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Atomic decomposition in regulated domains

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Abstract. We obtain atomic decomposition formulas for weighted Bergman spaces $A^p_\alpha(\Omega)$, $\alpha > -1$, $1 < p < \infty$, where $\Omega \subset \mathbb{C}$ belongs to the class of regulated domains. We are able to construct an atomic decomposition directly on a given regulated domain Ω by using its geometric properties.

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1 Introduction

The concept of atomic decomposition was first introduced by Coifman and Rochberg in [3], where they proved its existence in the standard Bergman-space $A^p_\alpha(\mathbb{D})$ on the unit disk $\mathbb{D}$ with radial weight functions. In principle it is easy to generalise this result to other simply connected domains $\Omega \in \mathbb{C}$ using the Riemann mapping theorem. However, the resulting atomic decomposition in the space $A^p_\omega(\Omega)$ is very implicit and its connection to the geometry of Ω remains unclear.

In this paper we construct an atomic decomposition directly in the domain Ω using its geometric properties assuming that the domain belongs to the class of regulated domains (see [9]). We work in the Bergman-space $A^p_\alpha(\Omega)$, $1 < p < \infty$, where the weight function is of the form $z \mapsto (\operatorname{dist}(z, \partial\Omega))^\alpha$, with $\alpha > -1$.

An essential part of the study is to create a division of the regulated domain Ω into squares $Q_{n,k}$, which are small enough for the functions $f \in A^p_\alpha(\Omega)$ to be practically constant inside each square. This covering, constructed in Section 6, allows us to select the sequence of "sampling points" $\lambda_{n,k}$ in a way intrinsic to Ω. In Section 7 we show that the sequence $(\lambda_{n,k})$ does in fact define an atomic decomposition, the proof uses the same method as [11] and [8].

Important results for us are found in [2], where it was shown that the weight function $|\psi'|^{2-p}$, where $\psi : \mathbb{D} \to \Omega$ is a conformal mapping, satisfies a

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Muckenhoupt-type condition if and only if certain geometric conditions for the boundary of Ω hold. This was also found to be equivalent to the boundedness of the Bergman projection on the space $L^p(\Omega)$. These results were generalised for arbitrary regulated domains in [10] which then gave rise to this article. Also useful has been the treatment of duality between weighted Bergman spaces in [7].

2 Preliminary results

Let $\Omega \in \mathbb{C}$ be a simply connected, bounded domain with a locally connected boundary. Thus the conformal map $\psi : \mathbb{D} \to \Omega$ is continuous in $\overline{\mathbb{D}}$ and the curve $w(t) = \psi(e^{it}), 0 \leq t \leq 2\pi$ is well-defined.

A function is called *Dini-smooth* if its derivative exists and is Dini-continuous. We say that $\partial\Omega$ has a *Dini-smooth corner* at $\psi(\zeta)$ if there are two closed arcs in $\mathbb{T}$ ending at $\zeta \in \mathbb{T}$ and lying on opposite sides of ζ that are mapped onto two Dini-smooth Jordan-arcs forming the angle $\pi\gamma$ at $\psi(\zeta)$. Then we have by [9] that

1 Theorem. *If $\partial\Omega$ has a Dini-smooth corner of opening $\pi\gamma$, $0 < \gamma \leq 2$, at $\psi(\zeta) \neq \infty$, then the functions*

$$\frac{\psi(z) - \psi(\zeta)}{(z-\zeta)^\gamma} \qquad \text{and} \qquad \frac{\psi'(z)}{(z-\zeta)^{\gamma-1}} \tag{1}$$

are continuous and bounded away from 0 in $\overline{\mathbb{D}} \cap D(\zeta, \rho)$ for some $\rho > 0$.

For a conformal mapping $f : \Omega \to \Omega'$ the Koebe distortion theorem has the form

$$\frac{1}{4}\frac{\operatorname{dist}(f(z), \partial\Omega')}{\operatorname{dist}(z, \partial\Omega)} \leq |f'(z)| \leq 4\frac{\operatorname{dist}(f(z), \partial\Omega')}{\operatorname{dist}(z, \partial\Omega)}, \quad z \in \Omega. \tag{2}$$

As a corollary to (2) we have that if ψ maps $\mathbb{D}$ conformally into $\mathbb{C}$ then

$$\frac{1}{4}(1-|z|^2)|\psi'(z)| \leq \operatorname{dist}(\psi(z), \partial\psi(\mathbb{D})) \leq (1-|z|^2)|\psi'(z)| \qquad \text{for } z \in \mathbb{D}. \tag{3}$$

To combine (3) with Theorem 1 set $\varphi := \psi^{-1} : \Omega \to \mathbb{D}$, $z \in \Omega$, where $\varphi(z) \in D(\zeta, \rho) \subset \overline{\mathbb{D}}$, $\rho > 0$, and $\pi\gamma$ is the corner of opening nearest to z. Now if $0 < \gamma < 1$, we get

$$d(z) := \operatorname{dist}(z, \partial\Omega) \leq C(1-|\varphi(z)|^2)^\gamma. \tag{4}$$

According to [9], Section 3.5, Ω is called a regulated domain if each point on $\partial\Omega$ is attained only finitely often by ψ, and if

$$\beta(t) = \lim_{\tau \to t+} \arg\big(w(\tau) - w(t)\big) \tag{5}$$

exists for all t and defines a regulated function. Recall that β is regulated if it can be uniformly approximated by step functions, i.e. if for every $\varepsilon > 0$ there exist $0 = t_0 < t_1 < \cdots < t_n = 2\pi$ and constants $\gamma_1, \ldots, \gamma_n$ such that

$$|\beta(t) - \gamma_\nu| < \varepsilon \qquad \text{for} \quad t_{\nu-1} < t < t_\nu, \ \ \nu = 1, \ldots, n. \tag{6}$$

Geometrically $\beta(t) = \beta(t+)$ and $\beta(t-)$ are the direction angles of the forward and backward tangents (respectively) of $\partial\Omega$ at $w(t)$. Specifically, if $w(t)$ is a corner we determine the argument by

$$\beta(t+) - \beta(t-) = \pi(1-\gamma), \qquad \pi\gamma \ \text{opening angle.} \tag{7}$$

For more details, see [9].

3 Bergman spaces

Let $dA(z)$ be the area measure on $\mathbb{D}$ normalised so that the area of $\mathbb{D}$ is 1, thus $dA(z) = \frac{1}{\pi} dxdy$ and let $\Omega \subset \mathbb{C}$ be a regulated domain. We denote by

$$d(z)^\alpha = (\operatorname{dist}(z, \partial\Omega))^\alpha \qquad \text{for some} \ \alpha > -1 \tag{8}$$

the power of boundary distance, the simplest possible type of weight function on Ω. For $1 < p < \infty$ let the Bergman space $A^p_\alpha(\Omega)$ be the space of analytic functions $f : \Omega \to \mathbb{C}$ with the norm

$$\|f\|_{\alpha,p} = \left(\int_\Omega d(z)^\alpha |f(z)|^p \, dA(z) \right)^{\frac{1}{p}} = \left(\int_\Omega |f|^p \, dA_\alpha(z) \right)^{\frac{1}{p}} < \infty,$$

By (3) the weights (8) correspond on the open unit disk $\mathbb{D}$ to the weights

$$\nu(z) = (1 - |z|^2)^\alpha |\psi'(z)|^{2+\alpha}, \tag{9}$$

where $\psi : \mathbb{D} \to \Omega$ is a Riemann conformal map and $\varphi := \psi^{-1} : \Omega \to \mathbb{D}$. Obviously ν is in general a nonradial weight on $\mathbb{D}$, although it is easily obtained from a very natural class of weights on Ω.

In the Hilbert space A^2_α, by definition, there exists an orthogonal Bergman projection $P_\alpha : L^2_\alpha(\Omega) \to A^2_\alpha(\Omega)$ defined by the formula

$$P_\alpha f(z) = (\alpha + 1) \int_\Omega K_\alpha(z, \zeta) f(\zeta) \, dA(\zeta) \qquad z \in \Omega, \tag{10}$$

where $K_\alpha(z, \zeta)$ is the Bergman kernel of Ω or the reproducing kernel of $A^2_\alpha(\Omega)$

$$K_\alpha(z, \zeta) = \frac{(1 - |\varphi(\zeta)|^2)^\alpha |\varphi'(\zeta)|^2}{(1 - \varphi(z)\overline{\varphi(\zeta)})^{2+\alpha}}. \tag{11}$$

The projection (10) can be extended as a bounded projection from $L^p_\alpha(\Omega)$ onto $A^p_\alpha(\Omega)$, see e.g. [10].

