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Economic Trend Free Blocked Full 2^k Factorial Experiments

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This article considers blocking full 2^k factorial experiments one-block-at-a-time into $\{1, 2, 4, 8, \dots 2^{k-1}\}$ blocks such that all factor main effects $(A_1, A_2, A_3, \dots A_k)$ are resistant to the polynomial time trend, which might be present in the sequentially generated responses while keeping blocking factors confounded with negligible high order interactions. Cost of factor level changes between successive runs within blocks is also minimized. For each blocked 2^k design we provide the following: (i) the independent blocking factors as negligible high order interactions (ii) the number of factor level changes (i.e., cost) in each block and the total cost of factor level changes in all blocks and (iii) the k independent Generalized Foldover run generators to sequence all 2^k runs of the full 2^k factorial experiment within each block one-run-at-a-time, where $(k - i)$ generators are within blocks generators while i are between blocks generators $(i = 0, 1, 2, \dots k - 1)$, where $i = 0$ for no blocking, $i = 1$ for blocking into 2 blocks, $i = 2$ for blocking into 4 blocks, $\dots i = (k - 1)$ for blocking into 2^{k-1} blocks. Proposed general blocking results have been derived inductively using extensive computer work on blocking trend free full 2^k factorial experiments for $k = 4, 5, 6, \dots 15$ factors.

keywords: Systematic factorial experiments, Generalized Foldover Scheme for runs sequencing, Cost of factor level changes, Factors' time trend resistance, Blocking factors and their confounding structure.

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1 Introduction, literature review and statement of the problem

Two-level full factorial experiments are the most frequently conducted experiments in exploratory investigations for the identification of significant factors affecting the experimental outcome. These experiments require blocking into small incomplete blocks when there are not enough homogeneous experimental units, where blocking reduces experimental error and provides better estimates for each factor effect if higher order factor interactions are negligible. Blocking may confound factor main effects or their low order interactions with blocking factors. But care should be taken to confound blocking factors with negligible higher order factor interactions not with factor main effects.

Experimental runs within blocks of full or fractional factorial experiments could be carried out randomly or systematically one-run-at-a-time. Complete runs randomization may result in large number of factor level changes between successive runs, hence rendering experimentation expensive, especially if the full or fractional factorial experiment involves hard-to-vary factors (like: oven temperature or vessel pressure). The alternative is then systematic implementation of runs of these experiments (i.e. one-run-at-a-time) but this could bias or confound factor effects with the time trend effect, which might be present in the successively collected experimental responses. Therefore, runs of blocked full or fractional factorial experiments should be sequenced within blocks such that factor effects are unaffected by (or orthogonal to) the time trend while keeping factor level changes between successive runs minimal, to economize experimentation cost. In addition, care should be taken not to confound factor main effects with blocking factors but with negligible high order factor interactions.

Several algorithms are available to systematically sequence runs of full or fractional factorial experiments (blocked or unblocked). Most blocking algorithms of full or fractional factorial experiments have concentrated on estimation of factor effects and their low-order interactions unbiased by the blocking factors, while fewer blocking algorithms have concentrated on either: (i) minimizing factor level changes between successive runs within and between blocks or on (ii) securing factor' resistance to the time trend effect. Next is a literature review on this topic along with some comments. For blocking full or fractional factorial experiments where major concern is on securing factor main effects estimable unbiased by blocking factors following are some researches: Bisgaard (1994), Sitter et al. (1997), Zhang and Park (2000), Das et al. (2003),Evangelaras and Koukouvinos (2003), Ye and Li (2003), Wang (2004), Cheng et al. (2004), Butler (2006), Jacroux (2009), Ai et al. (2010), Ou et al. (2011), Bailey (2011), Yang and Li (2013) and Garba et al. (2014). Bounds on the number of factors and on block sizes were other main concerns.

On the other hand, For blocking full or fractional factorial experiments to minimizing cost of factor level changes and/or to securing factors' time trend resistance, we start with Bradley and Yeh (1980) who initiated the idea of trend free block designs for single factor experiments in complete and incomplete blocks, where each block has the same polynomial time trend and where the three linear model components: treatment factors, blocking factors and the polynomial time trend are all orthogonal to each other. Lin and Dean (1991) considered blocking single factor and full factorial experiments using cyclic, generalized cyclic block designs (complete and incomplete blocks), where factors and/or their interactions are orthogonal to the polynomial trend, but allowing runs duplication within blocks.

