertices assigned as an endpoint of an element from S. Let us recall some of O'Rourke's results.

3 Lemma. [10]

a) *Every triangulation graph of a pentagon can be dominated by a single diagonal guard with one endpoint at any selected vertex.*

b) *Every triangulation graph of a hexagon or a septagon can be dominated by a single diagonal guard.* QED

As far as k-guards are considered, than it is easy to see that the lemma above implies the following corollary.

4 Corollary.

a) *Every triangulation graph of a pentagon can be dominated by $k+1$ combinatorial k-guarded guards with k guards placed at any selected vertex.*

b) *Every triangulation graph of a hexagon or a septagon can be dominated by $k+1$ combinatorial k-guards.* QED

Although the above corollary settles the case $n = 6$, let us establish more powerful property which will be used throughout this paper.

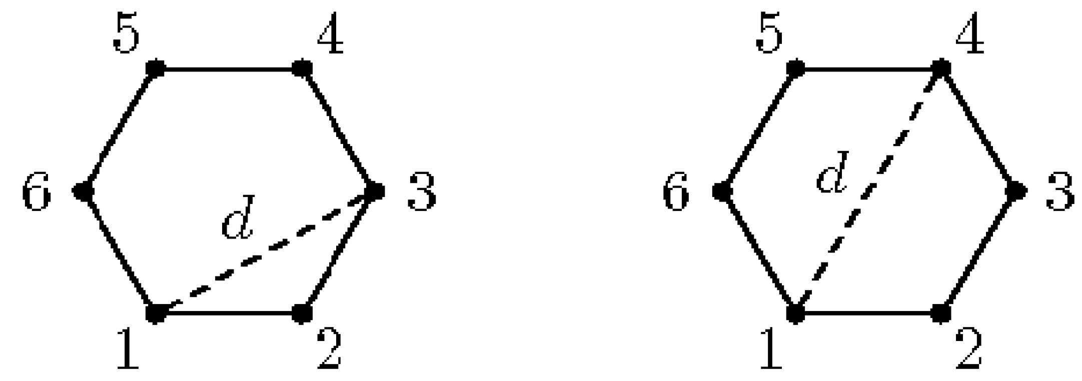

Figure 3. A triangulation graph of a hexagon can be dominated by $k+1$ combinatorial k-guards, with k-guards placed at any vertex of degree at least 3.

5 Lemma. *Let G_T be any triangulation graph of a hexagon, and let x be any vertex of degree at least* 3. *Then G_T can be dominated by $k+1$ combinatorial k-guards with k guards placed at x.*

PROOF. Let the selected vertex be labeled 1. As vertex 1 is of degree 3, there is a diagonal d with one of its endpoints at x. This diagonal partitions the six boundary edges of G_T according to either $2+4=6$ or $3+3=6$.

Case 1: $2+4=6$. Let $d=\{1,3\}$. Then $(1,3,4,5,6)$ is a triangulation graph of a pentagon (see Fig. 3), and by Corollary 4, this graph can be dominated by $k+1$ combinatorial k-guards with k guards placed at 1; the guards at 1 dominate triangle $(1,2,3)$.

Case 2: $3+3=6$. Let $d=\{1,4\}$. Then $(1,2,3,4)$ and $(1,4,5,6)$ are triangulation graphs of quadrilaterals (see Fig. 3). No matter how these quadrilaterals are triangulated, k guards at vertex 1 and one guard at vertex 4 dominate all triangles. QED

By the same method as in the proof above, we get a sequence of the following lemmas.

6 Lemma. *Let G_T be any triangulation graph of an octagon, and let x be any vertex of degree at least* 3. *Then G_T can be dominated by $k+2$ combinatorial k-guards with a guard placed at x.*

PROOF. Let the selected vertex be labeled 1. As the degree of vertex 1 is at least 3, there is a diagonal d with one of its endpoints at x. This diagonal partitions the eight boundary edges of G according to either $2+6=8$, $3+5=8$ or $4+4=8$.

Case 1: $2+6=8$. Let $d=\{1,3\}$. Then (1, 3, 4, 5, 6, 7, 8) is a triangulation graph of a septagon (see Fig. 4(a)), and by Corollary 4, this septagon can be dominated by $k+1$ combinatorial k-guards. One of these guards dominates vertex 1. Moreover, we can swap our k guards in such a way that k guards dominates vertex 1. Placing one additional guard at 1 gives a domination of triangle $T=(1,2,3)$ and all of G_T as well. The guard set is k-guarded.

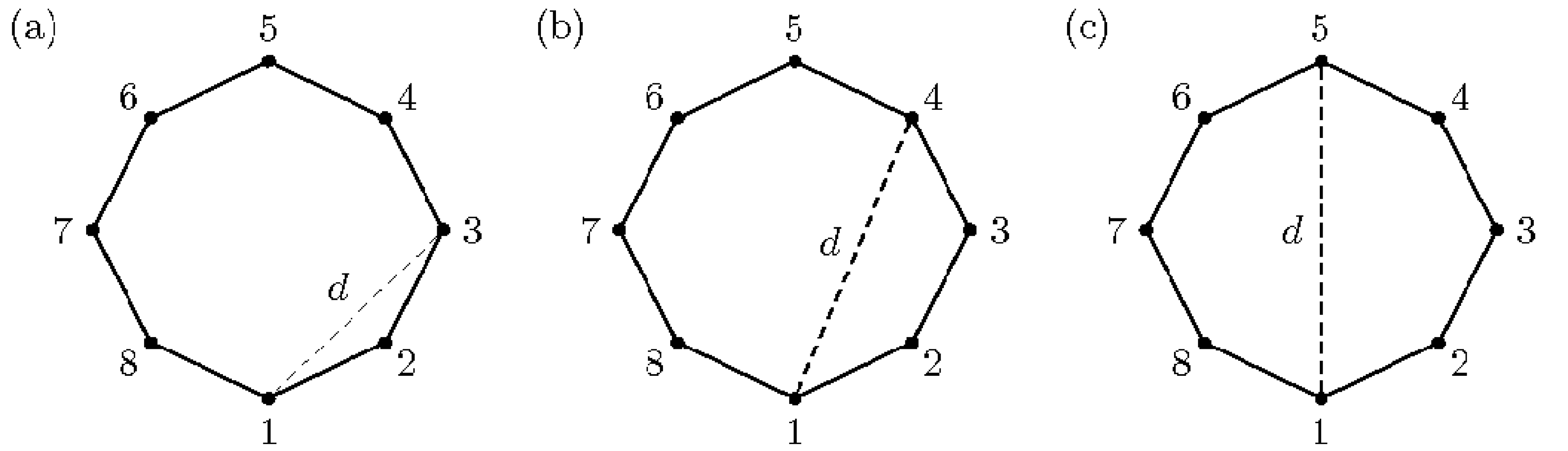

Figure 4. An octagon can be dominated by $k+2$ combinatorial k-guards, with a guard placed at any vertex of degree at least 3.