Through the weight function $v(z) = |\psi'(z)|^{2+\alpha}$ we get a connection between the geometry of Ω and the boundedness of the Bergman projection on $L^p_\alpha(\Omega)$. It follows that the weight v satisfies the condition in Theorem 3.1 of [10]:

$$\sup_S \int_S |v|\,dA_\alpha \left(\int_S |v|^{-q/p}\,dA_\alpha\right)^{p/q} \le C m_\alpha(S)^p, \tag{12}$$

where $S(\theta,\rho) = \{\, re^{it} \in \mathbb{D} \mid 1-\rho < r < 1, |\theta - t| < 2\pi\rho \,\}$, with $0 \le \theta \le 2\pi$ and $0 < \rho < 1$, if

$$\eta_1 < \pi \qquad \text{and} \qquad \eta_2 > -\frac{p}{q}\pi, \tag{13}$$

where

$$\eta_1 = \sup_{t\in[0,2\pi]} (\beta(t+) - \beta(t-)) \qquad \text{and} \qquad \eta_2 = \inf_{t\in[0,2\pi]} (\beta(t+) - \beta(t-)). \tag{14}$$

Conversely, if $\eta_1 > \pi$ or $\eta_2 < -\frac{p}{q}\pi$, then (12) fails. Combining this with [1], Théorème 1, yields that the Bergman projection (10) is bounded on $L^p_\alpha(\Omega)$ if (13) holds.

Since η_1 and η_2 represent the biggest changes in the argument of the boundary curve (η_1 for the smallest angle, η_2 for the widest), we get from (7) that

$$\eta_1 < \pi \;\Leftrightarrow\; \gamma > 0 \qquad \text{and} \qquad \eta_2 = \pi(1-\gamma) > -\frac{p}{q}\pi \;\Leftrightarrow\; \gamma < p. \tag{15}$$

Hence outward-pointing cusps are excluded. The second condition, however, does not pose any restriction on the widest possible angle if $p > 2$, so inward-pointing cusps are allowed in these cases.

By [1] and [7] we may now derive the dual space $(A^p_\nu)^*(\mathbb{D}) = A^q_{\nu^*}(\mathbb{D})$, dual pairing with respect to $dA_\alpha(z)$ with the weight $\nu^*(z) = (1-|z|^2)^\alpha |\psi'(z)|^{-\frac{q}{p}(2+\alpha)}$. Mapping $A^q_{\nu^*}(\mathbb{D})$ back onto Ω we get the dual space $(A^p_\alpha)^*(\Omega) = A^q_{\sigma^*}(\Omega)$ with the weight function

$$\sigma^*(z) = d(z)^\alpha \omega(z), \qquad \text{where } \omega(z) = |\varphi'(z)|^{q(2+\alpha)}.$$

In the dual space $A^q_{\sigma^*}(\Omega)$ we set

$$m_{\sigma^*}(A) = \int_A d(z)^\alpha |\varphi'(z)|^{q(2+\alpha)}\,dA(z) =: m_{\alpha,\omega}(A). \tag{16}$$

For other powers of $|\varphi'(z)|$ we denote the measures with a weight of the type $d(z)^\alpha|\varphi'(z)|^p$ by m_{α,φ^p}. Also for every $f \in A^p_\alpha(\Omega), g \in A^q_{\sigma^*}(\Omega)$

$$\langle f\circ\psi \mid g\circ\psi\rangle_{\alpha,\mathbb{D}} = \int_\Omega (1-|\varphi(z)|^2)^\alpha |\varphi'(z)|^2 f(z)\overline{g(z)}\,dA(z) = \langle f \mid g\rangle_{\alpha,\Omega}. \tag{17}$$

Using the notations above, setting additionally $A^p_{\alpha,v}(\mathbb{D}) := A^p_v(\mathbb{D})$, we have

2 Lemma. *If $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha > -1$ and $f \in A^p_{\alpha,v}(\mathbb{D})$, then $f' \in A^p_{p+\alpha,v}(\mathbb{D})$ and $\|f'\|^p_{\alpha+p,v} \le C\|f\|^p_\alpha$.*

PROOF. Differentiating the identity $f(z) = P_\alpha f(z), z \in \mathbb{D}$, once under the integral sign we get the identity

$$f'(z) = (\alpha+1)(\alpha+2)\int_{\mathbb{D}} \frac{\overline{w} f(w)}{(1-z\overline{w})^{\alpha+3}}\, dA_\alpha(w). \tag{18}$$

By using the normal dual pairing (in the sense of [7]) of the spaces $A^p_{\alpha,v}(\mathbb{D})$, we see that

$$\begin{aligned} \|f'\|^p_{\alpha+p,v} &= \sup\{\, |\langle f' \mid g\rangle_{\alpha+p}| \mid \|g\|_{\alpha+p,v^{-q/p},q} \le 1\,\} \\ &= \sup\{\, |\langle f' \mid g\rangle_{\alpha+p}| \mid \|g(z)(1-|z|^2)^{p/q}\|_{\alpha,v^{-q/p},q} \le 1\,\}. \end{aligned} \tag{19}$$

Here $\|g\|_{\alpha+p,v^{-q/p},q} = \left(\int_{\mathbb{D}} |g(z)|^q (1-|z|^2)^{\alpha+p} v(z)^{-q/p}\, dA(z)\right)^{\frac{1}{q}}$ and the weight $v(z) = |\psi'(z)|^{2+\alpha}$. Using (18) for $f'(z)$ we may write

$$\begin{aligned} \langle f' \mid g\rangle_{\alpha+p} &= (\alpha+1)(\alpha+2)\int_{\mathbb{D}} f'(z)\overline{g(z)}(1-|z|^2)^{p/q}(1-|z|^2)^{p-p/q}\, dA_\alpha(z) \\ &= (\alpha+1)(\alpha+2)\int_{\mathbb{D}}\left(\int_{\mathbb{D}} \frac{\overline{w} f(w)}{(1-z\overline{w})^{\alpha+3}}\, dA_\alpha(w)\right)\overline{g(z)}(1-|z|^2)^{p/q}\, dA_{\alpha+1}(z) \\ &= (\alpha+1)\int_{\mathbb{D}} \overline{w} f(w)\left((\alpha+2)\int_{\mathbb{D}} \frac{\overline{g(z)}(1-|z|^2)^{p/q}}{(1-z\overline{w})^{\alpha+3}}\, dA_{\alpha+1}(z)\right) dA_\alpha(w) \\ &= (\alpha+1)\left\langle \overline{z} f(z) \mid P_{\alpha+1}\left(g(z)(1-|z|^2)^{p/q}\right)\right\rangle_\alpha. \end{aligned}$$

Set $G(z) = g(z)(1-|z|^2)^{p/q}$. Now the above, combined with (19) and the boundedness of the operator $P_{\alpha+1}$ (see [1]), yields

$$\begin{aligned} \|f'\|^p_{\alpha+p,v} &\le C' \sup\{\, |\langle \overline{z} f(z) \mid P_{\alpha+1}(G(z))\rangle_\alpha| \ :\ \|G(z)\|_{\alpha,v^{-q/p},q} \le 1\,\} \\ &\le C \sup\{\, |\langle \overline{z} f(z) \mid G(z)\rangle_\alpha| \ :\ \|G(z)\|_{\alpha,v^{-q/p},q} \le 1\,\} \le C\|f\|^p_\alpha, \end{aligned}$$

which proves the initial statement. QED

Lemma 2 has been taken from [8], an unpublished manuscript. By applying the Riemann mapping $\psi : \mathbb{D} \to \Omega$ a simple change of variables now shows that

3 Corollary. *If $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha > -1$ and $f \in A^p_\alpha(\Omega)$, then $f' \in A^p_{\alpha+p,\varphi^p}(\Omega)$ and moreover $\|f'\|^p_{\alpha+p,\varphi^p} \le C\|f\|^p_\alpha$.*

4 Domain Ω and related definitions

The presentation in the following sections is very detailed, since we want to control the various constants. This is needed in the example of Section 8.

