A New Generalized Gamma Type II Exponentiated Half Logistic-Topp-Leone-G Family of Distributions with Applications

Appendix

R Codes

Datasets

Dataset 1: Italy COVID-19 Data

The data are: 52,26, 36, 63, 52, 37, 35, 28, 17, 21, 31, 30, 10, 56, 40, 14, 28, 42, 24, 21, 28, 22, 12, 31, 24, 14, 13, 25, 12, 7, 13, 20, 23, 9, 11, 13, 3, 7, 10, 21, 15, 17, 5, 7, 22, 24, 15, 19, 18, 16, 5, 20, 27, 21, 27, 24, 22, 11, 22, 31, 31.

Dataset 2: Earthquake Data

The data are: 1163, 3258, 323, 159, 756, 409, 501, 616, 398, 67, 896, 8592, 2039, 217, 9, 633, 461, 1821, 4863, 143, 182, 2117, 3709, 979.

Series Expansion and Representation

Let $y = W_G(z;\alpha,\vartheta,\underline{\zeta}) = \left[\frac{1-[1-\bar{G}^2(z;\underline{\zeta})]^{\vartheta}}{1+[1-\bar{G}^2(z;\underline{\zeta})]^{\vartheta}}\right]^{\alpha}$, we can express the RB-TII-EHL-TL-G density function as

$$
\begin{aligned}
f(z;\sigma,\alpha,\vartheta,\underline{\zeta}) &= \frac{4\alpha\vartheta}{\Gamma(\sigma)}\left(-\log[1-y]\right)^{\sigma-1} \\
&\times \frac{g(z;\underline{\zeta})[1-\bar{G}^2(z;\underline{\zeta})]^{\vartheta-1}\bar{G}(z;\underline{\zeta})\left(1-[1-\bar{G}^2(z;\underline{\zeta})]^{\vartheta}\right)^{\alpha-1}}{(1+[1-\bar{G}^2(z;\underline{\zeta})]^{\vartheta})^{\alpha+1}} \\
&= \frac{4\alpha\vartheta}{\Gamma(\sigma)}\sum_{m,s=0}^{\infty}\binom{\sigma-1}{m}b_{s,m}W_G(z;\alpha,\vartheta,\underline{\zeta})^{(m+s+\sigma-1)} \\
&\times \frac{g(z;\underline{\zeta})[1-\bar{G}^2(z;\underline{\zeta})]^{\vartheta-1}\bar{G}(z;\underline{\zeta})\left(1-[1-\bar{G}^2(z;\underline{\zeta})]^{\vartheta}\right)^{\alpha-1}}{(1+[1-\bar{G}^2(z;\underline{\zeta})]^{\vartheta})^{\alpha+1}} \\
&= \frac{4\alpha\vartheta}{\Gamma(\sigma)}\sum_{m,s=0}^{\infty}\binom{\sigma-1}{m}b_{s,m}\frac{(1-[1-\bar{G}^2(z;\underline{\zeta})]^{\vartheta})^{\alpha(m+s+\sigma)-1}}{(1+[1-\bar{G}^2(z;\underline{\zeta})]^{\vartheta})^{\alpha(m+s+\sigma)+1}} \\
&\times g(z;\underline{\zeta})[1-\bar{G}^2(z;\underline{\zeta})]^{\vartheta-1}\bar{G}(z;\underline{\zeta}) \\
&= \frac{4\alpha\vartheta}{\Gamma(\sigma)}\sum_{m,s,j,k=0}^{\infty}(-1)^{j}b_{s,m}\binom{\sigma-1}{m}\binom{\alpha(m+s+\sigma)-1}{j} \\
&\times \binom{-\alpha(m+s+\sigma)-1}{k}g(z;\underline{\zeta})[1-\bar{G}^2(z;\underline{\zeta})]^{\vartheta(j+k+1)-1}\bar{G}(z;\underline{\zeta}) \\
&= \frac{4\alpha\vartheta}{\Gamma(\sigma)}\sum_{m,s,j,k,l=0}^{\infty}(-1)^{j+l}b_{s,m}\binom{\sigma-1}{m}\binom{\alpha(m+s+\sigma)-1}{j} \\
&\times \binom{-\alpha(m+s+\sigma)-1}{k}\binom{\vartheta(j+k+1)-1}{l}g(z;\underline{\zeta})\bar{G}^{2l+1}(z;\underline{\zeta}) \\
&= \frac{4\alpha\vartheta}{\Gamma(\sigma)}\sum_{m,s,j,k,l,p=0}^{\infty}(-1)^{j+l+m}b_{s,m}\binom{\sigma-1}{m}\binom{\alpha(m+s+\sigma)-1}{j} \\
&\times \binom{-\alpha(m+s+\sigma)-1}{k}\binom{\vartheta(j+k+1)-1}{l}\binom{2l+1}{p} \\
&\times \left(\frac{p+1}{p+1}\right)g(z;\underline{\zeta})G^{p}(x;\underline{\zeta}).
\end{aligned}
$$

