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THE JOINT ASYMPTOTIC NORMALITY OF THE CONDITIONAL QUANTILES

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Abstract: *Let (X, Y) be a two dimensional random variable with a joint distribution function F(x.y). This paper studies the kernel estimation of the conditional quantiles of Y for a given value of X based on a random sample from the above distribution, which was proposed by [12]. In this paper, the joint asymptotic normality of the conditional quantile estimated at a finite number of distinct points is established under some regularity conditions. Moreover, the performance of the conditional quantile estimation in constructing prediction intervals is tested through two applications. The first application deals with simulated data set and the second deals with real life data set.*

Keywords: *Kernel estimation, conditional distribution, conditional quantile, multivariate distribution.*

2000 MSC: *primary 62G07, 62G08, secondary 62E20*.

1. Introduction

Although most investigation of the conditional distribution function $F(y\,|\,x)$ of a response variable Y given values x of a predictor variable X are concerned with the conditional mean function, other aspects of the conditional distribution function are also often of interest. A new insight about the conditional distribution function can be gained by considering the conditional quantiles functions. Although some individual quantiles, such as the conditional median, are sometimes of interest in practice, but more often one wishes to obtain a collection of conditional quantiles that can characterize the entire conditional distribution of Y given X . Alternatively,

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pairs of extreme conditional quantiles give a conditional prediction intervals within which one expects the majority of individuals points to lie. For more details, see [1] and [15].
Because of their useful applications, estimation of the conditional quantiles has gained particular attention during the recent three decades. [5] used the idea of conditional quantile technique to study salary data. [6] were the first to introduce conditional quantiles and provide detailed background and motivation from econometrics.
The asymptotic properties of nonparametric estimation of conditional quantiles, using kernel or nearest neighbor methods have been studied by [12], [14], [10] and [11]. Recently some new methods of estimating conditional quantiles have been proposed. The first, an approach using a check function is presented by [2], [15] and [16]. An alternative procedure is first to estimate the conditional distribution functions using the double kernel local linear technique, and then to invert it to produce an estimator of the conditional quantile, which is called Yu and Jones estimator. [4] discussed the Reweighted Nadaraya- Watson (RNW) estimator of the conditional distribution function. [1] inverted the RNW estimator of the conditional distribution function to drive a nonparametric estimator for the conditional quantiles of time series data. [1] established the asymptotic normality and weak consistency for the conditional distribution RNW estimator for α- mixing processes, at both boundary and interior points. It was shown that, to the first order, the RNW estimator enjoys the same convergence rates as those for the double kernel local linear estimator of [15]. A generalization of the work of Yu and Jones for a time series is found in [3].
Schuster [13] considered the problem of estimating the conditional regression function of [7] and established the joint asymptotic normality of the conditional regression function estimated at a finite number of distinct points. In this paper, we considered the conditional quantile functions. Following the same techniques of [13] and [12] and under some regularity conditions, the joint asymptotic normality of the conditional quantile estimated at two distinct conditional points is established.
Moreover, the performance of the conditional quantile estimation in constructing prediction intervals is tested through two applications. The first application deals with simulated data set and the second deals with real life data set.
The paper is organized as follows. In Section 2, some basic definitions are introduced. In Section 3, the conditions that allow us to derive the main result are stated. Also, some preliminaries theorems and lemmas are presented. The main result, Theorem 6 is presented and proved in Section 4. In Section 5, the performance of the conditional quantile estimator is tested. The technical arguments and proofs are collected in the Appendix.

2. Basic Definitions

In this section, the basic definitions that are needed throughout this paper are introduced. Let $(X_1,Y_1),(X_2,Y_2)\cdots,(X_n,Y_n)$ be i.i.d. two dimensional continuous random variables with a joint density function $f(x,y)$ and a joint distribution function:

$$F(x,y)=\int_{-\infty}^{y}\int_{-\infty}^{x} f(u,v)\,du\,dv\ .$$

The marginal density function of X is:

$$g(x) = \int_{-\infty}^{\infty} f(x, y)\, dy \ .$$

Definition 1. The conditional density function and the conditional distribution function of Y given $X = x$ are respectively denoted as $f(y \mid x)$, $F(y \mid x)$ and defined by:

$$f(y \mid x) = \frac{f(x, y)}{g(x)}, \quad F(y \mid x) = \int_{-\infty}^{y} f(u \mid x)\, du = \frac{\int_{-\infty}^{y} f(x, u)\, du}{g(x)}.$$

Definition 2. The αth order quantile of the conditional distribution function $F(y \mid x)$ is denoted as $q_\alpha(x)$ and defined by the root of the equation $F(\xi \mid x) = \alpha, 0 < \alpha < 1.$

Definition 3. The kernel estimators of the functions $f(x, y)$, $g(x)$, $f(y \mid x)$ and $F(y \mid x)$ are denoted as $f_n(x, y)$, $g_n(x)$, $f_n(y \mid x)$ and $F_n(y \mid x)$ respectively, and defined by:

$$f_n(x, y) = \frac{1}{n\, a_n^2} \sum_{i=1}^{n} \phi\left(\frac{x - X_i}{a_n}\right) \phi\left(\frac{y - Y_i}{a_n}\right),$$

$$g_n(x) = \int_{-\infty}^{\infty} f_n(x, y)\, dy = \frac{1}{n\, a_n} \sum_{i=1}^{n} \phi\left(\frac{x - X_i}{a_n}\right),$$

$$f_n(y \mid x) = \frac{f_n(x, y)}{g_n(x)}, \text{ and } F_n(y \mid x) = \int_{-\infty}^{y} f_n(u \mid x)\, du = \frac{B_n(x, y)}{g_n(x)},$$

where ϕ is a probability density function, $\{a_n\}$ is a sequence of positive numbers converging to zero and

$$B_n(x, y) = \frac{1}{n\, a_n} \sum_{i=1}^{n} \phi\left(\frac{x - X_i}{a_n}\right) \Phi\left(\frac{y - Y_i}{a_n}\right), \quad \Phi(y) = \int_{-\infty}^{y} \phi(u)\, du.$$