Let Ω be a bounded, regulated and simply connected domain such that $\partial\Omega$ consists of finitely many arcs $C_n \in C^\infty$ and every point on $\partial\Omega$ is attained only once. The angles $\pi\gamma$ (or corners of opening (7)) between adjacent arcs C_n may vary within $0 < \gamma < \min\{2, p\}$ as in (15) including equality in the right-hand side if $p > 2$. Thus, as mentioned before, no outward-pointing cusps are allowed in Ω, but if $p > 2$, then inward-pointing cusps are accepted. Let Ω also be such that it fits into a disk of radius 1.

Let $\alpha > -1$ be fixed and set $\pi\gamma_s$ to be the sharpest and $\pi\gamma_w$ the widest opening of a corner on $\partial\Omega$. These exist, because the number of corners is finite. Then set

$$\gamma_0 = \min\{\gamma_s, 1 - \frac{\gamma_w}{p}\}, \tag{20}$$

where clearly $0 < \gamma_0 < 1$, and let b be a constant for which

$$b = \frac{\gamma_0}{2 - \gamma_0}, \tag{21}$$

hence $0 < b < 1$ for all possible values of γ_0. The reason for defining γ_0 and b as above will become clear later in (41). Intuitively γ_0 tells how "close" the "worst" (sharpest or widest) corner is to the critical values of $\gamma = 0$ and $\gamma = p$. The constant b makes the set of belts Γ_n (22) more dense accordingly.

To construct a sequence in Ω connected to its geometry we will divide Ω in smaller sets (see Section 6). For this purpose we define the sets

$$\Gamma_n = \{\, z \in \Omega \mid (n+1)^{-b} \le d(z) \le n^{-b} \}, \qquad 0 < b < 1, \tag{22}$$

to be a decreasing sequence of belts in Ω parallel to the boundary $\partial\Omega$.

We call δ_n the "width" of a belt Γ_n and define

$$\delta_n = n^{-b} - (n+1)^{-b}. \tag{23}$$

Although the definition is not precise and the actual width of the belts may locally differ even significantly from δ_n when n is small, these problems do not arise anymore when n grows. Since we are mainly concerned with what happens near the border $\partial\Omega$, the concept of "width" as defined above is justified.

There is a correspondence between the width of a belt δ_n and the distance to the border $d(z)$ as

$$\frac{b}{4} d(z)^{\frac{b+1}{b}} \le \delta_n \le 4(b+1) d(z)^{\frac{b+1}{b}}. \tag{24}$$

By (4) we may also pronounce (24) as

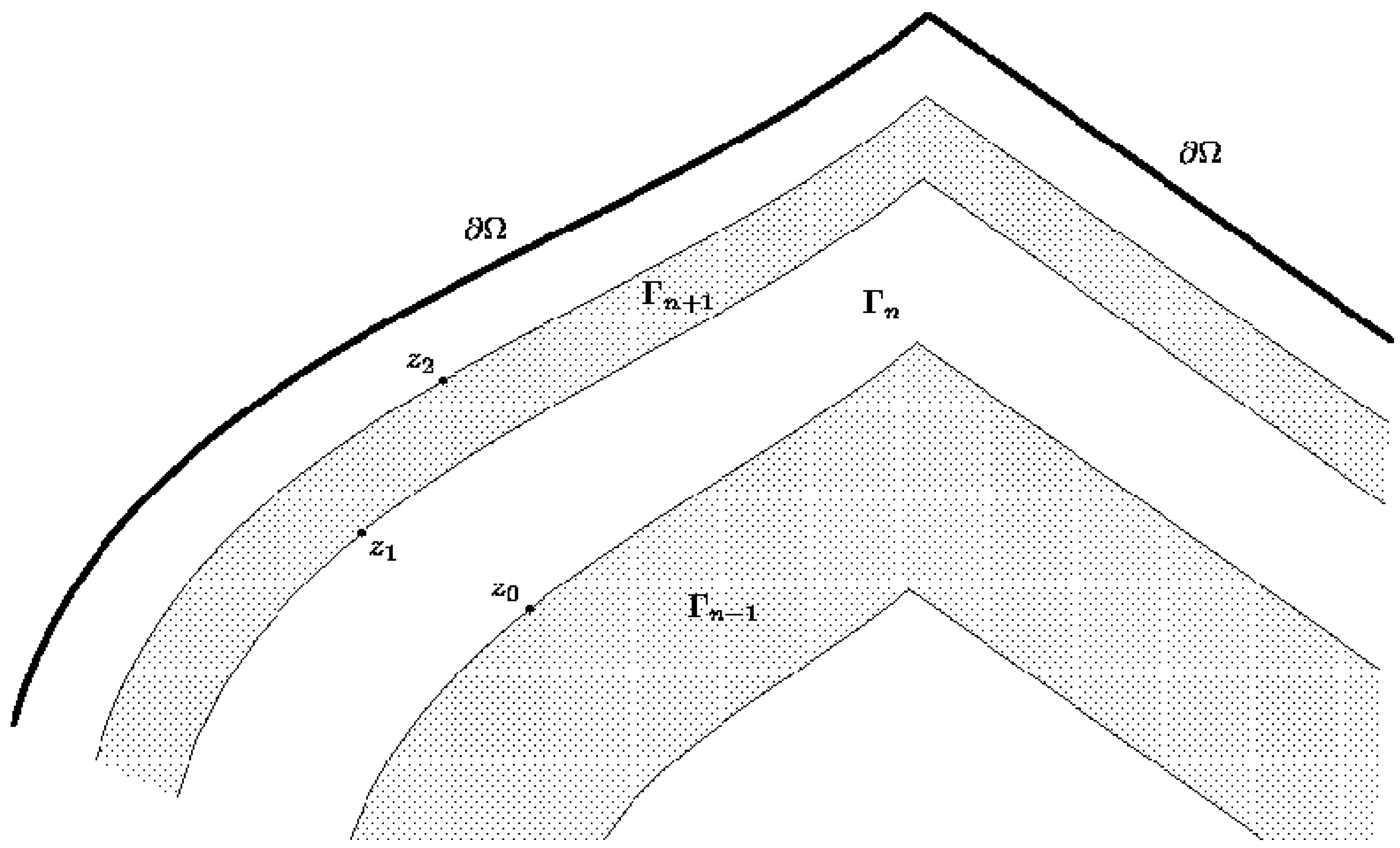

Figure 1. Belts Γ_{n-1},Γ_n and Γ_{n+1} in the domain Ω. Now $d(z_0) = n^{-b}$, $d(z_1) = (n+1)^{-b}$ and $d(z_2) = (n+2)^{-b}$.

4 Lemma. *Let Ω be as above. There exists a constant $C_2' > 0$ such that*

$$d(z) \le C_2' |\varphi'(z)|^{\frac{\gamma}{1-\gamma}}, \qquad \gamma \ne 1 \tag{25}$$

(and $d(z) \le C_2'$, when $\gamma = 1$) locally near every corner with opening $\pi\gamma$.

PROOF. From Theorem 1 we see that there exist such constants C and C' that when $\pi\gamma$, $0 < \gamma < 2$, is the opening of a corner at $\psi(\zeta) \in \partial\Omega$, then $C'|\varphi'(z)| \le |\varphi(z) - \zeta|^{1-\gamma} \le C|\varphi'(z)|$ in $\overline{\mathbb{D}} \cap D(\zeta, \rho)$ for some $\rho > 0, z \in \Omega$. By (3) and (4) we now get that

$$d(z) \le (1 - |\varphi(z)|^2)|\varphi'(z)|^{-1} \le C_2' |\varphi'(z)|^{\frac{\gamma}{1-\gamma}}, \tag{26}$$

which proves the statement for $\gamma \ne 1$.

When $\gamma = 1$ Theorem 1 implies that $0 < C' < |\varphi'(z)| < C$, which proves the claim. QED

Lemma 4 is valid only locally, every corner having its own constant C_2'. Fortunately there are only a finite number of corners in $\partial\Omega$, which enables us to choose the maximum out of all candidates. This makes it feasible to say that

5 Corollary. *There exists a constant $C_2 > 0$ (depending on Ω) such that*

$$d(z) \le C_2 |\varphi'(z)|^{\frac{\gamma_0}{1-\gamma_0}} \qquad \textit{for all } z \in \Omega, \tag{27}$$

where γ_0 is as in (20).