Case 2: $3+5=8$. Let $d=\{1,4\}$. Then $(1,2,3,4)$ and $H_6=(1,4,5,6,7,8)$ are triangulation graphs of a quadrilateral and a hexagon, respectively (see Fig. 4(b)). In H_6 either vertex 1 or 4 is of degree at least 3, without loss of generality, we can assume vertex 4 to be that one. By Lemma 5, $k+1$ combinatorial k-guards will dominate H_6 with k guards at 4. Now place one additional guard at 1. Regardless of how the quadrilateral is triangulated, the guards at 1 and 4 will dominate it.

Case 3: $4+4=8$. Let $d=\{1,5\}$. Then $P_5^1=(1,2,3,4,5)$ and $P_5^2=(1,5,6,7,8)$ are triangulation graphs of pentagons (see Fig. 4(c)). Dominate P_5^1 by $k+1$ combinatorial k-guards with k guards placed at vertex 1, and dominate P_5^2 by $k+1$ combinatorial k-guards with k guards placed at vertex 1, thus getting the domination of G by $k+2$ combinatorial k-guards with at least one guard placed at vertex 1. QED

7 Lemma. *Let G_T be any triangulation graph of an octagon and let x be any degree 2 vertex. Then one guard g at vertex x with an additional $k+1$ combinatorial k-guards are sufficient to dominate G_T (but, perhaps, guard g is not adjacent to any other guard).*

PROOF. Let the vertices of an octagon be labeled $1,\ldots,8$, in a counterclockwise manner, and assume vertex 1 to be of degree 2. By placing a guard at vertex 1 and cutting off triangle $(1,2,8)$ from G_T, we get a triangulation graph G_T^* of a septagon. By Corollary 4, graph G_T^* can be dominated by $k+1$ combinatorial k-guards, and the thesis follows. QED

8 Lemma. *Let G_T be any triangulation graph of an enneagon. Then G_T can be dominated by $k+2$ combinatorial k-guards.*

PROOF. In any triangulation graph of a polygon, there is at least one vertex of degree 2 (Meister's Two Ears Theorem 1975). Let the vertices of an enneagon

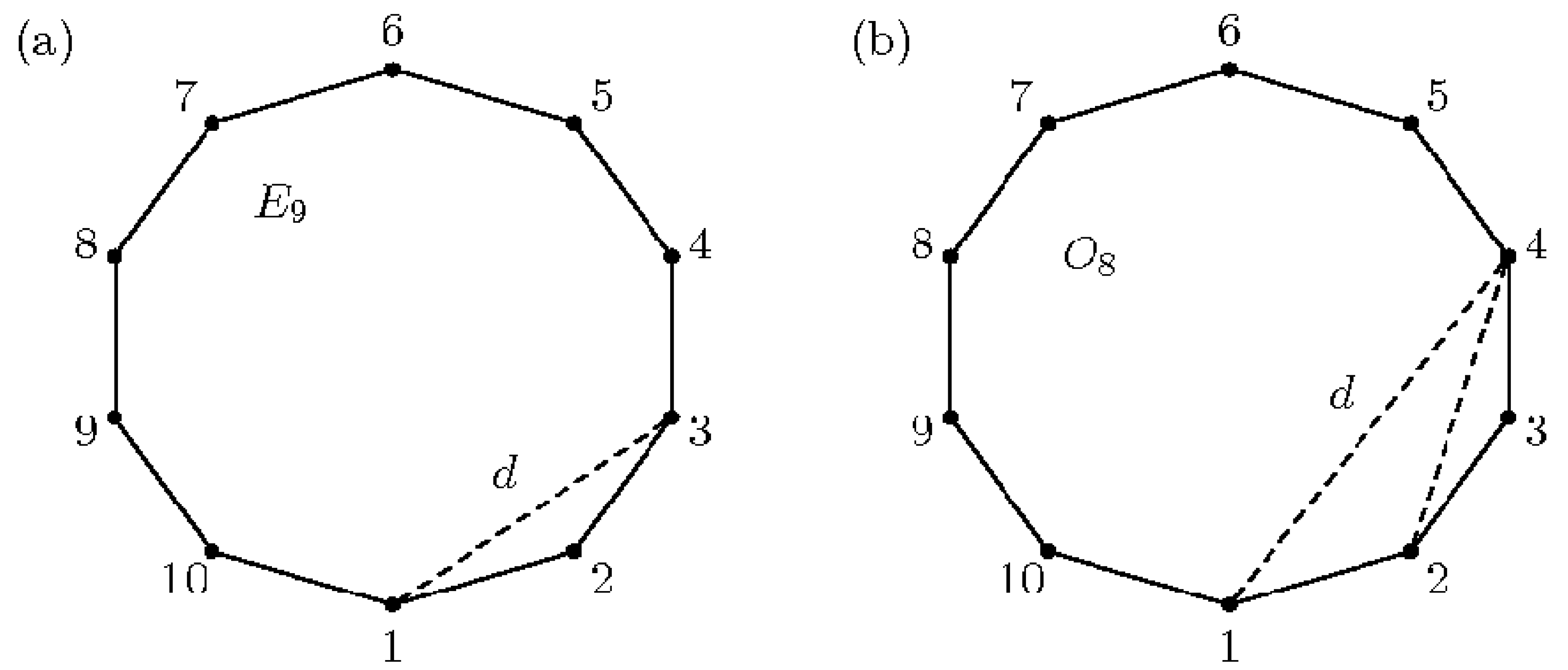

Figure 5. A domination of a 10-vertex polygon – Case 1 and 2.

be labeled $1, \ldots, 9$, in a counterclockwise manner, and assume vertex 1 to be of degree 2 in G_T. Cutting off triangle $T = (9, 1, 2$ from G_T results in a triangulation graph G_T^* of an octagon. By Lemma 6, graph G_T^* can be dominated by $k + 2$ combinatorial k-guards with one guard placed either at vertex 9 or 2. This yields a domination of G_T by $k + 2$ combinatorial k-guards, as triangle T is dominated, too. QED

9 Lemma. *Let G_T be any triangulation graph of a 10-vertex polygon. Then G_T can be dominated by $2k + 2$ combinatorial k-guards with one guard placed at any selected vertex.*

Proof. Let the vertices of a decagon be labeled in a counterclockwise manner, assuming that vertex 1 is the selected vertex. First, suppose that vertex 1 is of degree at least 3. Then there is a diagonal d with one of its endpoints at 1. This diagonal partitions the ten boundary edges of G_T according to either $2 + 8 = 10$, $3 + 7 = 10$, $4 + 6 = 10$ or $5 + 5 = 10$. Assume that d cuts off the minimal number of vertices.