Definition 4. The estimator of $q_\alpha(x)$ is denoted as $q_{\alpha,n}(x)$ and it is defined as the root of the equation $F_n(\xi \mid x) = \alpha, \ 0 < \alpha < 1.$

3. Preliminaries Theorems and Lemmas

In this section, some regularity conditions, notations and basic theorems that will be used in proving the lemmas and the main result are gathered together for easy reference. Also, four lemmas which are necessary to prove the main result in this paper are stated. The technical arguments and proofs are collected in the Appendix.
Firstly, the following conditions from [12], which are needed to prove the lemmas and the main result in this paper, are considered.

1. $F^{(i,j)}(x,y)=\dfrac{\partial^{i+j}F(x,y)}{\partial x^i \partial y^j}$ exist and are bounded for $(i,j)=(1,2),(2,0),(2,1),(3,0)$.
2. $g^{(i)}(x)=\int_{-\infty}^{\infty}\dfrac{\partial^i f(x,y)}{\partial x^i}\,dy$ exist for $i=1,2$.
3. Both $h(x)=\int_{-\infty}^{\infty}\left|\dfrac{\partial f(x,y)}{\partial x}\right|dy$ and $g^{(2)}(x)$ are bounded.
4. The conditional population quantiles $q_\alpha(x)$ are unique and defined by:

$$F(q_\alpha(x)\,|\,x)=\frac{F^{(1,0)}(x,q_\alpha(x))}{g(x)}=\alpha .$$

5. $\phi(u)$ is a function of bounded variation.
6. $\int_{-\infty}^{\infty}u\phi(u)\,du=0.$
7. $\int_{-\infty}^{\infty}u^2\phi(u)\,du<\infty.$
8. $a_n=n^{-\delta},\ \dfrac{1}{5}<\delta<\dfrac{1}{4}.$

Now, some notations, which are needed in the remaining of this paper, are introduced. Define for $i=1,2,$ for $j=1,2,\dots,n,$ the following:

$$U_{nj}^*(x_i)=\frac{1}{a_n}\phi\left(\frac{x_i-X_j}{a_n}\right),\quad V_{nj}^*(x_i)=\frac{1}{a_n}\Phi\left(\frac{q_\alpha(x_i)-Y_j}{a_n}\right)\phi\left(\frac{x_i-X_j}{a_n}\right),$$

$$U_{nj}(x_i)=a_n^{\frac{1}{2}}\left[U_{nj}^*(x_i)-E\left\{U_{nj}^*(x_i)\right\}\right],\ V_{nj}(x_i)=a_n^{\frac{1}{2}}\left[V_{nj}^*(x_i)-E\left\{V_{nj}^*(x_i)\right\}\right],$$

$$U_n(x_i)=\sum_{j=1}^{n}U_{nj}(x_i),\ V_n(x_i)=\sum_{j=1}^{n}V_{nj}(x_i),$$

$$\mathbf{W}_{nj} = \begin{bmatrix} U_{nj}(x_1) \\ U_{nj}(x_2) \\ V_{nj}(x_1) \\ V_{nj}(x_2) \end{bmatrix}, \quad n^{\frac{1}{2}}\,\mathbf{Z}_n = \begin{bmatrix} U_n(x_1) \\ U_n(x_2) \\ V_n(x_1) \\ V_n(x_2) \end{bmatrix},$$

$$w(x_i) = F^{(1,0)}(x_i, q_\alpha(x_i)), \quad n^{\frac{1}{2}}\,\mathbf{Z}_n^* = a_n^{\frac{1}{2}} \begin{bmatrix} \sum_{j=1}^{n}\left[U_{nj}^*(x_1) - g(x_1)\right] \\ \sum_{j=1}^{n}\left[U_{nj}^*(x_2) - g(x_2)\right] \\ \sum_{j=1}^{n}\left[V_{nj}^*(x_1) - w(x_1)\right] \\ \sum_{j=1}^{n}\left[V_{nj}^*(x_2) - w(x_2)\right] \end{bmatrix}.$$

Now, we state Bochner Theorem, Liapounov's Theorem, Cramér-Wold Theorem and Slutsky Theorem, since they play an important role in this paper.

Theorem 1 (Bochner Theorem). Suppose $K(y)$ is a Borel function satisfying the following conditions:

i. $\sup_{-\infty<y<\infty} |K(y)| < \infty.$

ii. $\int_{-\infty}^{\infty} |K(y)|dy < \infty.$

iii. $\lim_{y\to\infty} |y\,K(y)|\,dy = 0.$

Let $g(y)$ satisfying $\int_{-\infty}^{\infty} |g(y)|dy < \infty$, and let a_n be a sequence of positive constants satisfying $\lim_{n\to\infty} a_n = 0.$

Define $g_n(x) = \frac{1}{a_n}\int_{-\infty}^{\infty} K\left(\frac{y}{a_n}\right) g(x-y)dy$. Then, at every point x of continuity of $g(\cdot)$,

$\lim_{n\to\infty} g_n(x) = g(x)\int_{-\infty}^{\infty} K(y)dy$.