5 Integral estimates in Ω

According to [11] analytic functions on the unit disk $\mathbb{D}$ are "subharmonic" in the hyperbolic metric, that is, there exists a constant $C > 0$, independent of r and p, such that

$$|f(z)|^p \leq \frac{C}{|D(z,r)|} \int_{D(z,r)} |f(w)|^p \, dA(w) \tag{28}$$

for all analytic f, $z \subset \mathbb{D}$, $p > 0$ and $r \leq 1$, $|D(z,r)|$ is the area of a hyperbolic disk $D(z,r)$.

We will prove that (28) is valid also for an analytic $f \in \Omega$ with the integral taken over a small square Q. The outline of the proof is to map Ω onto the unit disk and to show that the squares approximate the disks $|D(z,r)|$ closely enough. We begin by showing how much the images of sets in Ω (under the Riemann mapping φ) are distorted.

6 Lemma. *If $Q \subset \Omega$ is a small square far from $\partial\Omega$, that is, if for its sidelength ℓ we have*

$$\ell \leq c d(z)^{1+\frac{1}{b}} \quad \textit{for all} \quad z \in Q, \quad 0 < c << 1 \tag{29}$$

and if $\sup_{\xi \in Q} d(\xi) \leq (1+c) \inf_{\xi \in Q} d(\xi)$, again with $0 < c << 1$, then its image $\varphi(Q) \subset \mathbb{D}$ is small. Especially then

$$\sup_{z \in Q}(1 - |\varphi(z)|^2) \leq (1+c) \inf_{z \in Q}(1 - |\varphi(z)|^2), \tag{30}$$

and

$$\sup_{w \in Q} |\varphi'(w)| \leq (1+c)^2 \inf_{w \in Q} |\varphi'(w)|, \qquad \textit{with } 0 < c << 1. \tag{31}$$

In condition (29) we actually set $\ell << \delta_n$, where δ_n is the "width" of the belt Γ_n (23) in Ω such that $Q \cap \Gamma_n \neq \varnothing$. Thus by (24) we have (29).

PROOF. The result follows by basic use of the Koebe distortion theorem (2). QED

An immediate consequence of Lemma 6 is the following:

7 Proposition. *Let Q be as in Lemma 6, $r > 0$. Then there exist disks $D \subset \mathbb{D}$ such that*

$$D(z, \frac{r}{2}) \subset \varphi(Q) \subset D(z, 2r), \tag{32}$$

where $z = \varphi(x), x \in Q$ and x is the center of Q.

From (28) and Proposition 7 it now follows that

8 Corollary. *If $Q \subset \Omega$ is such that Proposition 7 holds, then*

$$|f(a)|^p \le \frac{C_0}{m(Q)} \int_Q |f(w)|^p \, dA(w) \tag{33}$$

for all analytic $f : \Omega \to \mathbb{C}$, $a \in Q$, $p > 0$.

PROOF. Using the previous results, we get for $a, \lambda, \zeta \in \Omega$ and $b, z, w \in \mathbb{D}$, $b = \varphi(a)$ that

$$\begin{aligned}|f(\psi(b))|^p &\le \frac{C}{|D(z, \frac{r}{2})|} \int_{D(z,\frac{r}{2})} |f \circ \psi(w)|^p \, dA(w) \\ &\le \frac{C}{|D(z, 2r)|} \int_{\varphi(Q)} |f \circ \psi(w)|^p \, dA(w) \le \frac{C}{m(\varphi(Q))} \int_{\varphi(Q)} |f \circ \psi(w)|^p \, dA(w).\end{aligned}$$

Naturally

$$m(\varphi(Q)) = \int_{\varphi(Q)} dA(w) = \int_Q |\varphi'(\zeta)|^2 \, dA(\zeta) \ge \min_{\lambda \in Q} |\varphi'(\lambda)|^2 m(Q),$$

which implies that

$$\begin{aligned}&\frac{C}{m(\varphi(Q))} \int_{\varphi(Q)} |f \circ \psi(w)|^p \, dA(w) \\ &\qquad \le \frac{C}{\inf_{\lambda \in Q} |\varphi'(\lambda)|^2 m(Q)} \int_Q |f(\zeta)|^p |\varphi'(\zeta)|^2 \, dA(\zeta) \\ &\qquad \le \frac{(1+c)^4 C}{m(Q)} \int_Q |f(\zeta)|^p \, dA(\zeta),\end{aligned}$$

by (31). Now (33) follows by fixing the constant C_0. QED

Now we are in a position to formulate

9 Lemma. *Let $\alpha > -1$ be the one fixed before and $1 \le p < \infty$. Then there exists a constant $C_1 = C_1(p, \alpha) \ge 1$ as follows: If $Q \subset \Omega$ is a square as in Lemma 6, then for all analytic f in Ω and for all $a \in \Omega$, we have*

$$|f(a)|^p \le C_1 \int_Q |f(z)|^p \, dA_{\alpha+p,\varphi^p}(z) \Big/ m_{\alpha+p,\varphi^p}(Q), \quad a \in Q. \tag{34}$$

This fixes the constant C_1.

PROOF. Since, by definition, the images $\varphi(Q)$ are included in hyperbolic disks $D(z, 2r) \subset \mathbb{D}$, we have by Corollary 8, that

$$|f(a)| \le C_0 \int_Q |f(z)| \, dA(z) \Big/ \int_Q dA(z),$$

where $a \in Q$, recall that $\int_Q dA(z) = m(Q)$. Here we can replace $dA(z)$ by $dA_{\alpha+p}(z)$, since for $z \in Q$, function $d(z)$ is equivalent to $d(a)$ within constants independent of a. The same is also true for $|\varphi'(z)|$ by (31). So we get, because $p = p/q + 1$, that

$$|f(a)|^p \le C_0 \left(\int_Q |f(z)|^p |\varphi'(z)|^1 |\varphi'(z)|^{-1} \, dA_{\alpha+p}(z) \right)^p \Big/ m_{\alpha+p}(Q)^p$$

$$\le C_0 \left(\int_Q |f(z)|^p |\varphi'(z)|^p |\varphi'(z)|^{-p} \, dA_{\alpha+p}(z) \right) \left(\int_Q dA_{\alpha+p}(z) \right)^{p/q} \Big/ m_{\alpha+p}(Q)^p$$

$$\le C_0 \left(\int_Q |f(z)|^p |\varphi'(z)|^p \, dA_{\alpha+p}(z) \right) |\varphi'(a)|^{-p} m_{\alpha+p}(Q)^{p/q} m_{\alpha+p}(Q)^{-\frac{p}{q}-1}$$

$$\le C_0 \left(\int_Q |f(z)|^p \, dA_{\alpha+p,\varphi^p}(z) \right) \Big/ \left(|\varphi'(a)|^p \int_Q dA_{\alpha+p}(z) \right)$$

$$\le C_0 C \int_Q |f(z)|^p \, dA_{\alpha+p,\varphi^p}(z) \Big/ m_{\alpha+p,\varphi^p}(Q),$$

where we used (31) and replaced $dA(z)$ by $dA_\alpha(z)$ as above. QED

Combining some of the specifically named constants created above we let

$$C_b = 8(b+1) C_2^{(1-b)/2b} (C_1^2 + C_1)^{1/p}. \tag{35}$$

6 Division of Ω and the sequence $(\lambda_{n,k})$

Let $Q_{n,k}$ be half-open squares with the sidelengths $l_n = l(Q_n)$ depending on n by

$$l_n = \frac{\delta_n}{C_b}, \tag{36}$$

where C_b is the constant in (35) and δ_n represents the "width" of a belt Γ_n defined in (23). For $n \in \mathbb{Z}$, $k \in \mathbb{Z}^2$ we define the squares by setting

$$Q_{n,k} = \{ (x_1, x_2) \mid l_n k_i \le x_i < l_n(k_i + 1), \ i = 1, 2 \} \tag{37}$$

and thus $\bigcup_k Q_{n,k} = \mathbb{C}$ for every fixed $n \in \mathbb{Z}$ and $Q_{n,m} \cap Q_{n,p} = \varnothing$, when $m \ne p$.

We may now use the squares $Q_{n,k}$ to cover Ω. For each Γ_n we assign respectively a collection $\mathbf{Q_n}$ of the squares $Q_{n,k}$ as follows

$$Q_{n,k} \in \mathbf{Q_n} \iff Q_{n,k} \cap \Gamma_n \ne \varnothing. \tag{38}$$

Obviously there are only finitely many squares which satisfy condition (38), so we assign J_n to be the number of squares in each $\mathbf{Q_n}$. Assume then that the sequence $\left(Q_{n,k}\right)_{k=1}^{J_n}$ is re-indexed such that for all $k = 1, \ldots, J_n$, we have $Q_{n,k} \in \mathbf{Q_n}$.