Case 1: $2 + 8 = 10$. Let $d = \{1, 3\}$. Then $E_9 = (1, 3, 4, 5, 6, 7, 8, 9, 10)$ is a triangulation graph of an enneagon, see Fig. 5(a). By Lemma 8, enneagon E_9 can be dominated by $k + 2$ combinatorial k-guards. One of these guards dominates vertex 1. By placing k additional guards at 1, we get a domination of triangle $(1, 2, 3)$, and the guard set is k-guarded.

Case 2: $3 + 7 = 10$. Let $d = \{1, 4\}$. Then $O_8 = (1, 4, 5, 6, 7, 8, 9, 10)$ is a triangulation graph of an octagon, and the minimality of d ensures that quadrilateral $(1, 2, 3, 4)$ has diagonal $\{2, 4\}$, see Fig. 5(b). By Lemma 6, octagon O_8 can be dominated by $k + 2$ combinatorial k-guards with one guard either at vertex 1 or at 4.

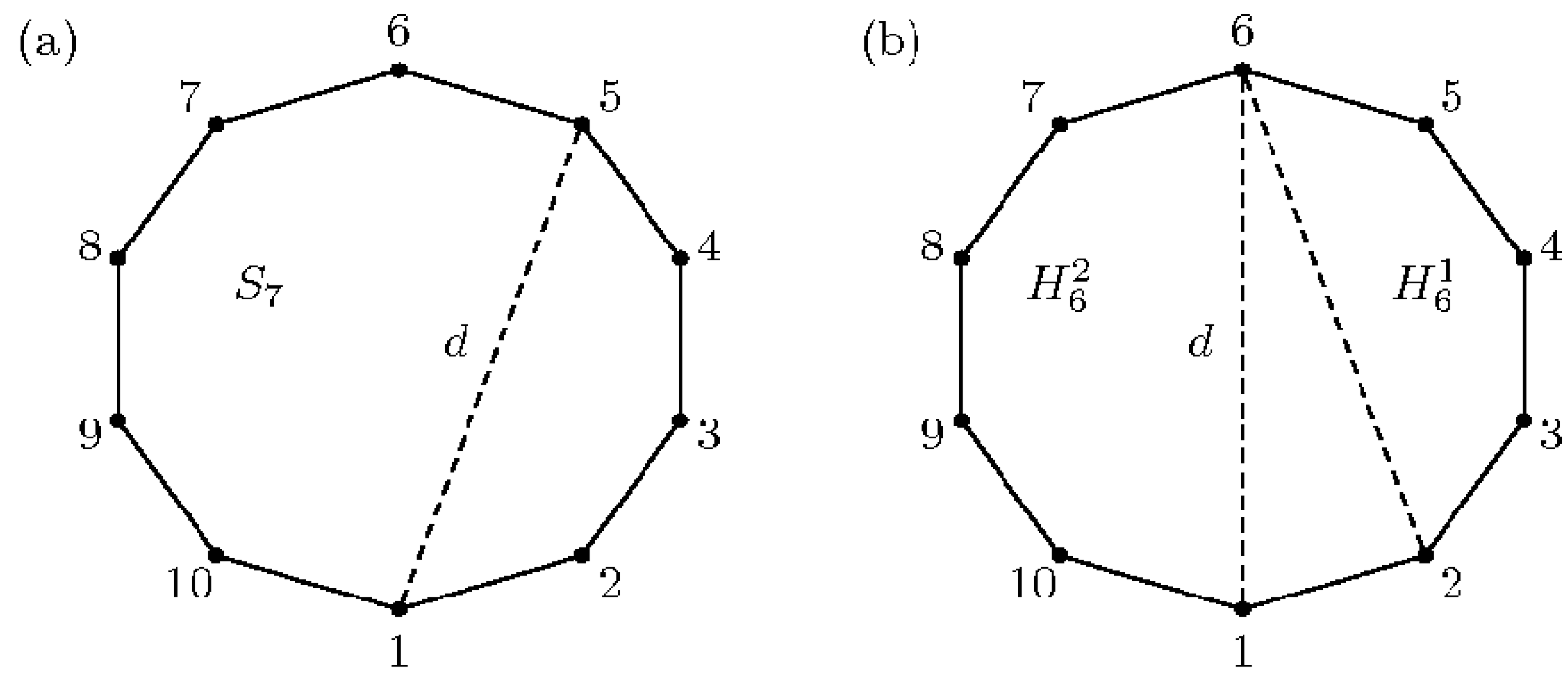

Figure 6. A domination of a 10-vertex polygon – Case 3 and 4.

Subcase 2/1: *there is a guard at* 1. By placing $k-1$ additional guards at 1 and one guard at 4, we get a domination of all triangles in G_T, and the guard set is k-guarded.

Subcase 2/4: *there is a guard at* 4. All triangles of G_T are dominated. Now place k additional guards at 1 – they are k-guarded, as there is a guard at 4.

Case 3: 4+6 = 10. Let $d = \{1, 5\}$. Then $S_7 = (1, 5, 6, 7, 8, 9, 10)$ is a triangulation graph of a septagon, see Fig. 6(a). By Corollary 4, septagon S_7 can be dominated by $k+1$ combinatorial k-guards, and pentagon $(1, 2, 3, 4, 5)$ can by dominated by $k+1$ combinatorial k-guards with k guards at 1.

Case 4: $5+5 = 10$. Let $d = \{1, 6\}$. Then $H_6^1 = (1, 2, 3, 4, 5, 6)$ and $H_6^2 = (1, 6, 7, 8, 9, 10)$ are triangulation graphs of hexagons, and the minimality of d ensure us that hexagon H_6^1 has a diagonal $\{2, 6\}$, see Fig. 6(b). By Lemma 5, this hexagon can be dominated by $k+1$ combinatorial k-guards with k guards at vertex 6. Place k guards at vertex 1. As either vertex 1 or 6 is of degree 3 in H_6^2, we need at most one additional guard for hexagon H_6^2 to be k-guarded by Lemma 5.

Thus the lemma holds for all vertices of degree at least 3. Now let us assume vertex 1 to be of degree 2. $E_9 = (2, 3, 4, 5, 6, 7, 8, 9, 10)$ is a triangulation graph of an enneagon. We proceed in four cases, depending on the triangle T in E_9 bounded by diagonal $\{2, 10\}$.