Proof. This theorem is Theorem 1A in [8].

Theorem 2 (Liapounov's Theorem). Let $X_k, k \geq 1$, be independent random variables such that $EX_k = \mu_k$ and $VarX_k = \sigma_k^2$, and for some $0 < \delta \leq 1$, $\nu_{2+\delta}^{(k)} = E\,|X_k - \mu_k|^{2+\delta} < \infty$, $k \geq 1$. Also, let $T_n = \sum_{k=1}^{n} X_k$, $\xi_n = ET_n = \sum_{k=1}^{n} \mu_k$, $s_n^2 = VarT_n = \sum_{k=1}^{n} \sigma_k^2$, $Z_n = (T_n - \xi_n)/s_n$ and $\rho_n = s_n^{-(2+\delta)} \sum_{k=1}^{n} \nu_{2+\delta}^{(k)}$. Then, if $\lim_{n\to\infty} \rho_n = 0$, we have $Z_n \xrightarrow{D} N(0,1)$.

Proof. This theorem is Theorem 3.3.2 in [9, p. 108].

Theorem 3 (Cramér-Wold Theorem). Let $\mathbf{X}, \mathbf{X}_1, \mathbf{X}_2, \ldots$ be random vectors in $\mathbf{R}^p$**;** then $\mathbf{X}_n \xrightarrow{D} \mathbf{X}$ if and only if, for every fixed $\mathbf{c} \in \mathbf{R}^p$**,** we have $\mathbf{c}^T \mathbf{X}_n \xrightarrow{D} \mathbf{c}^T \mathbf{X}$.

Proof. This theorem is Theorem 3.2.4 in [9, pp. 106].

Theorem 4 (Slutsky Theorem). Let { $\mathbf{X}_n$ } and { $\mathbf{Y}_n$ } be sequences of random p – vectors such that $\mathbf{X}_n \xrightarrow{D} \mathbf{X}$ and $\mathbf{Y}_n \xrightarrow{p} \mathbf{0}$; also let { $\mathbf{W}_n$ }be a sequence of random $\{w \times p\}$ matrices such that tr{($\mathbf{W}_n$ -**W**)^T ($\mathbf{W}_n$ -**W**)} $\xrightarrow{p}$ 0, where **W** is a nonstochastic matrix. Then

(i) $\mathbf{X}_n \pm \mathbf{Y}_n \xrightarrow{D} \mathbf{X}$,

(ii) $\mathbf{W}_n \mathbf{X}_n \xrightarrow{D} \mathbf{W}\,\mathbf{X}$.

Proof. This theorem is Theorem 3.4.3 in [9, pp. 130].

Theorem 5. Let { $\mathbf{T}_n$ } be a sequence of random p -vectors such that $\sqrt{n}(\mathbf{T}_n - \theta) \to N(\mathbf{0}, \Sigma)$ and consider a vector –valued function $\mathbf{g}: \mathbf{R}^p \to \mathbf{R}^q$ such that $\mathbf{G}(\theta) = \partial / \partial \mathbf{x}^T\, \mathbf{g}(\mathbf{x})|_{x=\theta}$ exists. Then $\sqrt{n}$ { $\mathbf{g}(\mathbf{T}_n)$-$\mathbf{g}(\theta)$} $\xrightarrow{D} N(\mathbf{0}, \mathbf{G}(\theta)\Sigma\,\mathbf{G}(\theta)^T)$.

Proof. This theorem is Theorem 3.4.6 in [9, pp. 136].

To prove our main result, the following four lemmas are needed:

Lemma 1. Under the conditions 1, 2, 3, 5, 6, 7 and 8, the following hold:

i. $\lim_{n\to\infty} E\left\{U_{nj}^2(x_i)\right\} = g(x_i)\int_{-\infty}^{\infty} \phi^2(u)\,du$, $i = 1,2$.

ii. $\lim_{n\to\infty} E\left\{V_{nj}^2(x_i)\right\} = w(x_i)\int_{-\infty}^{\infty} \phi^2(u)\,du$, $i = 1,2$.

iii. $\lim_{n\to\infty} E\left\{U_{nj}(x_i)\,V_{nj}(x_s)\right\} = \begin{cases} w(x_i)\int_{-\infty}^{\infty} \phi^2(u)\,du, & i = s = 1,2; \\ 0, & i \neq s,\ i = 1,2,\ s = 1,2. \end{cases}$

iv. $\lim_{n\to\infty} E\left\{V_{nj}(x_i)\,V_{nj}(x_s)\right\} = 0,\quad i \neq s,\ i = 1,2,\ s = 1,2.$

Lemma 2. Under the conditions 1, 2, 3,5, 6,7 and 8, $\mathbf{Z}_n$ as $n \to \infty$ converges in distribution to a four dimensional random variable with zero mean vector and a covariance matrix, **A**, where

$$\mathbf{A} = \int_{-\infty}^{\infty} \phi^2(u)\,du \cdot \begin{bmatrix} g(x_1) & 0 & w(x_1) & 0 \\ 0 & g(x_2) & 0 & w(x_2) \\ w(x_1) & 0 & w(x_1) & 0 \\ 0 & w(x_2) & 0 & w(x_2) \end{bmatrix}.$$

Now, the definition of the notation $o_p(1)$ is given, since it is needed in Lemma 3 and Lemma 4.