10 Remark. Every $Q_{n,k} \in \mathbf{Q_n}$ satisfies for all n, k the requirements of Lemma 6. This follows from (36) and the properties of δ_n and $d(z)$, see (24).

11 Definition. For every $Q_{n,k} \in \mathbf{Q_n}$ we denote by $\lambda_{n,k}$ the center of the square $Q_{n,k}$.

Since $Q_{n,l} \cap Q_{n,m} = \varnothing$ for all $n \in \mathbb{Z}, l \neq m$, it is clear that inside the family $\mathbf{Q_n}$ a point $\lambda_{n,k}$ may belong to only one $Q_{n,k}$. It is possible that the squares in the adjoining families $\mathbf{Q_{n \pm k}}$ would overlap the squares in $\mathbf{Q_n}$.

Assume $\lambda_{n,k} \in Q_{n+2,k}$ for some $k \in J_{n+2}$. According to Definition 11 there exists also a point $\lambda_{n+2,k} \in Q_{n+2,k}$. Clearly, as $\lambda_{n,k} \in Q_n$ and $\lambda_{n+2,k} \in \Gamma_{n+2}$

$$\sqrt{2} \cdot l_{n+2} = \sqrt{2} \cdot \frac{\delta_{n+2}}{C_b} < \delta_{n+2} < \delta_{n+1} \leq \operatorname{dist}(\lambda_{n,k}, \lambda_{n+2,k}), \tag{39}$$

but this is a contradiction since both points belong to the same square and thus we should have $\operatorname{dist}(\lambda_{n,k}, \lambda_{n+2,k}) \leq \sqrt{2} \cdot l_{n+2}$. Using the same argument for $\lambda_{n-2,k}$ and $\lambda_{n,k}$ we see that the points $\lambda_{n,k}$ belong to at most 3 squares.

Clearly now $\Omega = \bigcup_{n,k} Q_{n,k}$ and every $z \in \Omega$ belongs to at most 3 sets $Q_{n,k}$. From the definition above it is also easy to see that $(Q_{n,k} \cap \Gamma_n) \cap (Q_{m,l} \cap \Gamma_m) = \varnothing$ when $(n, k) \neq (m, l)$.

Remark 10 implies that for $z \in Q_{n,k}$ the functions $d(z)$ and $|\varphi'(z)|$ are equivalent to $d(\lambda_{n,k})$ and $|\varphi'(\lambda_{n,k})|$ respectively. Thus

$$C' d(\lambda_{n,k})^\alpha |\varphi'(\lambda_{n,k})| m(Q_{n,k}) \leq m_{\alpha,\varphi}(Q_{n,k}) \leq C d(\lambda_{n,k})^\alpha |\varphi'(\lambda_{n,k})| m(Q_{n,k}). \tag{40}$$

By Remark 10, Proposition 7 and Lemma 9 hold and we may prove that

12 Lemma. *Let $1 < p < \infty$, $\alpha > -1$ and $f \in A^p_\alpha(\Omega)$. For all $\lambda_{n,k} \in \Omega$ we have, that*

$$\sum_{n,k} d(\lambda_{n,k})^\alpha m_{p,\varphi^p}(Q_{n,k}) |f(\lambda_{n,k})|^p \leq C \int_\Omega |f(z)|^p \, dA_{\alpha+p,\varphi^p}(z)$$

holds for all $f \in A^p_{\alpha,\varphi}$.

Here and later $\sum_{n,k} = \sum_{n=1}^\infty \sum_{k=1}^{J_n}$.

Proof. Since the distance $d(\lambda_{n,k})$ is equivalent to $d(z)$ for all $z \in Q_{n,k}$, we have $d(\lambda_{n,k})^\alpha m_{p,\varphi^p}(Q_{n,k}) \leq C m_{\alpha+p,\varphi^p}(Q_{n,k})$. Thus we get by Lemma 9, that

$$d(\lambda_{n,k})^\alpha m_{p,\varphi^p}(Q_{n,k}) |f(\lambda_{n,k})|^p \leq C_1 \int_{Q_{n,k}} |f(z)|^p \, dA_{\alpha+p,\varphi^p}(z).$$

As any point in Ω belongs to at most 3 of the sets $Q_{n,k}$ and $Q_{n,k} \cap Q_{n,k'} = \varnothing$ for $k \neq k'$, the statement follows. QED

By combining (24) with Corollary 5 it follows that

$$\begin{aligned} l_n^{1/2} &\leq C'd(z)^{\frac{b+1}{2b}} = C'd(z)^{1/\gamma_0} = C'd(z)d(z)^{\frac{1-\gamma_0}{\gamma_0}} \\ &\leq 2C_b^{-1/2}(b+1)^{1/2}C_2^{\frac{1-b}{2b}}d(z)|\varphi'(z)|, \end{aligned} \tag{41}$$

with $C' = 2C_b^{-1/2}(b+1)^{1/2}$. On the other hand, for l_n it is also true, that

$$l_n = \delta_n C_b^{-1} \leq 4(b+1)C_b^{-1}n^{-(b+1)} \leq 4(b+1)C_b^{-1}. \tag{42}$$

Together, we have

$$l_n \leq \frac{4(b+1)}{C_b}C_2^{\frac{1-b}{2b}}d(z)|\varphi'(z)| \leq \frac{1}{2(C_1^2+C_1)^{1/p}}d(z)|\varphi'(z)|. \tag{43}$$

From Corollary 3 and (43) we have

13 Lemma. *Let $1 < p < \infty, \alpha > -1$. For all $f \in A^p_\alpha(\Omega)$, we have*

$$\sum_{n,k}\int_{Q_{n,k}}|f(z)-f(\lambda_{n,k})|^p\,dA_\alpha(z) \leq \frac{1}{C_1+1}\int_\Omega |f(z)|^p\,dA_\alpha(z), \tag{44}$$

where C_1 is the constant in (34).

PROOF. Since $|z-\lambda_{n,k}| \leq 2l_n$ for any $z \in Q_{n,k}$, using (43) we may estimate

$$\begin{aligned} |f(z)-f(\lambda_{n,k})| &\leq 2l_n \sup_{w\in Q_{n,k}}\{|f'(w)|\} \\ &\leq \frac{1}{(C_1^2+C_1)^{1/p}}d(\lambda_{n,k})|\varphi'(z)| \sup_{w\subset Q_{n,k}}\{|f'(w)|\}. \end{aligned}$$

Hence, using Lemma 9, we get for all n and k that

$$\begin{aligned} \int_{Q_{n,k}}|f(z)-f(\lambda_{n,k})|^p\,dA_\alpha(z) &\leq \frac{1}{C_1+1}m_{\alpha+p,\varphi}(Q_{n,k}) \sup_{w\in Q_{n,k}}\{|f'(w)|^p\} \\ &\leq \frac{1}{C_1+1}\int_{Q_{n,k}}|f'(z)|^p\,dA_{\alpha+p,\varphi}(z) \end{aligned}$$

Here we have used the fact that for $z \in Q_{n,k}$ the distance $d(\lambda_{n,k})$ is equivalent to $d(z)$. The result now follows from Corollary 3. QED

7 Atomic decomposition

In atomic decomposition every function $f \in A^p_\alpha$ is represented by a unique linear combination $f(z) = \sum_{n,k}(a_{n,k})k(z,\lambda_{n,k})$, z, $\lambda_{n,k} \in \Omega$, of atoms k which in A^p_α consist of the reproducing kernel $K_\alpha(z,\lambda_{n,k})$ and an appropriate weight function. The one-to-one correspondence between functions $f \in A^p_\alpha$ and the sequence of coefficients $(a_{n,k}) \in l^p$ means that the spaces l^p and A^p_α are isomorphic. By [6], Thm. 2.a.3, p.54, we have that

14 Theorem. *Let $1 \leq p < \infty$. Then every infinite-dimensional complemented subspace of l^p is isomorphic to l^p.*

Recall that a closed subspace G of a Banach space B is complemented if and only if there exists a continuous projection $P: B \to B$ such that $P(B) = G$.