Uncertainty Measure

$$
\begin{aligned}
f^v(z) &= \left(\frac{4\alpha\vartheta}{\Gamma(\sigma)}\right)^v \left(-\log\left[1-W_G(z;\alpha,\vartheta,\underline{\zeta})\right]\right)^{v(\sigma-1)} \\
&\times \frac{g^v(z;\underline{\zeta})[1-\bar{G}^2(z;\underline{\zeta})]^{v(\vartheta-1)}\bar{G}^v(z;\underline{\zeta})\left(1-[1-\bar{G}^2(z;\underline{\zeta})]^{\vartheta}\right)^{v(\alpha-1)}}{(1+[1-\bar{G}^2(z;\underline{\zeta})]^{\vartheta})^{v(\alpha+1)}} \\
&= \left(\frac{4\alpha\vartheta}{\Gamma(\sigma)}\right)^v \sum_{m,s=0}^{\infty} \binom{v(\sigma-1)}{m} b_{s,m} W_G(z;\alpha,\vartheta,\underline{\zeta})^{[m+s+v(\sigma-1)]} \\
&\times \frac{g^v(z;\underline{\zeta})[1-\bar{G}^2(z;\underline{\zeta})]^{v(\vartheta-1)}\bar{G}^v(z;\underline{\zeta})\left(1-[1-\bar{G}^2(z;\underline{\zeta})]^{\vartheta}\right)^{v(a-1)}}{(1+[1-\bar{G}^2(z;\underline{\zeta})]^{\vartheta})^{v(\alpha+1)}} \\
&= \left(\frac{4\alpha\vartheta}{\Gamma(\sigma)}\right)^v \sum_{m,s=0}^{\infty} \binom{v(\sigma-1)}{m} b_{s,m} \frac{(1-[1-\bar{G}^2(z;\underline{\zeta})]^{\vartheta})^{\alpha(m+s+v\sigma)-v}}{(1+[1-\bar{G}^2(z;\underline{\zeta})]^{\vartheta})^{\alpha(m+s+v\sigma)+v}} \\
&\times g^v(z;\underline{\zeta})[1-\bar{G}^2(z;\underline{\zeta})]^{v(\alpha-1)}\bar{G}^v(z;\underline{\zeta}) \\
&= \left(\frac{4\alpha\vartheta}{\Gamma(\sigma)}\right)^v \sum_{m,s,i,j=0}^{\infty} b_{s,m}(-1)^i \binom{v(\sigma-1)}{m} \binom{\alpha(m+s+v\sigma)-v}{i} \\
&\times \binom{-\alpha(m+s+v\sigma)-v}{j} g^v(z;\underline{\zeta})[1-\bar{G}^2(z;\underline{\zeta})]^{\vartheta(i+j)+v(\vartheta-1)}\bar{G}^v(z;\underline{\zeta}) \\
&= \left(\frac{4\alpha\vartheta}{\Gamma(\sigma)}\right)^v \sum_{m,s,i,j,k,r=0}^{\infty} b_{s,m}(-1)^{i+k+r} \binom{v(\sigma-1)}{m} \binom{\alpha(m+s+v\sigma)-v}{i} \\
&\times \binom{-\alpha(m+s+v\sigma)-v}{j} \binom{\vartheta(i+j)+v(b-1}{k} \binom{2k+v}{r} g^v(z;\underline{\zeta}) G^r(z;\underline{\zeta}).
\end{aligned}
$$

Elements of score vector

The partial derivatives of the log-likelihood function with respect to each component of the parameter vector are:

$$\begin{aligned}\frac{\partial \ell}{\partial \alpha} &= \frac{n}{\alpha} + (\sigma-1)\sum_{i=1}^{n} \frac{\left[\frac{1-[1-\bar{G}^2(z_i;\underline{\zeta})]^{\vartheta}}{1+[1-\bar{G}^2(z_i;\underline{\zeta})]^{\vartheta}}\right]^{\alpha} \ln\left[\frac{1-[1-\bar{G}^2(z_i;\underline{\zeta})]^{\vartheta}}{1+[1-\bar{G}^2(z;\underline{\zeta})]^{\vartheta}}\right]}{-\log\left[1-\left[\frac{1-[1-\bar{G}^2(z_i;\underline{\zeta})]^{\vartheta}}{1+[1-\bar{G}^2(z_i;\underline{\zeta})]^{\vartheta}}\right]^{\alpha}\right]\left[1-\left[\frac{1-[1-\bar{G}^2(z_i;\underline{\zeta})]^{\vartheta}}{1+[1-\bar{G}^2(z_i;\underline{\zeta})]^{\vartheta}}\right]^{\alpha}\right]} \\ &+ \sum_{i=1}^{n} \ln\left(1-[1-\bar{G}^2(z_i;\underline{\zeta})]^{\vartheta}\right) - \sum_{i=1}^{n} \ln(1+[1-\bar{G}^2(z_i;\underline{\zeta})]^{\vartheta}),\end{aligned}$$