Definition 5. For a sequence $\{X_n\}$ of random variables, if for every $\eta > 0,\ \varepsilon > 0,$ there exists a positive integer $n(\varepsilon, \eta)$, such that

$$P\{|X_n| > \eta\} < \varepsilon, \quad n \geq n(\varepsilon, \eta),$$

then we say that $X_n = o_p(1)$. In order words, $X_n = o_p(1)$ is equivalent to saying that $X_n \to 0$ in probability. The definition extends directly to the vector case by adapting the Euclidean norm. Here, we write $\mathbf{X}_n = o_p(\mathbf{1})$, see [9, pp. 37].

Lemma 3. Under the conditions 1, 2, 3, 5, 6, 7 and 8, $\mathbf{Z}_n^*$ as $n \to \infty$ converges in distribution to a four dimensional normal random variable with zero mean vector and a covariance matrix, **A**.

Lemma 4. Under the conditions 1-5, 7 and 8 if $g(x_i) > 0$ then as $n \to \infty$
$f_n(\xi_i \mid x_i) = f(q_\alpha(x_i) \mid x_i) + o_p(1),\ i = 1, 2.$

4. Main Result

In this section, the main theorem in this paper is presented and proved.
Theorem 6. Under the conditions 1-8, if $g(x_i) > 0$ and $f(x_i, q_\alpha(x_i)) > 0,\ i = 1, 2,$

then $(na_n)^{-\frac{1}{2}}(q_{\alpha,n}(x_1), q_{\alpha,n}(x_2))^T$ is asymptotically normally distributed with mean vector $(q_\alpha(x_1),\ q_\alpha(x_2))^T$ and a diagonal covariance matrix

$$\mathbf{B} = \int_{-\infty}^{\infty} \phi^2(u)\,du \begin{bmatrix} \dfrac{\alpha(1-\alpha)}{g(x_1) f^2(q_\alpha(x_1) \mid x_1)} & 0 \\ 0 & \dfrac{\alpha(1-\alpha)}{g(x_2) f^2(q_\alpha(x_2) \mid x_2)} \end{bmatrix}.$$

Proof. Let $H : \mathbf{R}^4 \to \mathbf{R}^2$, defined by:

$$H(\mathbf{y}) = \left(\frac{y_3}{y_1}, \frac{y_4}{y_2} \right)^T , \quad \mathbf{y} = (y_1, y_2, y_3, y_4).$$

Then the matrix of partial derivatives of H is given by:

$$\partial H(\mathbf{y}) / \partial \mathbf{y} = \begin{bmatrix} -\frac{y_3}{y_1^2} & 0 & \frac{1}{y_1} & 0 \\ 0 & -\frac{y_4}{y_2^2} & 0 & \frac{1}{y_2} \end{bmatrix}. \tag{1}$$

Let $\theta = \left(g(x_1), g(x_2), w(x_1), w(x_2) \right)^T$, $\mathbf{T}_n = \left(T_{n1}, T_{n2}, T_{n3}, T_{n4} \right)^T$,

where:

$$T_{ni} = \begin{cases} \frac{1}{n} \sum_{j=1}^{n} U_{nj}^*(x_i), & i = 1,2; \\ \frac{1}{n} \sum_{j=1}^{n} V_{nj}^*(x_i), & i = 3,4. \end{cases}$$

This implies that:

$$H(\theta) = \left(\frac{w(x_1)}{g(x_1)}, \frac{w(x_2)}{g(x_2)} \right)^T .$$

Let **D** denotes the matrix of partial derivatives of H, evaluated at θ. Then as in Equation (1), we obtain that:

$$\mathbf{D} = \begin{bmatrix} \frac{-w(x_1)}{g^2(x_1)} & 0 & \frac{1}{g(x_1)} & 0 \\ 0 & \frac{-w(x_2)}{g^2(x_2)} & 0 & \frac{1}{g(x_2)} \end{bmatrix}.$$

Now, $\mathbf{Z}_n^*$ can be written as:

$$\mathbf{Z}_n^* = (na_n)^{\frac{1}{2}} (\mathbf{T}_n - \theta).$$

Therefore, by an application of Lemma 3 and Theorem 5, we conclude that:

$$(na_n)^{\frac{1}{2}}\{H(\mathbf{T}_n) - H(\theta)\} = (na_n)^{\frac{1}{2}}\begin{bmatrix} F_n(q_\alpha(x_1)|x_1) - F(q_\alpha(x_1)|x_1) \\ F_n(q_\alpha(x_2)|x_2) - F(q_\alpha(x_2)|x_2) \end{bmatrix} = \mathbf{X}_n \xrightarrow{D} \mathbf{X},$$

where **X** is a bivariate normal random variable with zero mean vector and a covariance matrix, $\mathbf{DAD}^T$ **,** where:

$$\mathbf{DAD}^T = \begin{bmatrix} \dfrac{-w(x_1)}{g^2(x_1)} & 0 & \dfrac{1}{g(x_1)} & 0 \\ 0 & \dfrac{-w(x_2)}{g^2(x_2)} & 0 & \dfrac{1}{g(x_2)} \end{bmatrix}.$$

$$\int_{-\infty}^{\infty}\phi^2(u)du.\begin{bmatrix} g(x_1) & 0 & w(x_1) & 0 \\ 0 & g(x_2) & 0 & w(x_2) \\ w(x_1) & 0 & w(x_1) & 0 \\ 0 & w(x_2) & 0 & w(x_2) \end{bmatrix}.\begin{bmatrix} \dfrac{-w(x_1)}{g^2(x_1)} & 0 \\ 0 & \dfrac{-w(x_2)}{g^2(x_2)} \\ \dfrac{1}{g(x_1)} & 0 \\ 0 & \dfrac{1}{g(x_2)} \end{bmatrix}$$