Obviously we now need to define an isomorphism from A^p_α to a complemented subspace of l^p using the sequence $(\lambda_{n,k})$ from Definition 11. Thus we define operators $R: A^p_\alpha(\Omega) \to l^p$, $S: A^p_\alpha(\Omega) \to A^p_\alpha(\Omega)$, and $T: l^p \to A^p_\alpha(\Omega)$, as follows:

$$\begin{aligned} (Rf)_{n,k} &= r_{n,k} f(\lambda_{n,k}), \quad f \in A^p_\alpha(\Omega) \\ Sf(z) &= \sum_{n,k} s_{n,k} \frac{f(\lambda_{n,k})}{(1-\varphi(z)\overline{\varphi(\lambda_{n,k})})^{2+\alpha}}, \quad f \in A^p_\alpha(\Omega) \\ T\left((a_{n,k})\right)(z) &= \sum_{n,k} a_{n,k} \frac{t_{n,k}}{(1-\varphi(z)\overline{\varphi(\lambda_{n,k})})^{2+\alpha}}, \quad (a_{n,k}) \in l^p. \end{aligned} \tag{45}$$

The coefficients r, s and t in (45) are such that

$$\begin{aligned} r_{n,k} &= (1-|\varphi(\lambda_{n,k})|^2)^\alpha |\varphi'(\lambda_{n,k})|^2 \frac{m(Q_{n,k})}{m_{\alpha,\omega}(Q_{n,k})^{1/q}}, \\ s_{n,k} &= (1-|\varphi(\lambda_{n,k})|^2)^\alpha |\varphi'(\lambda_{n,k})|^2 m(Q_{n,k}), \\ t_{n,k} &= m_{\alpha,\omega}(Q_{n,k})^{1/q}, \end{aligned} \tag{46}$$

where $(\lambda_{n,k})$ is the sequence in Ω defined in Definition 11 and $Q_{n,k}$ is the disjoint decomposition of Ω. The coefficients (46) are equivalent, after a slight effort of tidying up, to the following:

$$\begin{aligned} r'_{n,k} &= d(\lambda_{n,k})^{\frac{1}{p}(\alpha+\kappa)}, \quad s'_{n,k}(z) = d(\lambda_{n,k})^{\kappa} K_\alpha(z,\lambda_{n,k}) \quad \text{and} \\ t'_{n,k}(z) &= d(\lambda_{n,k})^{\frac{\kappa}{q}-\frac{\alpha}{p}} K_\alpha(z,\lambda_{n,k}), \end{aligned} \tag{47}$$

where $\kappa = 2 + \frac{2}{b} = \frac{4}{\gamma_0}$.

Directly from the definition (45) it is easy to see, that $TRf(z) = Sf(z)$. To verify that the operators are continuous we get

15 Lemma. *The operators* $R : A^p_\alpha \to l^p, S : A^p_\alpha \to A^p_\alpha$ *and* $T : l^p \to A^p_\alpha$ *are bounded, when* $p > 1$.

Proof. Let $\frac{1}{p} + \frac{1}{q} = 1$ and $f \in H^\infty \subset A^q_{\sigma^*}$, where H^∞ stands for the space of bounded analytic functions. Since $d(z)^\alpha$ is approximately equal to $d(\lambda_{n,k})^\alpha$, it is fairly straightforward to see, using (17), that

$$\begin{aligned}\langle T((a_{n,k})) \mid f \rangle_\Omega &= (\alpha+1) \int_\Omega (1-|\varphi(z)|^2)^\alpha |\varphi'(z)|^2 \sum_{n,k} T((a_{n,k}))\overline{f(z)}\, dA(z) \\ &= \sum_{n,k} a_{n,k} m_{\alpha,\omega}(Q_{n,k})^{1/q} (\alpha+1) \int_\Omega \frac{(1-|\varphi(z)|^2)^\alpha |\varphi'(z)|^2}{(1-\overline{\varphi(\lambda_{n,k})}\varphi(z))^{q(2+\alpha)}} \overline{f(z)}\, dA(z) \\ &= \sum_{n,k} a_{n,k} m_{\alpha,\omega}(Q_{n,k})^{1/q} \overline{f(\lambda_{n,k})}.\end{aligned}$$

Using the inequality of Hölder and Lemma 9 and recalling that $m_{\alpha,\omega}(Q_{n,k})^{1/q} \le Cd(\lambda_{n,k})^{\alpha/q} m_{0,\omega}(Q_{n,k})^{1/q}$ by (40) we see that there exists a constant $C > 0$, such that

$$\begin{aligned}|\langle T((a_{n,k}))(z) \mid f \rangle_\Omega| &\le C \sum_{n,k} \left| a_{n,k} d(\lambda_{n,k})^{\frac{\alpha}{q}} m_{0,\omega}(Q_{n,k})^{1/q} \overline{f(\lambda_{n,k})} \right| \\ &\le \left(C \sum_{n,k} |a_{n,k}|^p \right)^{\frac{1}{p}} \left(C \sum_{n,k} d(\lambda_{n,k})^\alpha m_{0,\omega}(Q_{n,k}) \left| \overline{f(\lambda_{n,k})} \right|^q \right)^{\frac{1}{q}} \\ &\le C \|(a_{n,k})\|_{l^p} C \left(\int_\Omega d(z)^\alpha |\varphi'(\lambda_{n,k})|^{q(2+\alpha)} |f(z)|^q\, dA(z) \right)^{\frac{1}{q}} \\ &= C \|(a_{n,k})\|_{l^p} \|f\|_{A^q_{\sigma^*}}\end{aligned}$$

by Lemma 12. We have shown that $\sup \{ |\langle T((a_{n,k}))(z) \mid f \rangle| \}$ is finite, when $f \in H^\infty$ goes through the unit ball of $A^q_{\sigma^*}$. Because H^∞ is dense in $A^q_{\sigma^*}$, the same applies also when $f \in A^q_{\sigma^*}$. Thus $T : l^p \to A^p_\alpha$ is bounded, because $(A^p_\alpha)^* \cong A^q_{\sigma^*}$. For R we have

$$r_{n,k} \le C \frac{d(\lambda_{n,k})^\alpha |\varphi'(\lambda_{n,k})|^{2+\alpha} m(Q_{n,k})}{d(\lambda_{n,k})^{\alpha/q} |\varphi'(\lambda_{n,k})|^{2+\alpha} m(Q_{n,k})^{1/q}} = Cd(\lambda_{n,k})^{\frac{\alpha}{p}} m(Q_{n,k})^{\frac{1}{p}},$$

and therefore

$$\begin{aligned}\|Rf\|^p_{l^p} &\le C \sum_{n,k} d(\lambda_{n,k})^\alpha |f(\lambda_{n,k})|^p m(Q_{n,k}) \\ &\le C' \int_\Omega |f(z)|^p d(z)^\alpha\, dA(z) = C' \|f\|^p_\alpha.\end{aligned}$$

Thus R is also bounded. The boundedness of S follows from the fact that $\|S\| = \|TR\|$. QED

16 Lemma. *If $p > 1$, the operator S is invertible in A^p_α.*

PROOF. Let $\frac{1}{p} + \frac{1}{q} = 1$ and $A = I - S$, where I is the identity operator. For S to be invertible, by using the Neumann series it is enough to show that $\|A\| < 1$. Let $\langle \cdot \mid \cdot \rangle_\Omega = \langle \cdot \circ \psi \mid \cdot \circ \psi \rangle_{\mathbb{D}}$ be the normal inner product and let $f, g \in H^\infty$ be fixed. Thus it follows that

$$
\begin{aligned}
\langle Af \mid g \rangle_\Omega &= \int_\Omega (1 - |\varphi(z)|^2)^\alpha |\varphi'(z)|^2 f(z)\overline{g(z)}\, dA(z) \\
&\quad - \sum_{n,k} (1 - |\varphi(\lambda_{n,k})|^2)^\alpha |\varphi'(\lambda_{n,k})|^2 m(Q_{n,k}) f(\lambda_{n,k}) \\
&\qquad \cdot (\alpha + 1) \int_\Omega \frac{(1 - |\varphi(z)|^2)^\alpha |\varphi'(z)|^2}{(1 - \overline{\varphi(\lambda_{n,k})}\varphi(z))^{2+\alpha}} \overline{g(z)}\, dA(z) \\
&= \int_\Omega (1 - |\varphi(z)|^2)^\alpha |\varphi'(z)|^2 f(z)\overline{g(z)}\, dA(z) \\
&\quad - \sum_{n,k} \int_{Q_{n,k}} f(\lambda_{n,k})\overline{g(\lambda_{n,k})}(1 - |\varphi(\lambda_{n,k})|^2)^\alpha |\varphi'(\lambda_{n,k})|^2\, dA(z),
\end{aligned}
$$