$$\begin{aligned}\frac{\partial \ell}{\partial \vartheta} &= \frac{n}{\vartheta} - 2(\sigma-1)\sum_{i=1}^{n} \frac{\alpha\left[\frac{1-[1-\bar{G}^2(z_i;\underline{\zeta})]^{\vartheta}}{1+[1-\bar{G}^2(z_i;\underline{\zeta})]^{\vartheta}}\right]^{\alpha-1}}{-\log\left[1-\left[\frac{1-[1-\bar{G}^2(z_i;\underline{\zeta})]^{\vartheta}}{1+[1-\bar{G}^2(z_i;\underline{\zeta})]^{\vartheta}}\right]^{\alpha}\right]\left[1-\left[\frac{1-[1-\bar{G}^2(z_i;\underline{\zeta})]^{\vartheta}}{1+[1-\bar{G}^2(z_i;\underline{\zeta})]^{\vartheta}}\right]^{\alpha}\right]} \\ &\times \frac{[1-\bar{G}^2(z_i;\underline{\zeta})]^{\vartheta}\ln[1-\bar{G}^2(z_i;\underline{\zeta})]}{(1+[1-\bar{G}^2(z_i;\underline{\zeta})]^{\vartheta})^2} + \sum_{i=1}^{n} \ln[1-\bar{G}^2(z_i;\underline{\zeta})] \\ &- (\alpha-1)\sum_{i=1}^{n} \frac{[1-\bar{G}^2(z_i;\underline{\zeta})]^{\vartheta}\ln[1-\bar{G}^2(z_i;\underline{\zeta})]}{(1-[1-\bar{G}^2(z_i;\underline{\zeta})]^{\vartheta})} \\ &- (\alpha+1)\sum_{i=1}^{n} \frac{[1-\bar{G}^2(z_i;\underline{\zeta})]^{\vartheta}\ln[1-\bar{G}^2(z_i;\underline{\zeta})]}{(1+[1-\bar{G}^2(z_i;\underline{\zeta})]^{\vartheta})},\end{aligned}$$

$$\frac{\partial \ell}{\partial \sigma} = \sum_{i=1}^{n} \ln\left(-\log\left[1-\left[\frac{1-[1-\bar{G}^2(z_i;\underline{\zeta})]^{\vartheta}}{1+[1-\bar{G}^2(z_i;\underline{\zeta})]^{\vartheta}}\right]^{\alpha}\right]\right) - \frac{n\Gamma'(\sigma)}{\Gamma(\sigma)},$$

$$\begin{aligned}\frac{\partial \ell}{\partial \underline{\zeta}_k} &= (\sigma-1)\sum_{i=1}^{n} \frac{\frac{\partial\left(-\log\left[1-W_G(z;\alpha,\vartheta,\underline{\zeta})\right]\right)}{\partial \underline{\zeta}_k}}{\left(-\log\left[1-W_G(z;\alpha,\vartheta,\underline{\zeta})\right]\right)} + \sum_{i=1}^{n} \frac{\frac{\partial g(z_i;\underline{\zeta})}{\partial \underline{\zeta}_k}}{g(z_i;\underline{\zeta})} \\ &+ (\vartheta-1)\sum_{i=1}^{n} \frac{\frac{\partial[1-\bar{G}^2(z_i;\underline{\zeta})]}{\partial \underline{\zeta}_k}}{[1-\bar{G}^2(z_i;\underline{\zeta})]} + \sum_{i=1}^{n} \frac{\frac{\partial \bar{G}(z_i;\underline{\zeta})}{\partial \underline{\zeta}_k}}{\bar{G}(z_i;\underline{\zeta})} \\ &+ (\alpha-1)\sum_{i=1}^{n} \frac{\frac{\partial\left(1-[1-\bar{G}^2(z_i;\underline{\zeta})]^{\vartheta}\right)}{\partial \underline{\zeta}_k}}{\left(1-[1-\bar{G}^2(z_i;\underline{\zeta})]^{\vartheta}\right)} - (\alpha+1)\sum_{i=1}^{n} \frac{\frac{\partial\left(1+[1-\bar{G}^2(z_i;\underline{\zeta})]^{\vartheta}\right)}{\partial \underline{\zeta}_k}}{\left(1+[1-\bar{G}^2(z_i;\underline{\zeta})]^{\vartheta}\right)}\end{aligned}$$