Case 5: $T = (2, 3, 10)$, see Fig. 7(a). $O_8 = (3, 4, 5, 6, 7, 8, 9, 10)$ is a triangulation graph of an octagon. Place k guards at vertex 10 and one guard at 1. By Lemma 7, k guards at 10 permits the remainder of O_8 to be dominated by at most $k+1$ k-guards.

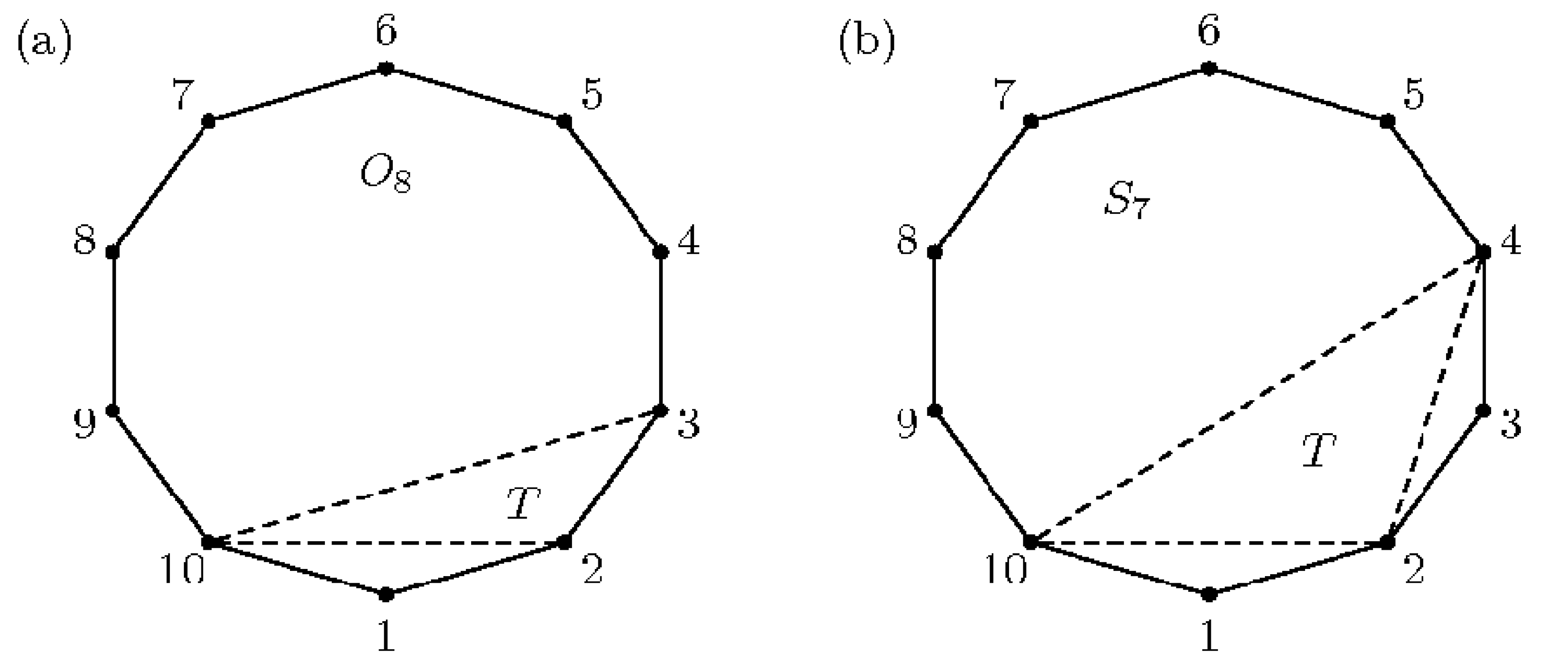

Figure 7. A domination of a 10-vertex polygon – Case 5 and 6.

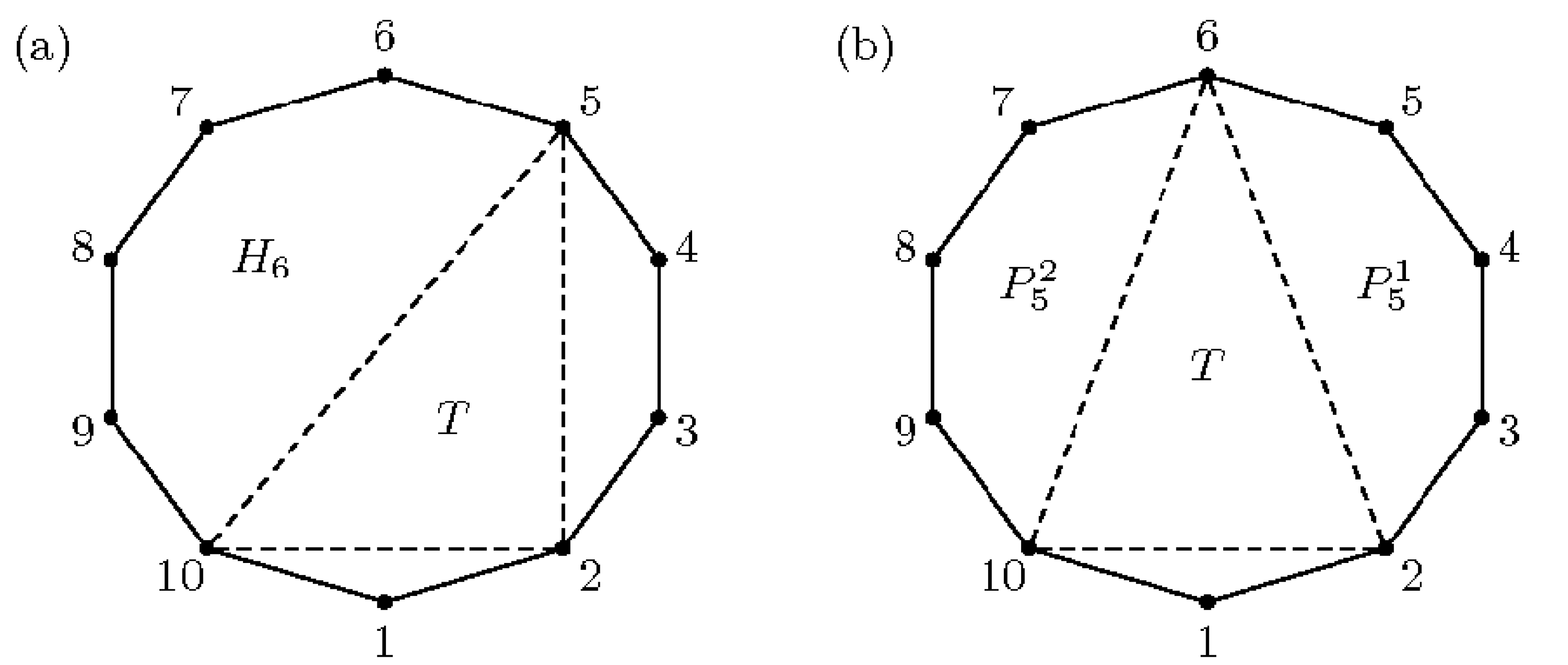

Figure 8. A domination of a 10-vertex polygon – Case 7 and 8.