because $m(Q_{n,k})$ is the area of $Q_{n,k}$. We then choose a $Q'_{n,k} \subset Q_{n,k}$ for every n, k, such that $\bigcup_{n,k} Q'_{n,k} = \Omega$ is a disjoint union. (This is always possible, in cases where $\lambda_{n,k} = \lambda_{n+1,k'}$ we put simply $Q'_{n,k} = Q_{n,k} \setminus Q_{n+1,k}$ and $Q'_{n+1,k'} = Q_{n-1,k}$.) Then

$$
\begin{aligned}
\langle Af \mid g \rangle_\Omega = \sum_{n,k} \int_{Q'_{n,k}} \Big(& f(z)\overline{g(z)}(1 - |\varphi(z)|^2)^\alpha |\varphi'(z)|^2 \\
& - f(\lambda_{n,k})\overline{g(\lambda_{n,k})}(1 - |\varphi(\lambda_{n,k})|^2)^\alpha |\varphi'(\lambda_{n,k})|^2 \Big)\, dA(z),
\end{aligned} \tag{48}
$$

Set $W(z) = (1 - |\varphi(z)|^2)^\alpha |\varphi'(z)|^2$. The integral in (48) may now be split into three different integrals by adding and subtracting appropriate terms. Then

$$
\begin{aligned}
\langle Af \mid g \rangle_\Omega &= \sum_{n,k} \int_{Q'_{n,k}} f(z) \left(\overline{g(z)} - \overline{g(\lambda_{n,k})} \right) W(\lambda_{n,k})\, dA(z) \\
&\quad + \sum_{n,k} \int_{Q'_{n,k}} \overline{g(\lambda_{n,k})} \big(f(z) - f(\lambda_{n,k}) \big) W(\lambda_{n,k})\, dA(z) \\
&\quad + \sum_{n,k} \int_{Q'_{n,k}} f(z)\overline{g(z)} \big(W(z) - W(\lambda_{n,k}) \big)\, dA(z) \\
&= I_1 + I_2 + I_3.
\end{aligned}
$$

We apply the Hölder inequality twice for I_1 and get, because $(1-|\varphi(z)|^2)^\alpha \leq Cd(z)^\alpha|\varphi'(z)|^\alpha$ and hence $W(\lambda_{n,k}) \leq CW(z) \leq Cd(z)^\alpha|\varphi'(z)|^{2+\alpha}$, that

$$|I_1| \leq 10\left(\sum_{n,k}\int_{Q_{n,k}}|f(z)|^p\,dA_\alpha(z)\right)^{\frac{1}{p}} \cdot \left(\sum_{n,k}\int_{Q_{n,k}}|g(z)-g(\lambda_{n,k})|^q|\varphi'(z)|^{q(2+\alpha)}\,dA_\alpha(z)\right)^{\frac{1}{q}}.$$

Now, with the help of Lemma 13, we are able to find a constant $C_4=(C_1+1)^{-1}$, $0<C_4<<1$, such that

$$\left(\sum_{n,k}\int_{Q_{n,k}}|g(z)-g(\lambda_{n,k})|^q\,dA_{\alpha,\omega}(z)\right)^{\frac{1}{q}} \leq C_4\left(\int_\Omega|g(z)|^q\,dA_{\alpha,\omega}(z)\right)^{\frac{1}{q}}, \quad (49)$$

which means

$$|I_1| \leq C\|f\|_{A^p_\alpha}\|g\|_{A^q_{\sigma*}}. \quad (50)$$

Again using Hölder twice we get

$$\begin{aligned}|I_2| &\leq \left(\sum_{n,k}\int_{Q_{n,k}}|f(z)-f(\lambda_{n,k})|^p\,dA_\alpha(z)\right)^{\frac{1}{p}}\left(\sum_{n,k}\int_{Q_{n,k}}|g(\lambda_{n,k})|^q\,dA_{\alpha,\omega}(z)\right)^{\frac{1}{q}}\\ &= \left(\sum_{n,k}\int_{Q_{n,k}}|f(z)-f(\lambda_{n,k})|^p\,dA_\alpha(z)\right)^{\frac{1}{p}}\left(\sum_{n,k}m_{\alpha,\omega}(Q_{n,k})|g(\lambda_{n,k})|^q\right)^{\frac{1}{q}}.\end{aligned} \quad (51)$$

According to Lemma 9

$$|g(\lambda_{n,k})|^q \leq \frac{C_1}{m_{\alpha,\omega}(Q_{n,k})}\int_{Q_{n,k}}|g(z)|^q\,dA_{\alpha,\omega}(z) \quad (52)$$

for all $n \geq 1$. Since every point in Ω belongs to at most 3 sets $Q_{n,k}$, we thus have

$$\sum_{n,k}m_{\alpha,\omega}(Q_{n,k})|g(\lambda_{n,k})|^q \leq 3C_1\int_\Omega|g(z)|^q\,dA_{\alpha,\omega}(z). \quad (53)$$

The factor $\sum_{n,k}\int_{Q_{n,k}}|f(z)-f(\lambda_{n,k})|^p\,dA_\alpha(z)$ in (51) is again (similarly to (49)) bounded by $(C_1+1)^{-1}\int_\Omega|f(z)|^p\,dA_\alpha(z)$. Hence

$$|I_2| \leq C\|f\|_{A^p_\alpha}\|g\|_{A^q_{\sigma*}}. \quad (54)$$

Applying Lemma 6 and Proposition 7 to I_3 yields $|W(z) - W(\lambda_{n,k})| \leq \frac{1}{100}W(z)$. Hence, adopting the same routine we used for I_1 and I_2, we get

$$
\begin{aligned}
|I_3| &\leq \frac{1}{100}\sum_{n,k}\int_{Q_{n,k}} |f(z)||g(\lambda_{n,k})|d(z)^\alpha|\varphi'(z)|^{2+\alpha}\,dA(z) \\
&\leq \frac{1}{100}\left(\sum_{n,k}\int_{Q_{n,k}}|f(z)|^p\,dA_\alpha(z)\right)^{\frac{1}{p}}\left(\sum_{n,k}\int_{Q_{n,k}}|g(z)|^q\,dA_{\alpha,\omega}(z)\right)^{\frac{1}{q}} \qquad (55)\\
&\leq \frac{1}{100}\|f\|_{A^p_\alpha}\|g\|_{A^q_{\sigma^*}}.
\end{aligned}
$$

Combining (50), (54) and (55) we find also a constant $0 < C_7 < 1$, for which

$$\langle Af \mid g\rangle_\Omega \leq C_7\|f\|_{A^p_\alpha}\|g\|_{A^q_{\sigma^*}},$$

for all $f, g \in H^\infty$. On the basis of the duality between A^p_α and $A^q_{\sigma^*}$ it is easy to see that there exists a constant $C > 0$, for which $\|A\| = \|I - S\| \leq C < 1$. QED

Since S is invertible, we may define an operator $RS^{-1}T : l^p \to l^p$. Now

$$(RS^{-1}T)^2 = RS^{-1}(TR)S^{-1}T = RS^{-1}SS^{-1}T = RS^{-1}T,$$

which makes $RS^{-1}T =: P$ a projection operator from l^p onto a complemented subspace of l^p. Because $S = TR$ is invertible, S^{-1} and T are surjective and R is bounded from below. Thus $P(l^p) = R(A^p_\alpha(\Omega))$, making $R : A^p_\alpha(\Omega) \to P(l^p)$ an isomorphism to a complemented subspace of l^p. By Theorem 14 now $A^p_\alpha \cong l^p$.

We can now prove our main result, the atomic decomposition for functions in $A^p_\alpha(\Omega)$ using the sequence $(\lambda_{n,k})$ constructed directly in Ω.