Coster (1993) proposed a catalog of trend free blocked two- and three-level fractional factorial experiments, where runs within blocks are sequenced by the Generalized Foldover Scheme of Coster and Cheng (1988) and where the cost of factor level changes between successive runs is minimum (or near minimum), while factor effects are orthogonal to the polynomial time trend. The maximum number of factors considered was 17 while the highest fractionation level was 4 (i.e. only up to 1/16 fractions of the full factorial experiment). The (n-p) independent GFS generators to sequence all runs of these 2^{n-p} and 3^{n-p} fractional factorial experiments were provided but neither the Defining Contrasts nor the Resolution of each fractional design were given nor also given the blocking structure. Jacroux et al. (1995) extended the work of Bradley and Yeh (1980) for single factor trend free blocked experiments allowing for different polynomial time trends within blocks. Jacroux (2005) proposed an algorithm to sequence fractional factorial 2^{m-k} designs in 2^i blocks for the early estimation of factor main effects, where factor main effects and their lower order factor interactions are estimable before higher order interactions, hence allowing for the termination of the experiment after any block. However, this block sequencing did not consider the cost of factor level changes within blocks nor did it consider orthogonality of factor effects to the time trend. Sarkar et al. (2009) proposed an algorithm to block full 2^k factorial experiments into only 2 blocks where factor main effects are trend free but without regard to minimizing the cost of factor level changes between successive runs.

Wang and Wu (2013) utilized the Interactions-Main Effects Assignment of **?** for selecting effects columns from the saturated orthogonal array and assigning them as new factor main effects of trend free blocked fractional factorial $s^{(n-p)}$ designs (s: prime and $p < n$). But no algorithm was given nor the minimization of the cost of factor level changes between successive runs was considered and also not considering the blocks-interaction confounding structure.

Bhowmik et al. (2015) proposed an algorithm to sequence runs of blocked and unblocked full sk factorial experiment (s: prime) in minimum cost of factor level changes but without regard to factors' time trend resistance and without considering the blocks -interaction confounding structure. Singh et al. (2016) utilized the generator matrices of linear codes (cyclic and noncyclic) for the construction of trend free blocked and unblocked systematic fractional factorial designs, where rows of these generator matrices are the independent Generalized Foldover generator runs, but no general theory or algorithm was given. Thapliyal and Budhraja (2020) used rows of incidence matrices of Balanced Incomplete Block Designs as GFS generator runs to sequence runs of full (not fractional) factorial experiments such that factor main effects are time trend free but without considering blocking. Guiatni and Hilow (2023) proposed a catalog of minimum cost trend free 2-level fractional factorial designs of resolution IV derivable from Sylvester-Hadamard matrices (of size 2^k) but without considering blocking, where designs' GFS generators were given but designs' Defining Contrasts were not provided.

From this review, it is clear that literature lacks trend free blocked full 2^k and fractional 2^{k-p} factorial experiments with reasonable blocking structure and in minimum cost of factor level changes between successive runs where factor main effects are orthogonal to the time trend present in the sequentially generated responses. It is the objective of this research to undertake this task. The remainder of this paper is as follows: Section 2 starts by illustrating the proposed blocking scheme using blocked trend free full 2^5 factorial experiments into 1,2,4,8,16 blocks, then generalizes these illustrative blocking results inductively to blocking trend free full 2^k factorial experiments into 1,2,4, $\ldots 2^{k-1}$ blocks. Section 3 provides a conclusion and suggests recommendations.

2 Economic trend free blocked full 2^k factorial experiment in 1,2,4,8, $\ldots 2^{k-1}$ blocks

This section starts by considering blocking the full 2^5 factorial experiment into 1, 2, 4, 8, and ($16 = 2^{5-1}$) blocks such that all five factor main effects (A, B, C, D, E) are orthogonal to the linear time trend and such that factor level changes between the $32 = 2^5$ successive runs within blocks are kept to a minimum while confounding blocking factors with negligible high order interactions (not with factor main effects). The Generalized Foldover Scheme of Coster and Cheng (1988) is used to sequence runs within and between all blocks. Then these illustrative blocking results are generalized inductively to a class of economic trend free blocked full 2^k factorial experiments in $\{1, 2, 4, 8, \ldots, 2^{k-1}\}$ blocks.

The detailed full 2^5 factorial experiment under the standard run order can be laid out as in Table (1), where the standard run order is given in the second column of this table. All five factor columns (A, B, C, D, E) and their 26 interactions (of orders 2, 3, 4, 5) are listed in increasing level changes from 1 up to $31 = 2^5 - 1$ as follows: {E, DE, D, CD, CDE, CE, C, BC, BCE, BCDE, BCD, BD, BDE, BE, B, AB, ABE, ABDE, ABD, ABCD, ABCDE, ABCE, ABC, AC, ACE, ACDE, ACD, AD, ADE, AE, A}. The respective number of level changes (i.e., cost) for the five main effect factors (A, B, C, D, E) are: {31, 15, 7, 3, 1}, totaling to the cost (57 = 31 + 15 + 7 + 3 + 1), which is 26 above the minimal total cost $= 31 = 2^5 - 1$. The minimal cost makes only one level change between any two of the 32 successive runs of the full 2^5 experiment.