$$= \int_{-\infty}^{\infty}\phi^2(u)du\begin{bmatrix} \dfrac{1}{g(x_1)}.\dfrac{w(x_1)}{g(x_1)}\left(1-\dfrac{w(x_1)}{g(x_1)}\right) & 0 \\ 0 & \dfrac{1}{g(x_2)}.\dfrac{w(x_2)}{g(x_2)}\left(1-\dfrac{w(x_2)}{g(x_2)}\right) \end{bmatrix}$$

$$= \int_{-\infty}^{\infty}\phi^2(u)du\begin{bmatrix} \dfrac{\alpha(1-\alpha)}{g(x_1)} & 0 \\ 0 & \dfrac{\alpha(1-\alpha)}{g(x_2)} \end{bmatrix},$$

since by Condition 4, we have:

$$\frac{w(x_i)}{g(x_i)} = \frac{F^{(1,0)}(x_i, q_\alpha(x_i))}{g(x_i)} = \alpha, \quad i = 1, 2.$$

Now, using Taylor expansion of $F_n(q_{\alpha,n}(x_i)|x_i)$ around $q_\alpha(x_i), \ i = 1, 2$, the following holds

$$F(q_\alpha(x_i)|x_i) \approx F_n(q_{\alpha,n}(x_i)|x_i) \approx F_n(q_\alpha(x_i)|x_i) + (q_{\alpha,n}(x_i) - q_\alpha(x_i))f_n(\xi_i|x_i),$$

where ξ_i is some random point between $q_{\alpha,n}(x_i)$ and $q_\alpha(x_i)$, $i = 1, 2$.

$$q_{\alpha,n}(x_i)-q_\alpha(x_i)\approx\frac{F(q_\alpha(x_i)\,|\,x_i)-F_n(q_\alpha(x_i)\,|\,x_i)}{f_n(\xi_i\,|\,x_i)},\ i=1,2.$$

Therefore,

$$(na_n)^{\frac{1}{2}}(q_{\alpha,n}(x_i)-q_\alpha(x_i))\approx\frac{(na_n)^{\frac{1}{2}}\left\{F_n(q_\alpha(x_i)\,|\,x_i)-F(q_{\alpha,n}(x_i)\,|\,x_i)\right\}}{f_n(\xi_i\,|\,x_i)}. \tag{2}$$

Equation (2) implies that:

$$(na_n)^{\frac{1}{2}}\begin{bmatrix}q_{\alpha,n}(x_1)-q_\alpha(x_1)\\ q_{\alpha,n}(x_2)-q_\alpha(x_2)\end{bmatrix}\approx(na_n)^{\frac{1}{2}}\begin{bmatrix}\dfrac{F_n(q_\alpha(x_1)\,|\,x_1)-F(q_{\alpha,n}(x_1)\,|\,x_1)}{f_n(\xi_1\,|\,x_1)}\\ \dfrac{F_n(q_\alpha(x_2)\,|\,x_2)-F(q_{\alpha,n}(x_2)\,|\,x_2)}{f_n(\xi_2\,|\,x_2)}\end{bmatrix}=\mathbf{W}_n\ \mathbf{X}_n,$$

where:

$$\mathbf{W}_n=\begin{bmatrix}\dfrac{1}{f_n(\xi_1\,|\,x_1)} & 0\\ 0 & \dfrac{1}{f_n(\xi_2\,|\,x_2)}\end{bmatrix}.$$

Now, let:

$$\mathbf{W}=\begin{bmatrix}\dfrac{1}{f\ (\xi_1\,|\,x_1)} & 0\\ 0 & \dfrac{1}{f\ (\xi_2\,|\,x_2)}\end{bmatrix}.$$

Then by Lemma 4, we get that tr$\{(\mathbf{W}_n-\mathbf{W})^T(\mathbf{W}_n-\mathbf{W})\}\xrightarrow{p}0$.

Next, an application of Slutsky Theorem implies that:

$$(na_n)^{\frac{1}{2}}\begin{bmatrix}q_{\alpha,n}(x_1)-q_\alpha(x_1)\\ q_{\alpha,n}(x_2)-q_\alpha(x_2)\end{bmatrix}=\mathbf{W}_n\ \mathbf{X}_n\xrightarrow{D}\mathbf{W}\ \mathbf{X},$$

which completes the proof of the theorem.

5. Applications

In this section, the performance of the conditional quantile estimator in constructing prediction intervals is tested upon two applications. The first application deals with simulated data set and the second deals with real life data set.

5.1 A simulation study

A sample of size 300 from the model $y = \sin 2\pi(1 - x^2) + \varepsilon,$ where $\varepsilon \sim N(0,1)$ and $x \sim Uniform[0,1]$ is simulated. A scatter plot of the simulated data is shown in Figure1. Then the first 295 observation were used to construct a 90% and 95% prediction intervals $(q_{0.05,n}(x_i), q_{0.95,n}(x_i))$ and $(q_{0.025,n}(x_i), q_{0.975,n}(x_i))$, respectively, $i = 296, \ldots, 300$, for the last five values of the response variable y.

The results are recorded in Table 1. We note that all the prediction intervals contain the corresponding true values. Moreover, to test the good performance of the conditional quantile estimator, the average lengths of the prediction intervals were calculated, and they were 3.66 and 4.77 respectively, which are %48.4 and %56.5 of the range of the original data.