Order Statistics

$$
\begin{aligned}
f(z)[F(z)]^{r+i-1} &= \frac{4\alpha\vartheta}{\Gamma(\sigma)}\left(-\log\left[1-W_G(z;\alpha,\vartheta,\underline{\zeta})\right]\right)^{\sigma-1} \\
&\times \frac{g(z;\underline{\zeta})[1-\bar{G}^2(z;\underline{\zeta})]^{\vartheta-1}\bar{G}(z;\underline{\zeta})\left(1-[1-\bar{G}^2(z;\underline{\zeta})]^{\vartheta}\right)^{\alpha-1}}{(1+[1-\bar{G}^2(z;\underline{\zeta})]^{\vartheta})^{\alpha+1}} \\
&\times \left[1-\frac{\gamma\left(\sigma,-\log\left[1-W_G(z;\alpha,\vartheta,\underline{\zeta})\right]\right)}{\Gamma(\sigma)}\right]^{r+i-1} \\
&= \sum_{j=0}^{\infty}(-1)^j\binom{r+j-1}{j}\frac{4\alpha\vartheta}{[\Gamma(\sigma)]^{j+1}} \\
&\times \left(-\log\left[1-W_G(z;\alpha,\vartheta,\underline{\zeta})\right]\right)^{\sigma-1} \\
&\times \frac{g(z;\underline{\zeta})[1-\bar{G}^2(z;\underline{\zeta})]^{\vartheta-1}\bar{G}(z;\underline{\zeta})\left(1-[1-\bar{G}^2(z;\underline{\zeta})]^{\vartheta}\right)^{\alpha-1}}{(1+[1-\bar{G}^2(z;\underline{\zeta})]^{\vartheta})^{\alpha+1}} \\
&\times \left[\gamma\left(\sigma,-\log\left[1-W_G(z;\alpha,\vartheta,\underline{\zeta})\right]\right)\right]^{j} \\
&= \sum_{j=0}^{\infty}(-1)^j\binom{r+i-1}{j}\left(\sum_{p=0}^{\infty}\frac{(-1)^p}{(p+\sigma)p!}\right)^j\frac{4\alpha\vartheta}{[\Gamma(\sigma)]^{j+1}} \\
&\times \left(-\log\left[1-W_G(z;\alpha,\vartheta,\underline{\zeta})\right]\right)^{j(p+\sigma)+\sigma-1} \\
&\times \frac{g(z;\underline{\zeta})[1-\bar{G}^2(z;\underline{\zeta})]^{\vartheta-1}\bar{G}(z;\underline{\zeta})\left(1-[1-\bar{G}^2(z;\underline{\zeta})]^{\vartheta}\right)^{\alpha-1}}{(1+[1-\bar{G}^2(z;\underline{\zeta})]^{\vartheta})^{\alpha+1}} \\
&= \sum_{j,m,s=0}^{\infty}b_{s,m}(-1)^j\left(\sum_{p=0}^{\infty}\frac{(-1)^p}{(p+\sigma)p!}\right)^j\frac{4\alpha\vartheta}{[\Gamma(\sigma)]^{j+1}}\binom{r+i-1}{j} \\
&\times \binom{j(p+\sigma)+\sigma-1}{m}[W_G(z;\alpha,\vartheta,\underline{\zeta})]^{[m+s+j(p+\sigma)+\sigma-1]} \\
&\times \frac{g(z;\underline{\zeta})[1-\bar{G}^2(z;\underline{\zeta})]^{\vartheta-1}\bar{G}(z;\underline{\zeta})\left(1-[1-\bar{G}^2(z;\underline{\zeta})]^{\vartheta}\right)^{\alpha-1}}{(1+[1-\bar{G}^2(z;\underline{\zeta})]^{\vartheta})^{\alpha+1}} \\
&= \sum_{j,m,s=0}^{\infty}b_{s,m}(-1)^j\left(\sum_{p=0}^{\infty}\frac{(-1)^p}{(p+\sigma)p!}\right)^j\frac{4\alpha\vartheta}{[\Gamma(\sigma)]^{j+1}}\binom{r+i-1}{j} \\
&\times \binom{j(p+\sigma)+\sigma-1}{m}\frac{(1-[1-\bar{G}^2(z;\underline{\zeta})]^{\vartheta})^{\alpha[m+s+j(p+\sigma)+\sigma]-1}}{(1+[1-\bar{G}^2(z;\underline{\zeta})]^{\vartheta})^{\alpha[m+s+j(p+\sigma)+\sigma]+1}} \\
&\times g(z;\underline{\zeta})[1-\bar{G}^2(z;\underline{\zeta})]^{\vartheta-1}\bar{G}(z;\underline{\zeta}).
\end{aligned}
$$