Case 6: $T = (2, 4, 10)$, see Fig. 7(b). Then $S_7 = (4, 5, 6, 7, 8, 9, 10)$ is a triangulation graph of a septagon, and by Corollary 4, this septagon can be dominated by $k + 1$ combinatorial k-guards. Place k guards at vertex 1 and one guard at vertex 2 – all of G_T is k-guarded.

Case 7: $T = (2, 5, 10)$, see Fig. 8(a). Then $H_6 = (5, 6, 7, 8, 9, 10)$ is a triangulation graph of a hexagon, and by Lemma 5, hexagon H_6 can be dominated by $k + 1$ combinatorial k-guards with k guards either at 5 or at 10.

Subcase 7/5: *there are k guards at* 5. By placing k additional guards at 1 and one guard at 2, we get a domination of all triangles in G_T – no matter how quadrilateral $(2, 3, 4, 5)$ is triangulated; the guard set is k-guarded.

Subcase 7/10: *there are k guards at* 10. By placing k additional guards at 1 and one guard at either at 2 or 5, depending on how quadrilateral $(2, 3, 4, 5)$ is triangulated, we get a domination of all triangles in G_T, and the guard set is k-guarded.

Case 8: $T = (2, 6, 10)$, $P_5^1 = (2, 3, 4, 5, 6)$ and $P_5^2 = (6, 7, 8, 9, 10)$ are triangulation graphs of pentagons, see Fig. 8(b). By placing one guard at vertex 1, $k-1$ guards at vertex 2, $k-1$ guards at vertex 10, and one guard at vertex 6, with one additional guard for P_5^1, and one additional guard for P_5^2, we get a domination of G_T, and the guard set is k-guarded. QED

10 Lemma. *Let G_T be any triangulation graph of an 11-vertex polygon. Then G_T can be dominated by $2k+2$ combinatorial k-guards.*

PROOF. In any triangulation graph of a polygon, there is at least one vertex of degree 2. Let the vertices of an 11-vertex polygon be labeled $1, \ldots, 11$, in a counterclockwise manner, and assume vertex 1 to be of degree 2 in G_T. Cutting off triangle $T = (1, 2, 11)$ from G_T results in a triangulation graph G_T^* of a 10-vertex polygon. By Lemma 9, graph G_T^* can be dominated by $2k+2$ combinatorial k-guards with one guard placed at vertex 2. This yields a domination of G_T by $2k+2$ combinatorial k-guards, as triangle T is dominated, too. QED

Before we proceed with the case $n = 12$, let us recall the following theorem that establishes the existence of a special diagonal.

11 Theorem (the Cutting Diagonal Theorem). [12] *Given a polygon triangulation graph G_T of n vertices and some positive integer $t \leq n-2$, there exists an edge d of G_T which separates G into two pieces G_T^1 and G_T^2 (with d in both pieces) such that G_1 has between t and $2t-1$ triangles, inclusive. The degenerate case G_T^2 is allowed.* QED

12 Lemma. *Let G_T be a triangulation graph of a 12-vertex polygon. Then for all $k \geq 2$, G_T can be dominated by $2k+2$ combinatorial k-guards.*

PROOF. Theorem 11 guarantees the existence of a diagonal d that partitions G_T into two graphs G_T^1 and G_T^2, where G_T^1 contains l boundary edges of G_T with $5 \leq l \leq 8$. Assume that l is minimal. We consider each value of l separately.

Case 1: $l = 5$. Let $d = \{1, 6\}$. Then G_T^1 and G_T^2 are triangulation graphs of a hexagon $(1, 2, 3, 4, 5, 6)$ and an octagon $(1, 6, 7, 8, 9, 10, 11, 12)$, respectively. By Lemma 5, G_T^1 can be dominated by $k+1$ k-guards with k guards either at 1 or at 6. By Lemma 7, k guards either at 1 or at 6 permit the remainder of O_8 to be dominated by at most $k+1$ k-guards.

Case 2: $l = 6$. Let $d = \{1, 7\}$. Then G_T^1 and G_T^2 are triangulation graphs of septagons, and by Corollary 4, both of them dominated by $2k+2$ combinatorial k-guards.

Case 3: $l = 6$. This case is equivalent to Case 1.

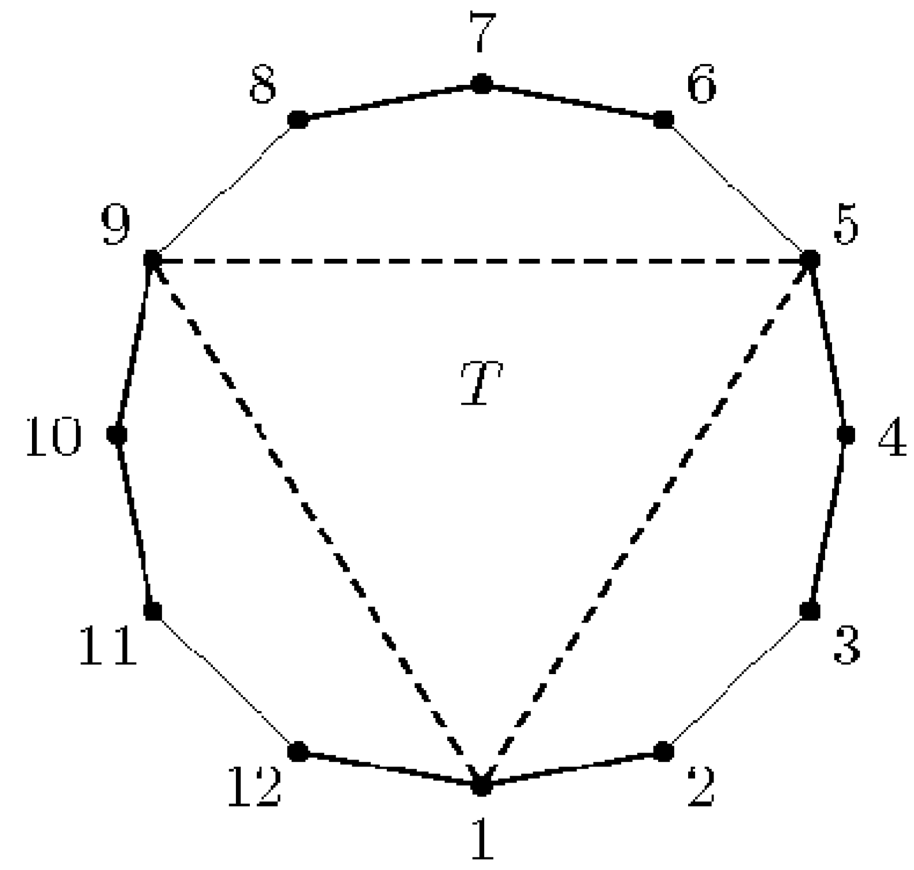

Figure 9. A domination of a 12-vertex polygon Case 4.