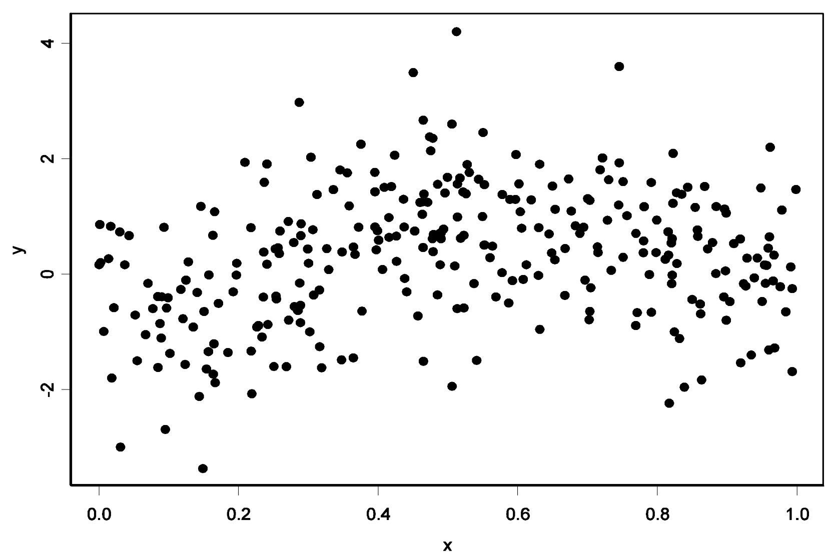

Figure 1. Scatter plot of the simulation data.

Table 1. %90 and %95 prediction intervals for the simulation data.

i	x_i	y_i	$(q_{0.05,n}(x_i), q_{0.95,n}(x_i))$	$(q_{0.025,n}(x_i), q_{0.975,n}(x_i))$
296	0.52	0.67	(-1.66, 2.02)	(-1.97, 2.34)
297	0.43	0.66	(-1.63, 2.05)	(-1.97, 2.45)
298	0.90	-0.39	(-1.49, 2.37)	(-1.62, 2.66)
299	0.24	0.39	(-1.09, 2.45)	(-1.51, 2.66)
300	0.29	0.67	(-1.54, 2.02)	(-1.84, 2.34)

5.2 Real Data

In this subsection, the air pollution data which is built in S-Plus 6.1 program data is considered. It consists of 111 observations, and we interest in the regression of the ozone concentration on the temperature. A scatter plot of ozone against temperature is shown in Figure 2. From the scatter plot, we hypothesize a linear relationship between temperature and ozone concentration. As in Subsection 5.1, the first 106 observation from the data were used to construct 90% and 95% prediction intervals for the last five values of the ozone.

The results are recorded in Table 2. We note that all the prediction intervals contain the corresponding true values. The average lengths of the prediction intervals were 2.05 and 2.39 respectively, which are 46.0% and 52.0% of the range of the original data.

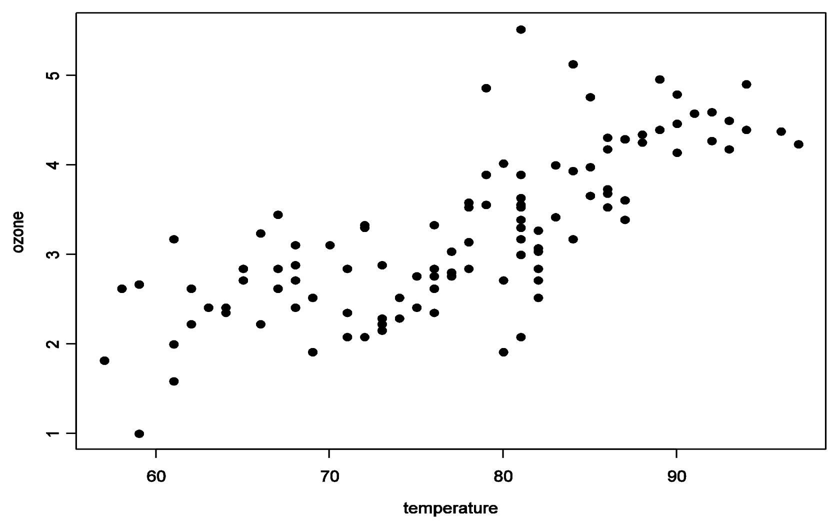

Figure 2. Scatter plot of the ozone against the temperature.

Table 2. %90 and %95 prediction intervals for the ozone data

i	x_i	y_i	$(q_{0.05,n}(x_i), q_{0.95,n}(x_i))$	$(q_{0.025,n}(x_i), q_{0.975,n}(x_i))$
107	63	2.41	(1.27, 3.45)	(1.03, 3.64)
108	70	3.11	(1.71, 3.61)	(1.59, 3.77)
109	75	2.41	(1.79, 3.88)	(1.69, 4.11)
110	76	2.62	(1.84, 4.06)	(1.73, 4.33)
111	68	2.71	(1.74, 3.62)	(1.60, 3.76)

6. Conclusion

In this paper, we studied the conditional quantile estimator which was proposed by [12]. Under some regularity conditions, the joint asymptotic normality of the conditional quantile estimated at a finite number of distinct points is established. Moreover, the good performance of the conditional quantile estimation in constructing prediction intervals is tested upon two applications.

7. Appendix

Here, just the details of the proof of the four lemmas from Section 3 will be given.