$$\begin{aligned} f(z)[F(z)]^{r+i-1} &= \sum_{j,m,s,k,l=0}^{\infty} b_{s,m}(-1)^{j+k}\left(\sum_{p=0}^{\infty}\frac{(-1)^p}{(p+\sigma)p!}\right)^j \frac{4\alpha\vartheta}{[\Gamma(\sigma)]^{j+1}}\binom{r+i-1}{j} \\ &\times \binom{j(p+\sigma)+\sigma-1}{m}\binom{\alpha[m+s+j(p+\sigma)+\sigma]-1}{k}\binom{-\alpha[m+s+j(p+\sigma)+\sigma]-1}{l} \\ &\times g(z;\underline{\zeta})[1-\bar{G}^2(z;\underline{\zeta})]^{\vartheta(k+l+1)-1}\bar{G}(z;\underline{\zeta}) \\ &= \sum_{j,m,s,k,l,t,q=0}^{\infty} b_{s,m}(-1)^{j+k+t+q}\left(\sum_{p=0}^{\infty}\frac{(-1)^p}{(p+\sigma)p!}\right)^j \frac{4\alpha\vartheta}{[\Gamma(\sigma)]^{j+1}}\binom{r+i-1}{j} \\ &\times \binom{j(p+\sigma)+\sigma-1}{m}\binom{\alpha[m+s+j(p+\sigma)+\sigma]-1}{k}\binom{-\alpha[m+s+j(p+\sigma)+\sigma]-1}{l} \\ &\times \binom{\vartheta(k+l+1)-1}{t}\binom{2t+1}{q}\left(\frac{q+1}{q+1}\right) g(z;\underline{\zeta})G^q(z;\underline{\zeta}) \\ &= \sum_{q=0}^{\infty} a_{q+1}g_{q+1}(z;\underline{\zeta}) \end{aligned}$$

Competing Models

The gamma-Topp-Leone-type II-exponentiated half logistic-Weibull (RBTLTIIEHLW) distribution with pdf

$$\begin{aligned} f_{RBTLTIIEHLW}(z;\sigma,a,b,\lambda) &= \frac{4ab}{\Gamma(\sigma)}\left[-\log\left(\left[1-\left(\frac{\exp(-z^\lambda)}{1+(1-\exp(-z^\lambda))}\right)^{2a}\right]^b\right)\right]^{\sigma-1} \\ &\times \left[1-\left(\frac{\exp(-z^\lambda)}{1+(1-\exp(-z^\lambda))}\right)^{2a}\right]^{b-1}\left[\exp(-z^\lambda)\right]^{2a-1} \\ &\times \frac{\lambda z^{\lambda-1}\exp(-z^\lambda)}{[1+(1-\exp(-z^\lambda))]^{2(a+1)-1}}, \end{aligned}$$

for $\sigma, a, b, \lambda > 0$, gamma-generalized inverse Weibull (GGIW) distribution with pdf,

$$\begin{aligned} f_{GGIW}(z;k,\lambda,\sigma) &= \frac{k\vartheta z^{-\vartheta-1}e^{-kz^{-\vartheta}}}{\Gamma(\sigma)\lambda^\sigma} \\ &\times \left[-\log(1-e^{-kz^{-\vartheta}})\right]^{\sigma-1}\left[1-e^{-kz^{-\vartheta}}\right]^{\frac{1}{\lambda}-1}, \end{aligned}$$

for $k, \lambda, \sigma > 0$, exponentiated half-logistic odd Burr III-log-logistic (EHLOBI-IILLoG) distribution with pdf

$$
\begin{aligned}
f_{EHLOBIIILoG}(z;a,b,\alpha,\lambda) &= 2\alpha ab\left(\left[1+\left(\frac{1-(1+z^{\lambda})^{-1}}{(1+z^{\lambda})^{-1}}\right)^{-a}\right]\right)^{\alpha-1} \\
&\times \left(1+\left(\frac{1-(1+z^{\lambda})^{-1}}{(1+z^{\lambda})^{-1}}\right)^{-a}\right)^{-b-1} \\
&\times \left(1+\left[1-\left(1+\left[\frac{1-(1+z^{\lambda})^{-1}}{(1+z^{\lambda})^{-1}}\right]^{-a}\right)^{-b}\right]\right)^{-(\alpha+1)} \\
&\times \left(\frac{1-(1+z^{\lambda})^{-1}}{(1+z^{\lambda})^{-1}}\right)^{-a-1}\frac{\lambda z^{\lambda-1}(1+z^{\lambda})^{-2}}{((1+z^{\lambda})-1)^{2}} \\
&\times \left(1-\left[\frac{\left(1+\left(\frac{1-(1+z^{\lambda})^{-1}}{(1+z^{\lambda})^{-1}}\right)^{-a}\right)^{-b}}{1+\left[1-\left(1+\left(\frac{1-(1+z^{\lambda})^{-1}}{(1+z^{\lambda})^{-1}}\right)^{-a}\right)^{-b}\right]}\right]^{\alpha}\right)^{-1},
\end{aligned}
$$