Case 4: $l = 7$. Let $d = \{1, 9\}$. The minimality of l ensures that the triangle T in G_T^1, which is bounded by d, is $(1, 5, 9)$, see Fig. 9. By placing one guard at vertex 1, $k-1$ guards at vertex 5, $k-1$ guards at vertex 9, and one additional guard for $(1, 2, 3, 4, 5)$, one additional guard for $(5, 6, 7, 8, 9)$, and one additional guard for $(9, 10, 11, 12, 1)$, we get a domination of G_T, and the guard set is k-guarded. QED

Thus, with all preceding lemmas available, we get the following corollary.

13 Corollary. *For all $k \geq 2$, every triangulation graph G_T of a polygon with $5 \leq n \leq 12$ vertices can be dominated by $k\lfloor \frac{n}{5} \rfloor + \lfloor \frac{n+2}{5} \rfloor$ combinatorial k-guards.*

2 Arbitrary triangulation graphs

The induction proof of the sufficiency of $k\lfloor \frac{n}{5} \rfloor + \lfloor \frac{n+2}{5} \rfloor$ k-guards for triangulation graphs with at least 13 vertices is a nearly straightforward enumeration of cases. The idea of this proof follows the main outlines of O'Rourke's proof for mobile guards: we cut off a small piece for the induction step.

14 Theorem. *For all $k \geq 2$, every triangulation graph G_T of a polygon with $n \geq 5$ vertices can be dominated by $k\lfloor \frac{n}{5} \rfloor + \lfloor \frac{n+2}{5} \rfloor$ combinatorial k-guards.*

PROOF. Corollary 13 establishes the validity of the theorem for $n = 5, \ldots, 12$, so assume that $n \geq 13$, and that the theorem holds for all $5 \leq \hat{n} < n$. The following lemma follows by the same method as in [10].

15 Lemma. *Suppose that for all $m < n$, $f(m, k)$ combinatorial k-guards are always sufficient to dominate any m-vertex triangulation graph, with at most k guards at each vertex. Then if G'_T is any triangulation graph of a polygon with n' vertices and $n' < n$, then:*

a) *k guards $g_1, \ldots, g_k$ placed at any one of vertices of G'_T with an additional $f(n'-1,k)$ combinatorial k-guards are sufficient to dominate G'_T (but, perhaps, guards $g_1, \ldots, g_k$ are only $(k-1)$-guarded);*

b) *there are at most k guards at each vertex of G'_T.*

PROOF OF LEMMA 15. Suppose that for all $m < n$, $f(m,k)$ watched guards are always sufficient to dominate any m-vertex triangulation graph, with at most k guards at a vertex, and let G'_T be a triangulation graph of a polygon P' with n' vertices, where $n' < n$. Let u be the vertex at which k guards are placed, and let v be a vertex adjacent in G'_T to u across an edge corresponding to an edge e of P'. Edge-contraction G'_T across e produces the graph G^*_T on $n'-1$ vertices. As G^*_T is a triangulation graph (see Lemma 2 in [10]), it can be dominated by $f(n'-1,k)$ k-guards, as $n'-1 < n$. Let x be the vertex that replaced u and v. Suppose that no guard is placed at x in domination of G^*_T. Then the same guard placement with k guards at u will dominate all of G'_T, since the given guards at u dominate the triangle supported by e, and the remaining triangles of G'_T have dominated counterparts in G^*_T. Otherwise, if a guard is used at x in the domination of G^*_T, more precisely, by induction hypothesis there are used at most k guards, then these guards can be assigned to v in G_T, with the remaining guards maintaining their positions. Again with k guards at u, every triangle of G'_T is dominated. Note that all guards that were k-guarded in G^*_T are k-guarded in G'_T as well. The only guards that could be non-k-guarded are the ones at vertex u. And it is clear that there are at most k guards at any vertex of G'_T. QED

Let us go back to the proof of Theorem 14. Theorem 11 guarantees the existence of a diagonal d that partitions triangulation graph G_T into two graphs G^1_T and G^2_T, where G^1_T contains l boundary edges of G_T with $5 \leq l \leq 8$. Assume that l is minimal. We consider each value of l separately.

Case 1: $l = 5$. Let $d = \{0,5\}$. Then G^1_T is a triangulation graph of hexagon $(0,1,2,3,4,5)$. In G^1_T either vertex 0 or 5 is of degree at least 3, we can assume vertex 0 to be of degree at least 3. By Lemma 5, graph G^1_T can be dominated by $k+1$ combinatorial k-guards with k guards placed at 0. Next by Lemma 15, k guards at vertex 0 permit the remainder of G^2_T to be dominated by $f(n-4-1,k) = f(n-5,k)$ k-guards, where $f(\hat{n},k)$ specifies the number of k-guards that are always sufficient to dominate a triangulation graph on $\hat{n}$ vertices. By the induction hypothesis,

$$f(n-5,k) = k\lfloor\frac{n-5}{5}\rfloor + \lfloor\frac{(n-5)+2}{5}\rfloor = k\lfloor\frac{n}{5}\rfloor + \lfloor\frac{n+2}{5}\rfloor - k - 1$$

k-guards suffice to dominate the remainder of G_T^2. Together with $k+1$ k-guards allocated to G_T^1, all of G_T is dominated by at most $k\lfloor \frac{n}{5} \rfloor + \lfloor \frac{n+2}{5} \rfloor$ k-guards.

Case 2: $l = 6$. Let $d = \{0, 6\}$. Then G_T^1 is a triangulation graph of septagon $(0, 1, 2, 3, 4, 5, 6)$. By Corollary 4, graph G_T^1 can be dominated by $k+1$ k-guards. Since graph G_T^2 has $n - 5$ vertices, it can be dominated by $k\lfloor \frac{n-5}{5} \rfloor + \lfloor \frac{(n-5)+2}{5} \rfloor$ k-guards by the induction hypothesis. This yields a domination of graph G_T by $k\lfloor \frac{n}{5} \rfloor + \lfloor \frac{n+2}{5} \rfloor$ k-guards.