Proof of Lemma 1. The proof of this lemma is obtained by using Bochner Theorem.

i. $\left\{EU_{nj}^{2}(x_i)\right\} = a_n\left[\frac{1}{a_n^2}\int_{-\infty}^{\infty}\phi^2\left(\frac{x_i-u}{a_n}\right)g(u)du - \left(\frac{1}{a_n}\int_{-\infty}^{\infty}\phi\left(\frac{x_i-u}{a_n}\right)g(u)du\right)^2\right].$

$$\lim_{n\to\infty}\left\{EU_{nj}^{2}(x_i)\right\} = \lim_{n\to\infty}\frac{1}{a_n}\int_{-\infty}^{\infty}\phi^2\left(\frac{x_i-u}{a_n}\right)g(u)du - \lim_{n\to\infty}a_n\left(\frac{1}{a_n}\int_{-\infty}^{\infty}\phi\left(\frac{x_i-u}{a_n}\right)g(u)du\right)^2$$

$$= g(x_i)\int_{-\infty}^{\infty}\phi^2(u)\,du - 0 = g(x_i)\int_{-\infty}^{\infty}\phi^2(u)\,du,$$

By an application of Bochner Theorem.

ii. Using the definition of $V_{nj}(x_i)$ and Bochner Theorem, we obtain

$$\lim_{n\to\infty}\left\{EV_{nj}^{2}(x_i)\right\} = \lim_{n\to\infty}\frac{1}{a_n^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi^2\left(\frac{q_\alpha(x_i)-v}{a_n}\right)\phi^2\left(\frac{x_i-u}{a_n}\right)f(u,v)\,du\,dv$$

$$= \lim_{n\to\infty}\frac{1}{a_n^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi^2\left(\frac{q_\alpha(x_i)-v}{a_n}\right)f(v\,|\,x_i-a_nu)dv\,\phi^2(u)\,g(x_i-a_nu)\,du$$

$$= g(x_i)\int_{-\infty}^{\infty}\phi^2(u)du.\lim_{n\to\infty}\frac{1}{a_n^2}\int_{-\infty}^{\infty}\Phi^2\left(\frac{q_\alpha(x_i)-v}{a_n}\right)f(v\,|\,x_i-a_nu)dv$$

$$= g(x_i)\int_{-\infty}^{\infty}\phi^2(u)du.\int_{-\infty}^{q_\alpha(x_i)}f(v\,|\,x_i)\,dv$$

$$= g(x_i)F(q_\alpha(x_i)\,|\,x_i)\int_{-\infty}^{\infty}\phi^2(u)du.$$

iii. Similarly as in Part (ii), we have

$$\lim_{n\to\infty} a_n E\left\{U_{nj}(x_i)V_{nj}(x_i)\right\} = \lim_{n\to\infty}\frac{1}{a_n^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi\left(\frac{q_\alpha(x_i)-v}{a_n}\right)\phi^2\left(\frac{x_i-u}{a_n}\right)f(u,v)\,du\,dv$$
$$= g(x_i)F(q_\alpha(x_i)\,|\,x_i)\int_{-\infty}^{\infty}\phi^2(u)\,du.$$

On the other hand,

$$\lim_{n\to\infty} a_n E\left\{U_{nj}(x_i)\,V_{nj}(x_s)\right\} = \lim_{n\to\infty}\frac{1}{a_n^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi\left(\frac{q_\alpha(x_i)-v}{a_n}\right)\phi\left(\frac{x_i-u}{a_n}\right)\phi\left(\frac{x_s-u}{a_n}\right)f(u,v)\,du\,dv = 0,$$

since

$$\lim_{n\to\infty}\frac{1}{a_n}\int_{-\infty}^{\infty}\phi\left(\frac{x_i-u}{a_n}\right)\phi\left(\frac{x_s-u}{a_n}\right)g(u)\,du = 0.$$

iv. Similarly as part (iii).

Proof of Lemma 2. Let $\mathbf{c} = (c_1,c_2,c_3,c_4)^T$, $\mathbf{c}\neq\mathbf{0}$. For $j=1,2,\dots,n$, define

$\sigma_{nj}^2 = \mathrm{Var}\,\{\mathbf{c}^T\,\mathbf{W}_{nj}\}$, $\rho_{nj}^3 = \mathrm{E}|\,n^{-\frac{1}{2}}\{\mathbf{c}^T\,\mathbf{W}_{nj}\}|^3$.

$\sigma_n^2 = \dfrac{1}{n}\displaystyle\sum_{j=1}^{n}\sigma_{nj}^2$, $\quad\rho_n^3 = \displaystyle\sum_{j=1}^{n}\rho_{nj}^3$.

Note that $\mathbf{A}$ is a positive definite matrix whenever $g(x_i)>0$ and $w(x_i)>0$, $i=1,2.$
Using Lemma 1, we obtain for $j=1,2,\dots,n$

$$\lim_{n\to\infty}\sigma_{nj}^2 = \lim_{n\to\infty}\ \mathrm{Var}\left\{c_1U_{nj}(x_1)+c_2U_{nj}(x_2)+c_3V_{nj}(x_1)+c_4V_{nj}(x_2)\right\}$$
$$=\int_{-\infty}^{\infty}\phi^2(u)\,du\left\{c_1^2\,g(x_1)+2c_1c_3w(x_1)+c_2^2\,g(x_2)+2c_2c_4w(x_2)+c_3^2w(x_1)+c_4^2w(x_2)\right\}$$
$$=(c_1,c_2,c_3,c_4)\int_{-\infty}^{\infty}\phi^2(u)du.\begin{bmatrix} g(x_1) & 0 & w(x_1) & 0\\ 0 & g(x_2) & 0 & w(x_2)\\ w(x_1) & 0 & w(x_1) & 0\\ 0 & w(x_2) & 0 & w(x_2)\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\\c_4\end{bmatrix}$$
$$=\mathbf{c}^T\,\mathbf{A}\,\mathbf{c}>0,$$

since $\mathbf{c}^T\,\mathbf{A}\,\mathbf{c}$ is a quadratic form associated with the positive definite matrix $\mathbf{A}$.