for $a, b, \alpha, \lambda > 0$, exponentiated odd Weibull-Topp-Leone-Log logistic (EOWTL-LLoG) distribution with pdf

$$
\begin{aligned}
f_{EOWTLLLoG}(z;b,\alpha,\vartheta,c) &= \frac{2b\alpha\vartheta cz^{c-1}(1+z^{c})^{-3}[1-(1+z^{c})^{-2}]^{b\vartheta-1}}{(1-[1-(1+z^{c})^{-2}]^{b})^{\vartheta+1}} \\
&\times \exp\left\{-\left[\frac{[1-(1+z^{c})^{-2}]^{b}}{(1-[1-(1+z^{c})^{-2}]^{b})}\right]^{\vartheta}\right\} \\
&\times \left[1-\exp\left\{-\left[\frac{[1-(1+z^{c})^{-2}]^{b}}{(1-[1-(1+z^{c})^{-2}]^{b})}\right]^{\vartheta}\right\}\right]^{\alpha-1},
\end{aligned}
$$

for $b, \alpha, \vartheta, c > 0$, odd Weibull-Topp-Leone-log-logistic Poisson (OW-TL-LLoGP) distribution with pdf

$$
\begin{aligned}
f_{OWTLLLoGP}(z;\alpha,\lambda,\gamma,\theta) &= \frac{2\theta\gamma\alpha\lambda x^{\lambda-1}(1+x^{\lambda})^{-3}[1-(1+x^{\lambda})^{-2}]^{\gamma\alpha-1}}{[1-[1-(1+x^{\lambda})^{-2}]^{\gamma}]^{\alpha+1}} \\
&\times \exp\left\{-\left[\frac{[1-(1+x^{\lambda})^{-2}]^{\gamma}}{[1-[1-(1+x^{\lambda})^{-2}]^{\gamma}]}\right]^{\alpha}\right\} \\
&\times \frac{\exp\left(\theta\left(\exp\left\{-\left[\frac{[1-(1+x^{\lambda})^{-2}]^{\gamma}}{[1-[1-(1+x^{\lambda})^{-2}]^{\gamma}]}\right]^{\alpha}\right\}\right)\right)}{\exp(\theta)-1},
\end{aligned}
$$

for $\alpha, \lambda, \gamma, \theta > 0$, exponentiated half logistic odd Weibull-Topp-Leone-Log logistic (EHLOWTLLLoG) distribution with pdf

$$\begin{aligned} f_{EHLOWTLLLoG}(z;\alpha,\vartheta,\sigma,c) &= \frac{4\alpha\vartheta\sigma c z^{c-1}(1+z^c)^{-2}(1+z^c)^{-1}[1-(1+z^c)^{-2}]^{\alpha\vartheta-1}}{(1-[1-(1+z^c)^{-2}]^{\alpha})^{\vartheta+1}} \\ &\times \frac{\exp(-t)}{(1+\exp(-t))^2}\left[\frac{1-\exp(-t)}{1+\exp(-t)}\right]^{\sigma-1}, \end{aligned}$$

where $t = \left[\frac{[1-BG^2]^{\alpha}}{1-[1-BG^2]^{\alpha}}\right]^{\vartheta}$, for $\alpha, \vartheta, \sigma, c > 0$, type II Exponentiated half-logistic-Gompertz Topp-Leone-Weibull (TIIEHLGomTLW) distribution with pdf

$$\begin{aligned} f_{TIIEHLGomTLW}(x;\alpha,\gamma,b,\beta) &= 4\alpha b\left(1+\left[1-\exp\left(\frac{1}{\gamma}\left[1-\left[1-\left[1-\exp(-2x^{\beta})\right]^{b}\right]^{-\gamma}\right]\right)\right]\right)^{-(\alpha+1)} \\ &\times \left[1-\left(1-\exp(-2x^{\beta})\right)^{b}\right]^{-\gamma-1} \\ &\times \exp\left(\frac{\alpha}{\gamma}\left[1-\left[1-\left[1-\exp(-2x^{\beta})\right]^{b}\right]^{-\gamma}\right]\right) \\ &\times \left[1-\exp(-2x^{\beta})\right]^{b-1}\exp(-x^{\beta})\beta z^{\beta-1}\exp(-x^{\beta}), \end{aligned}$$

for $\alpha, \gamma, b, \beta > 0$.