17 Theorem. *If $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then there exists a sequence $(\lambda_{n,k})$ in Ω satisfying Definition 11 and a constant $C > 0$ with the following properties (where $t_{n,k}$ are as in (46)):*

(1) For any $(a_{n,k})$ in l^p, the function

$$f(z) = \sum_{n,k} a_{n,k}\frac{t_{n,k}}{(1 - \varphi(z)\overline{\varphi(\lambda_{n,k})})^{2+\alpha}}$$

is in $A^p_\alpha(\Omega)$ with

$$\|f\|_{A^p} \leq C\|(a_{n,k})\|_{l^p}.$$

(2) If $f \in A^p_\alpha(\Omega)$, then there is $(a_{n,k})$ in l^p such that

$$f(z) = \sum_{n,k} a_{n,k}\frac{t_{n,k}}{(1 - \varphi(z)\overline{\varphi(\lambda_{n,k})})^{2+\alpha}}$$

and

$$\|(a_{n,k})\|_{l^p} \leq C\|f\|_{A^p}.$$

PROOF. Statement (1) follows from Lemma 15, statement (2) from Lemma 16 and the second part of the proof of Lemma 15, i.e. the boundedness of S. QED

8 Numerical example

In order to check how the atomic decomposition obtained in Theorem 17 behaves, we have calculated a numerical example in a simple domain.

Let Ω be the first quadrant of the unit circle, that is $\Omega = \{ z \in \mathbb{C} \mid |z| < 1, \mathrm{Re}(z) > 0, \mathrm{Im}(z) > 0 \}$. Then a conformal mapping $\varphi : \Omega \to \mathbb{D}$,

$$\varphi(z) = i\frac{z^4 + 2iz^2 + 1}{z^4 - 2iz^2 + 1}, \qquad \text{with} \qquad \varphi'(z) = \frac{8z(z^4 - 1)}{(z^4 - 2iz^2 + 1)^2}$$

will map Ω onto the unit circle. Clearly Ω is regulated and $\partial\Omega$ has three corners with opening $\pi/2$. Thus

$$b = \frac{1}{3} \quad \text{making} \quad \delta_n = \frac{1}{\sqrt[3]{n}} - \frac{1}{\sqrt[3]{n+1}}.$$

Since in Ω the maximum distance to the border $\partial\Omega$ is $\sup_{z\in\Omega} d(z) = \sqrt{2} - 1$, the first belt that (partially) fits in Ω is Γ_{14} with width $\delta_{14} = 9.433 \cdot 10^{-3}$.

From [11] we get that the constant C in (28) is

$$C \geq \frac{(1 + s|z|)^4}{(1 - |z|^2 s^2)^2} \geq 54.598, \quad \text{when} \quad s = \tanh r < 1,\ 0 < r < 1.$$

Examining more closely the proof of Corollary 8 we find, using again [11], that since $|D(z, 2r)|/|D(z, r/2)| < 5.00985$ and since we may take $c = 1/10$ in (30) to be on the safe side, we have that

$$C_0 \simeq 54.598 \cdot 5.00985 \cdot 1.1^4 \simeq 400.471.$$

From the proof of Lemma (9) we see that $C_1 \simeq 1.1^2 \cdot C_0 \simeq 484.570$. To calculate C_2 in Corollary 5 we get, using *Mathematica*, that

$$\frac{d(z)}{|\varphi'(z)|} \leq \frac{1}{4\sqrt{2}} = C_2,$$

where the maximum occurs at the corners $z = 1$ and $z = i$.

Finally, investigating equation (4) more carefully we may optimize (35) such that, taking here $p = 2$ yields $C_b = \frac{1}{4}(b+1)C_2\sqrt{C_1^2 + C_1}$. Thus, rounding up to ease the calculations, we have $C_b = 30$ and hence the sidelength (36) of a square $Q_{14,k}$ is $l_{14} = 3.144 \cdot 10^{-4}$.

Recall from (37) that every division $Q_{n,k}$ of $\mathbb{C}$ starts from the origin. Thus knowing that the lower left corner of every Γ_n is at $(n+1)^{-1/3}(1+i)$, we get that the first index such that Q_{n,k_1} intersects Γ_n is

$$k_1(n) = (M_n, M_n) \in \mathbb{N}^2 \quad \text{with} \quad M_n = \left[l_n^{-1}(n+1)^{-1/3}\right],$$

where $[x] = \min\{\, k \in \mathbb{N} \mid k > x \,\}$. In the belt Γ_{14} we have $M_{14} = 1290$. Then let $\zeta_{n,1} = \operatorname{dist}(\lambda_{n,1}, \partial\Omega)$, which for the first point of the sequence in Γ_n yields $\lambda_{n,1} = \zeta_{n,1}(1+i)$ and (because $\lambda_{n,1}$ is at the center of the square Q_{n,k_1}) $\zeta_{14,1} = (1290 - \frac{1}{2}) \cdot l_{14} \simeq 0.40547$.

Thus for every n we have the first $\lambda_{n,1}$ at $\zeta_{n,1}(1+i)$ and all the other $\lambda_{n,k}$ are translations of $\lambda_{n,1}$ to x- and y- directions by l_n. Eventually we get a sequence of points $\lambda_{14,k}$, where

$$\begin{aligned}
\lambda_{14,1} &= \zeta_{14,1}(1+i) = (1290 - \tfrac{1}{2})(1+i)l_{14} \simeq 0,40547(1+i) \\
\lambda_{14,2} &= \zeta_{14,1}(1+i) + l_{14} \simeq 0.40578 + 0.40547i \\
&\vdots \\
\lambda_{14,67} &= \zeta_{14,1}(1+i) + 67 \cdot l_{14} \simeq 0.42654 + 0.40547i \\
\lambda_{14,68} &= \zeta_{14,1}(1+i) + l_{14}i \simeq 0.40547 + 0.40578i \\
&\vdots
\end{aligned}$$

Here 67 happens to be the number of squares $Q_{14,k}$ that cover the lower border of Γ_{14}, in total there are approximately 3600 squares that intersect Γ_{14}.

The number of squares needed to cover the belts Γ_n grows rapidly when the belts get closer to the border (Table 1). In this case it takes 1000 belts to get within the distance of 0.1 from $\partial\Omega$ and already there will be approximately 80 million points λ in the decomposition. As the smallest angle gets sharper, the situation becomes even more cumbersome to handle. For instance with the angle $\pi/4$ we get $b = 1/7$ and to get closer to the border than 0.1 would require 10 million belts and a large number of covering squares.

It seems that in order to be applied the method above would need to be optimized. In this study, however, the priority has been to ensure that the sequence $\lambda_{n,k}$ in Definition 11 does indeed meet the requirements of an atomic decomposition and accordingly many of the approximations have been done with an ample margin to ease the calculations.

n	δ_n	l_n	M_n	$\zeta_{n,1}$	$\#Q_{n,k}$
14	$9.4331 \cdot 10^{\ 3}$	$3.1444 \cdot 10^{\ 4}$	1290	0.40547	3 509
15	$8.6299 \cdot 10^{-3}$	$2.8766 \cdot 10^{-4}$	1380	0.39683	12 459
16	$7.9391 \cdot 10^{-3}$	$2.6464 \cdot 10^{-4}$	1470	0.38889	21 141
17	$7.3397 \cdot 10^{-3}$	$2.4466 \cdot 10^{-4}$	1560	0.38154	32 526
18	$6.8152 \cdot 10^{-3}$	$2.2717 \cdot 10^{-4}$	1650	0.37472	43 585
20	$5.9430 \cdot 10^{-3}$	$1.9810 \cdot 10^{-4}$	1830	0.36243	67 635
30	$3.4984 \cdot 10^{-3}$	$1.1661 \cdot 10^{-4}$	2730	0.31830	221 709
50	$1.7859 \cdot 10^{-3}$	$5.9528 \cdot 10^{-5}$	4530	0.26963	662 855
100	$7.1339 \cdot 10^{-4}$	$2.3780 \cdot 10^{-5}$	9030	0.21472	2 299 927
1000	$3.3311 \cdot 10^{-5}$	$1.1104 \cdot 10^{-6}$	90030	0.099966	77 727 082

Table 1. Some data of the belts Γ_n. Here n is the number of the belt, δ_n the width of a belt, l_n the sidelength of a square $Q_{n,k}$, M_n tells how many squares $Q_{n,k}$ are needed counting from the origin before the first intersects Γ_n, $\zeta_{n,1}$ the border-distance of the first point $\lambda_{n,1}$ of a decomposition and $\#Q_{n,k}$ the minimum number of squares $Q_{n,k}$ needed to cover a belt Γ_n. All other entries have been calculated as above, $\#Q_{n,k}$ by comparing the areas of the squares and Γ_n.

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