Therefore by the definition of σ_n^2, the following holds

$$\lim_{n\to\infty}\sigma_n^2 = \mathbf{c}^T\,\mathbf{A}\,\mathbf{c}>0. \qquad (3)$$

Now, using computations similar to those in Lemma 1 implies that:

$$E\,|U_{n1}(x_i)|^3 = O(a_n^{-\frac{1}{2}}),\quad E\,|V_{n1}(x_i)|^3 = O(a_n^{-\frac{1}{2}}),\ \ i=1,2.$$

This implies that:

$$\begin{aligned}\rho_{nj}^3 &= E\,|n^{-\frac{1}{2}}\mathbf{c}^T\,\mathbf{W}_{nj}|^3 \le |\ \mathbf{c}^T\ |^3\, n^{-\frac{3}{2}} E\,|\ \mathbf{W}_{nj}|^3 \\ &\le 4^{\frac{3}{2}}\,|\ \mathbf{c}^T\ |^3\, n^{-\frac{3}{2}} \max\left\{E\,|U_{n1}(x_i)|^3, E\,|V_{n1}(x_i)|^3,\ i=1,2\right\}.\end{aligned}$$

Therefore, from the definition of ρ_n^3, the following holds:

$$\rho_n^3 \le 4^{\frac{3}{2}} n\,|\ \mathbf{c}^T\ |^3\, n^{-\frac{3}{2}} \max\left\{E\,|U_{n1}(x_i)|^3,\ E\,|V_{n1}(x_i)|^3, i=1,2\right\} = O((n\,a_n)^{-\frac{1}{2}}).$$

Now, an application of Condition (8), implies that:

$$\lim_{n\to\infty}\rho_n^3 = 0. \tag{4}$$

Thus, a combination of Equation 3 and Equation (4) implies that

$$\lim_{n\to\infty}\frac{\rho_n^3}{\sigma_n^3} = 0.$$

Next, an application of Liapounov's Theorem, implies that $\mathbf{c}^T\,\mathbf{Z}_n = n^{-\frac{1}{2}}\sum_{j=1}^{n}\mathbf{c}^T\,\mathbf{W}_{nj}$ converges in distribution to a univariate normal random variable with zero mean and variance $\mathbf{c}^T\,\mathbf{A}\,\mathbf{c}$.

Let $\mathbf{Z}$ be a four dimensional random variable with zero mean vector and a covariance matrix $\mathbf{A}$. Then $\mathbf{c}^T\,\mathbf{Z}$ is a univariate normal random variable with zero mean and variance $\mathbf{c}^T\,\mathbf{A}\,\mathbf{c}$. From above, we have that $\mathbf{c}^T\,\mathbf{Z}_n \xrightarrow{D} \mathbf{c}^T\,\mathbf{Z}$ in distribution. Now, an application of the Cramér-Wold Theorem implies that $\mathbf{Z}_n \xrightarrow{D} \mathbf{Z}$ in distribution. This completes the proof of the lemma.

Proof of Lemma 3. Let:

$$\mathbf{C}_n = \begin{bmatrix} E\left\{U_{n1}^*(x_2)\right\} - g(x_1) \\ E\left\{U_{n2}^*(x_1)\right\} - g(x_2) \\ E\left\{V_{n1}^*(x_1)\right\} - w(x_1) \\ E\left\{V_{n2}^*(x_2)\right\} - w(x_2) \end{bmatrix}.$$

By Taylor expansion and conditions 6 and 7, we get that:

$$
\begin{aligned}
E\left\{U_{n1}^{*}(x_1)\right\}-g(x_1) &= \frac{1}{a_n}\int_{-\infty}^{\infty}\phi\left(\frac{x_1-u}{a_n}\right)g(u)du-g(x_1)\\
&=\int_{-\infty}^{\infty}\phi(u)g(x_1-a_n u)du-g(x_1)\\
&=\int_{-\infty}^{\infty}\phi(u)\left\{g(x_1)-a_n u\, g^{(1)}(x_1)+\frac{a_n^2u^2}{2}g^{(2)}(x_1)+o(a_n^2)\right\}du-g(x_1)\\
&=\frac{a_n^2}{2}g^{(2)}(x_1)\int_{-\infty}^{\infty}u^2\phi(u)du\le Ca_n^2=O(a_n^2).
\end{aligned}
$$

Similarly, for the other.

From the above and Conditions (8), the following holds:

$$
(na_n)^{\frac{1}{2}}\mathbf{C}_n=(na_n)^{\frac{1}{2}}\begin{bmatrix}O(a_n^2)\\O(a_n^2)\\O(a_n^2)\\O(a_n^2)\end{bmatrix}=\begin{bmatrix}(na_n^5)^{\frac{1}{2}}\\(na_n^5)^{\frac{1}{2}}\\(na_n^5)^{\frac{1}{2}}\\(na_n^5)^{\frac{1}{2}}\end{bmatrix}\to\begin{bmatrix}0\\0\\0\\0\end{bmatrix}.
$$

Now, from the definitions of $\mathbf{Z}_n^*$ and $\mathbf{Z}_n$, we obtain

$\mathbf{Z}_n^* = \mathbf{Z}_n + (\mathbf{Z}_n^* - \mathbf{Z}_n) = \mathbf{Z}_n + (na_n)^{\frac{1}{2}}\mathbf{C}_n$ **.**

This implies that $\mathbf{Z}_n^* - \mathbf{Z}_n = o_p(\mathbf{1})$.

Proof of Lemma 4. This lemma is Lemma 6 in [12].

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