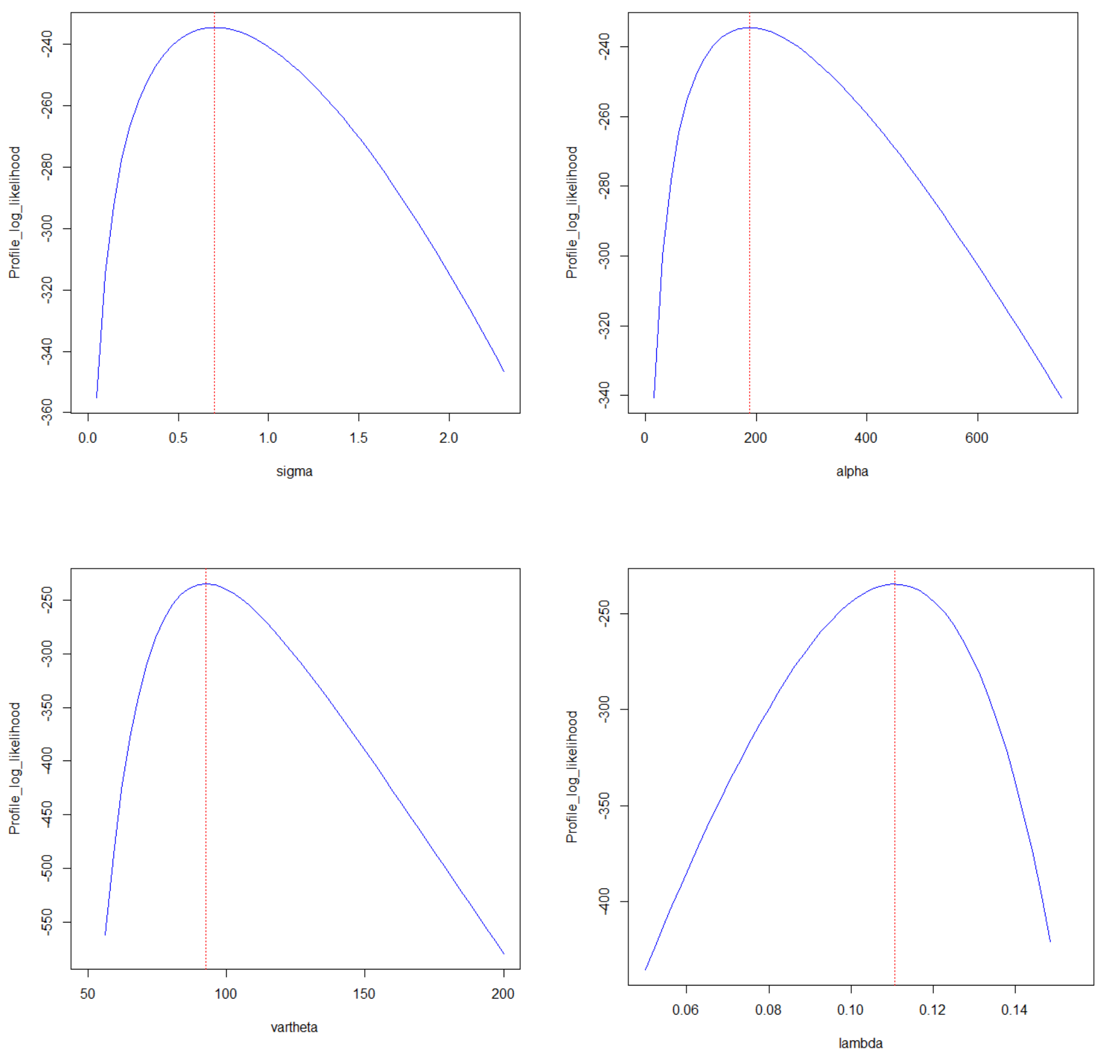

Figure 1: Profile Log-likelihood for $\sigma, \alpha, \vartheta$, and λ for Italy COVID-19 Data