Case 3: $l = 7$. Let $d = \{0, 7\}$, see Fig. 10(a). The presence of any of the diagonals $\{0, 6\}$, $\{1, 6\}$, $\{0, 5\}$, $\{2, 5\}$ would violate the minimality of l. Consequently, the triangle T in G_T^1 bounded by d is either $(0, 3, 7)$ or $(0, 4, 7)$, but without loss of generality, we can assume T to be the first one. Form graph G_T^0 by adjoining triangle T to G_T^2. G_T^0 has $n - 6 + 1$ vertices, and so can be dominated by $k\lfloor \frac{n-5}{5} \rfloor + \lfloor \frac{(n-5)+2}{5} \rfloor$ k-guards by the induction hypothesis. In such a domination, at least one of vertices of T must be assigned as a guard. There are three possibilities:

Subcase 3/0: *there is a guard at vertex* 0. By Corollary 4, $k + 1$ additional k-guards with k guards at vertex 3 suffice to dominate pentagon $(3, 4, 5, 6, 7)$. Regardless of how quadrilateral $(0, 1, 2, 3)$ is triangulated, guards at vertex 0 and 3 will dominate it.

Subcase 3/3: *there is a guard at vertex* 3. We can move this guard to vertex 0 without destroying the k-guarded domination, thus getting Subcase 3/0.

Subcase 3/7: *there is a guard at vertex* 7. Place one guard at vertex 0 and $k - 1$ guards at vertex 3. Regardless of how quadrilateral $(0, 1, 2, 3)$ is triangulated, guards at vertex 0 and 3 will dominate it. Next, it is easy to check that there is a vertex v in pentagon $(3, 4, 5, 6, 7)$ such that v is adjacent to both vertices 3 and 7, and such that by placing one guard at v, together with $k - 1$ guards at 3 and one guard at 7, we get a domination of the whole triangulation graph.

Thus all but quadrilateral $(0, 1, 2, 3)$ and pentagon $(3, 4, 5, 6, 7)$ can be dominated by $k\lfloor \frac{n}{5} \rfloor + \lfloor \frac{n+2}{5} \rfloor - k - 1$ k-guards, and the pentagon and the quadrilateral merely require together $k+1$ guards. As these $k+1$ guards with one guard either at vertex 0 or 7 in G_T^0 are k-guarded, all of G_T is dominated by $k\lfloor \frac{n}{5} \rfloor + \lfloor \frac{n+2}{5} \rfloor$ k-guards.

Case 4: $k = 8$. Let $d = \{0, 8\}$, see Fig. 10(b). The presence of any of the diagonals $\{0, 7\}$, $\{1, 8\}$, $\{0, 6\}$, $\{2, 8\}$, $\{0, 5\}$ or $\{3, 8\}$ would violate the minimality of k. Consequently, the triangle T in G_T^1 bounded by d is $(0, 4, 8)$. Dominate pentagon $(0, 1, 2, 3, 4)$ by $k+1$ k-guards with k guards at vertex 4, and dominate pentagon $(4, 5, 6, 7, 8)$ by $k+1$ k-guards with k guards at vertex 4, thus getting

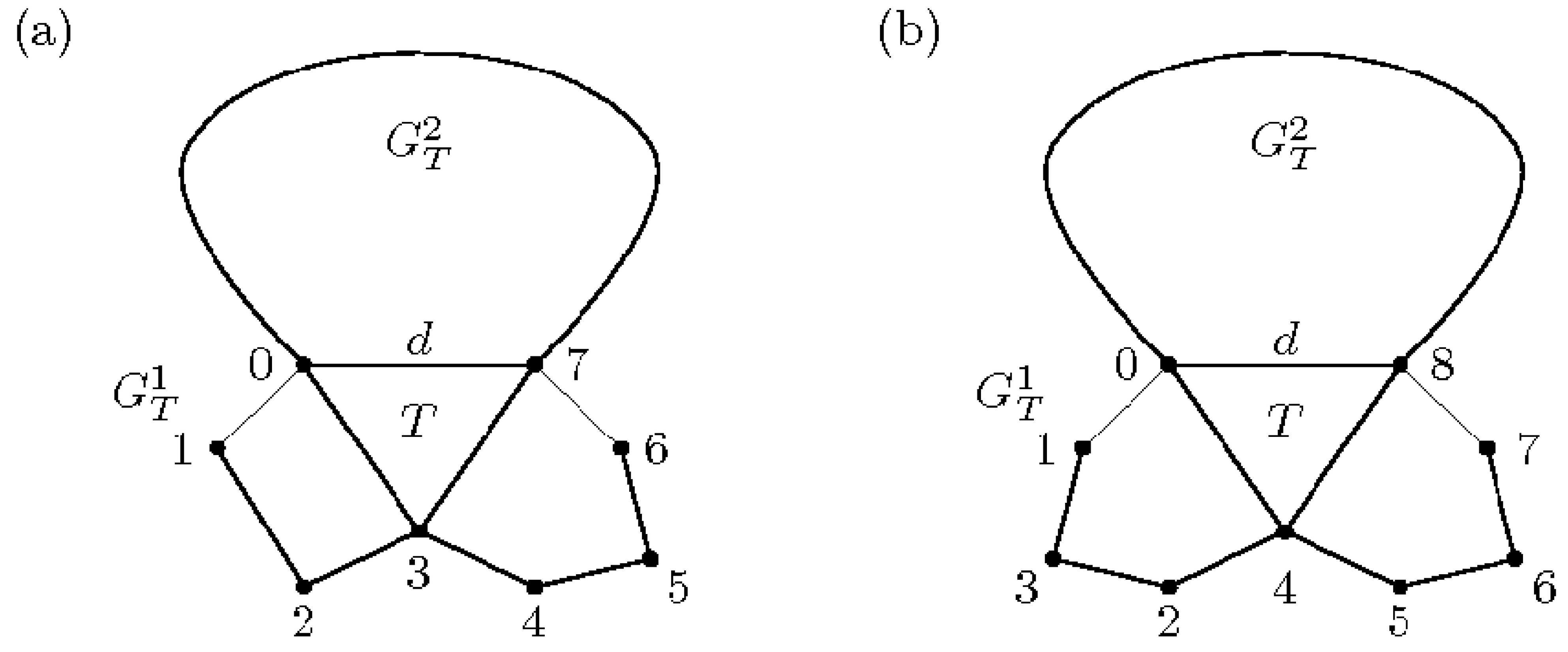

Figure 10. (a) Case $k = 7$, (b) Case $k = 8$.

a domination of G_T^1 by $k+2$ k-guards (triangle T is dominated by the guards placed at vertex 4). Next the proof proceeds in five cases, depending on the value of n mod 5.

Subcase 4/3: $n = 5t+3,\ t \geq 2$. Graph G_T^2 has $5(t-2)+3$ vertices, and it can be dominated by

$$k\lfloor\frac{5(t-1)+1}{5}\rfloor + \lfloor\frac{[5(t-1)+1]+2}{5}\rfloor = k(t-1)+t-1 = tk+t-k-1$$

k-guards by the induction hypothesis. Together with $k+2$ k-guards allocated to G_T^1, we get a domination of G_T by $tk+t+1 = k\lfloor\frac{n}{5}\rfloor + \lfloor\frac{n+2}{5}\rfloor$ k-guards.

Subcase 4/4: $n = 5t+4,\ t \geq 2$.

$$k\lfloor\frac{5(t-1)+2}{5}\rfloor + \lfloor\frac{[5(t-1)+2]+2}{5}\rfloor + k + 2 = tk+t+1 = k\lfloor\frac{n}{5}\rfloor + \lfloor\frac{n+2}{5}\rfloor.$$

Subcase 4/0: $n = 5t,\ t \geq 3$.

$$k\lfloor\frac{5(t-2)+3}{5}\rfloor + \lfloor\frac{[5(t-2)+3]+2}{5}\rfloor + k + 2 =$$

$$= tk+t-k+1 \leq tk+t = k\lfloor\frac{n}{5}\rfloor + \lfloor\frac{n+2}{5}\rfloor, \text{ as } k \geq 2.$$

Subcase 4/1: $n = 5t+1,\ t \geq 3$.

$$k\lfloor\frac{5(t-2)+4}{5}\rfloor + \lfloor\frac{[5(t-2)+4]+2}{5}\rfloor + k + 2 =$$

$$= tk+t-k+1 \leq tk+t = k\lfloor\frac{n}{5}\rfloor + \lfloor\frac{n+2}{5}\rfloor, \text{ as } k \geq 2.$$

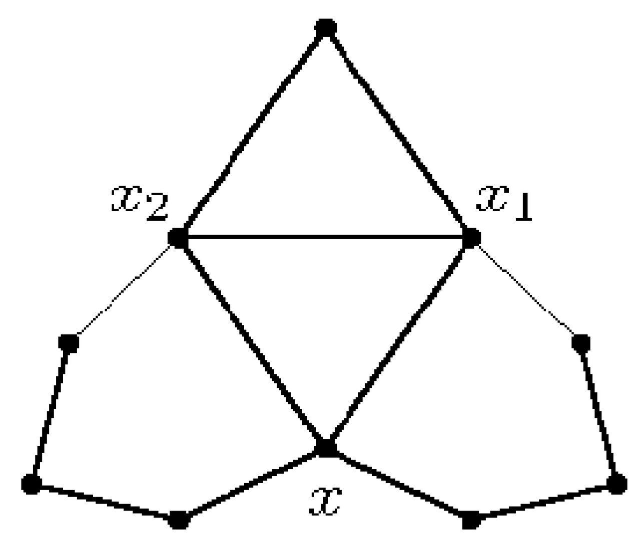

Figure 11. A 10-vertex triangulation graph without diagonals cutting off $4, 5$ or 6 vertices.

Subcase 4/2: $n = 5t + 2$, $t \geq 3$. First, let us consider the case $t = 3$, that is, $n = 17$. Then G_T^2 is a triangulation graph of a 10-vertex polygon, but the minimality of k ensures that G_T^2 has a form shown in Fig. 11. If we place $k - 1$ guards at vertex x, and one guard per each of vertices x_1 and x_2, then it is easy to see that together with at most two additional guards, we get a k-guarded guard set of G_T^2 of cardinality at most $k + 3$. This yields a domination of G_T by at most $2k + 5 \leq 3k + 3$ k-guards, as $k \geq 2$.

Now suppose that $t \geq 4$. The minimality of k and Shermer's proof of Theorem 11 (see [12]) give us more, namely there is a diagonal d' in G_T^2 such that d' partitions G_T^2 into two pieces, one of which contains 8 edges corresponding to external edges of G_T^2 and vertices 0 and 8 are left in the remainder of G_T^2. Again by minimality of k, cut off piece G_T^{21} can be dominated by $k + 2$ k-guards. Note that G_T^{22} – the remainder of G_T^2 – is now on $n - 14$ vertices. Thus we get

$$k\lfloor\frac{5(t-3)+3}{5}\rfloor + \lfloor\frac{[5(t-3)+3]+2}{5}\rfloor + 2k + 4 -$$

$$= tk + t - k + 2 \leq tk + t = k\lfloor\frac{n}{5}\rfloor + \lfloor\frac{n+2}{5}\rfloor, \text{ as } k \geq 2.$$

QED

Of course, when $n = 3$ or $n = 4$, the $(k\lfloor\frac{n}{5}\rfloor + \lfloor\frac{n+2}{5}\rfloor)$-bound fails. Clearly, $gg(3, k) = gg(4, k) = k + 1$.

3 Discussion

The k-guarded sets constructed in the previous section are multisets, and they are satisfactory if we consider the graph theory only. But geometrically speaking, it is not, as guards must not be placed at the same point. Nevertheless, we will show now that the guards at the same vertex can be always separated without destroying the k-guardness.

Let T be a non-degenerate[1] triangulation of a polygon P, let S be a k-guarded guard set for P (obtained by the method of the proof of Theorem 14), and let $V(S)$ denote the set of vertices of P at which our k-guards are placed. For a vertex v, define the $fan(v)$ as the union of all triangles of T which are incident to v. The following observations imply the existence of a disjoint set of k-guarded guard set of cardinality $k\lfloor\frac{n}{5}\rfloor + \lfloor\frac{n+2}{5}\rfloor$.

(i) We have actually proved that for any vertex $v \in V(S)$, there are only three possibilities: there is one guard located at v, or there are either $k-1$ or k guards located at v.

(ii) Let $n(v)$ denotes the number of guards located at v: we have to split up only guards at this vertex v for which $n(v) \geq k-1 \geq 2$.

(iii) Let C_v be a set of guards located at the same vertex v of $fan(v)$. Choose a vertex from C_v, let it be labeled $l(C_v)$, and call it the *leader* of C_v. Note that for any $v, w \in V(S)$, with $n(v) \geq k-1 \geq 2$, $v \neq w$, a guard $g \in C_v$ has to see at most $l(C_w) \in fan(v)$ to be k-guarded.

(iv) For a vertex v, as $fan(v)$ is a star-shaped polygon and the triangulation is non-degenerate, it easy to see that we can move and relocate all but $l(C_v)$ guards from C_v preserving the visibility to all vertices of $fan(v)$.

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