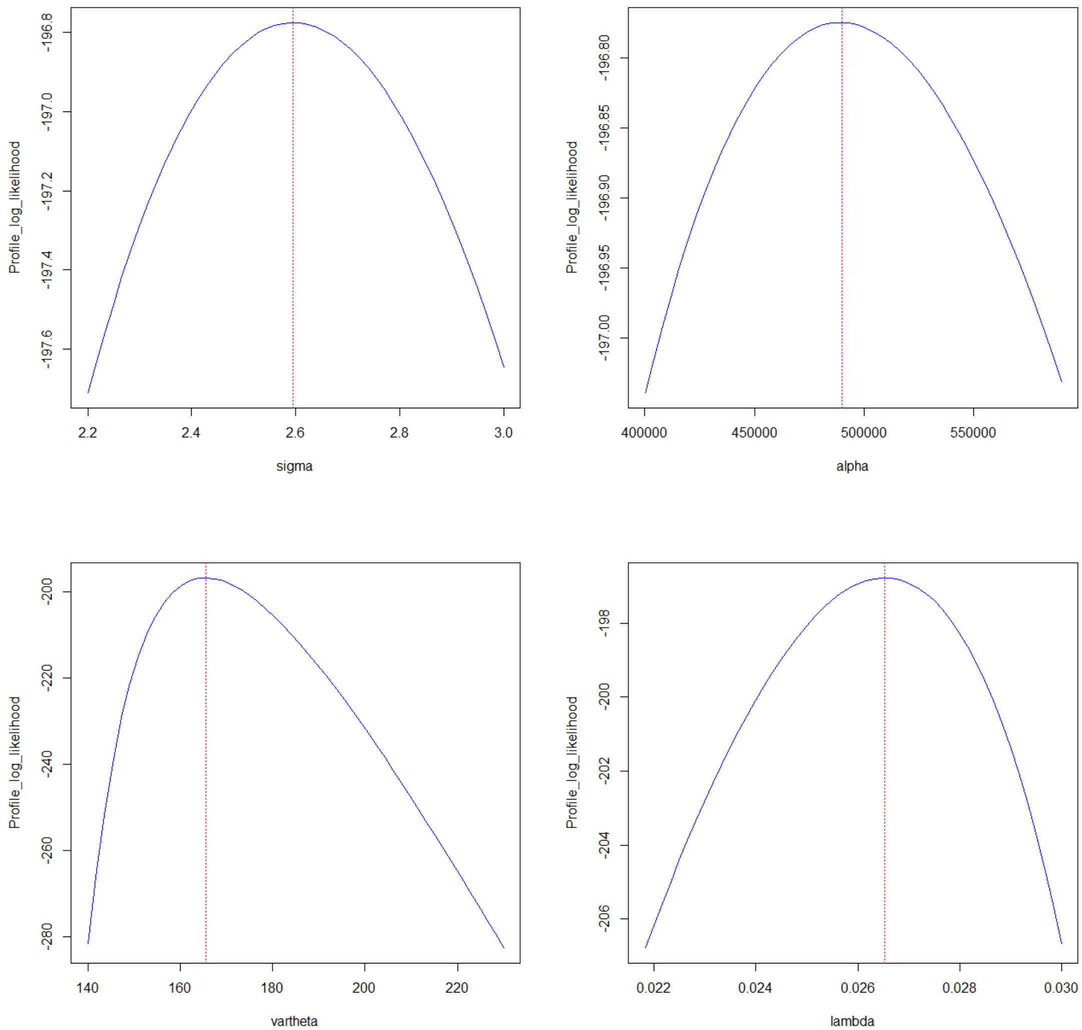

Figure 2: Profile Log-likelihood for $\sigma, \alpha, \vartheta$, and λ for Earthquake Data

R Codes

ML Estimation

```
RBTIIEHLTLG<-function(delta,a,b,lambda){-sum(log(
   ((4*a*b*(lambda*x^(lambda-1)*exp(-x^(lambda))))/(gamma(delta)))*
(-log(1-((1-((1-(exp(-x^(lambda)))^2))^b)/(1+
((1-(exp(-x^(lambda)))^2))^b))^a))^(delta-1)*
((((1-(exp(-x^(lambda)))^2))^(b-1)*
(exp(-x^(lambda)))*
```

```
(1-((1-(exp(-x^(lambda)))^2))^b)^(a-1))/(1+
((1-(exp(-x^(lambda)))^2))^b)^(a+1))
))}
mle.result<-mle2(RBTIIEHLTLG, hessian =
NULL, start=list(delta= 0.00,a=0.00 ,
b=0.00,lambda=0.00),optimizer="nlminb",lower=0)
summary(mle.result)
```

GoF

```
RBTIIEHLTLG_pdf<-function(par,x){
  delta=par[1]
  a=par[2]
  b=par[3]
  lambda=par[4]

  ((4*a*b*(lambda*x^(lambda-1)*exp(-x^(lambda))))/(gamma(delta)))*
(-log(1-((1-((1-(exp(-x^(lambda)))^2))^b)/(1+((1-
 (exp(-x^(lambda)))^2))^b))^a))^(delta-1)*
((((1-(exp(-x^(lambda)))^2))^(b-1)*(exp(-x^(lambda)))*
(1-((1-(exp(-x^(lambda)))^2))^b)^(a-1))/(1+
((1-(exp(-x^(lambda)))^2))^b)^(a+1))
}
RBTIIEHLTLG_cdf<-function(par,x){

  delta=par[1]
  a=par[2]
  b=par[3]
  lambda=par[4]
  1-pgamma(-log(1-((1-((1-(exp(-x^(lambda)))^2))^b)/(1+
((1-(exp(-x^(lambda)))^2))^b))^a),delta)

}
goodness.fit(pdf=RBTIIEHLTLG_pdf,
cdf=RBTIIEHLTLG_cdf,mle=c(0.1,0.2,0.3,0.4),data =x,method = "BFGS",
domain = c(0,1), lim_inf = c(0,0,0,0),
lim_sup = c(10000000000000000,100000000000000,
            10000000000000000,10000000000000000))
```