All 31 effects columns of the full 2^5 factorial experiment (main and interaction effects) can be generated from only 5 independent effect columns, where the other 26 effect columns are dot products of these 5 independent effects columns (once, twice, thrice, 4 and 5 times). One choice for these 5 independent columns is the 5 factor main effects (A, B, C, D, E). All the 31 effects columns of Table (1) and the identity column (one column of 32 ones) form an Abelian group. Similarly, from a row-wise perspective, the 32 rows of the full 2^5 factorial experiment (excepting the null treatment row) can be generated as products (once, twice, thrice, 4 and 5 times) of only five independent generator rows $\{g_1, g_2, g_3, g_4, g_5\}$, where all 32 rows (runs) of the full 2^5 factorial experiment can be sequenced by the GFS technique of Coster and Cheng (1988) using five generator rows $\{g_1, g_2, g_3, g_4, g_5\}$ as follows: (1), g_1, g_2, g_1g_2, g_3, g_1g_3, g_2g_3, $g_1g_2g_3$, g_4, g_1g_4,

g_2g_4, $g_1g_2g_4$, g_3g_4, $g_1g_3g_4$, $g_2g_3g_4$, $g_1g_2g_3g_4$, g_5, g_1g_5, g_2g_5, $g_1g_2g_5$, g_3g_5, $g_1g_3g_5$, $g_2g_3g_5$, $g_1g_2g_3g_5$, g_4g_5, $g_1g_4g_5$, $g_2g_4g_5$, $g_1g_2g_4g_5$, $g_3g_4g_5$, $g_1g_3g_4g_5$, $g_2g_3g_4g_5$, $g_1g_2g_3g_4g_5$.

These 5 generators $\{g_1, g_2, g_3, g_4, g_5\}$ are the $2^{nd}, 3^{rd}, 5^{th}, 9^{th}$, and 17^{th} runs in this GFS sequence. For the standard full 2^5 factorial experiment, the five independent GFS generator runs are: {a = g_1, b = g_2, c = g_3, d = g_4, e = g_5}, where the entire GFS sequence for this standard order is as given in the second column of Table (1) and as follows: {(1), a, b, ab, c, ac, bc, abc, d, ad, bd, abd, cd, acd, bcd, abcd, e, ae, be, abe, ce, ace, bce, abce, de, ade, bde, abde, cde, acde, bcde, abcde}.

Table 1: Full 2^5 factorial experiment under the standard order with its column level changes are in increasing order from 1 up 31.

Run order	Standard Run Order	The 5 factors (A, B, C, D, E) and their 26 interactions of orders (2, 3, 4 and 5)																														
		E	I_{10}	D	I_8	I_{20}	I_9	C	I_5	I_{18}	I_{25}	I_{17}	I_6	I_{19}	I_7	B	I_1	I_{13}	I_{23}	I_{12}	I_{21}	I_{26}	I_{22}	I_{11}	I_2	I_{15}	I_{24}	I_{14}	I_3	I_{16}	I_4	A
1	(1)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	a	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	b	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
4	ab	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
5	c	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0
6	ac	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1
7	bc	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0
8	abc	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
9	d	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0
10	ad	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1
11	bd	0	1	1	1	1	0	0	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	0	0	1	1	1	1	0	0
12	abd	0	1	1	1	1	0	0	1	1	0	0	0	0	1	1	0	0	1	1	1	1	0	0	1	1	0	0	0	0	1	1
13	cd	0	1	1	0	0	1	1	1	1	0	0	1	1	0	0	0	0	1	1	0	0	1	1	1	1	0	0	1	1	0	0
14	acd	0	1	1	0	0	1	1	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	0	0	1	1	0	0	1	1
15	bcd	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0
16	abcd	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
17	e	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0
18	ae	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1
19	be	1	1	0	0	1	1	0	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	0	1	1	0	0	1	1	0
20	abe	1	1	0	0	1	1	0	1	0	0	1	1	0	0	1	0	1	1	0	0	1	1	0	1	0	0	1	1	0	0	1
21	ce	1	1	0	1	0	0	1	1	0	0	1	0	1	1	0	0	1	1	0	1	0	0	1	1	0	0	1	0	1	1	0
22	ace	1	1	0	1	0	0	1	1	0	0	1	0	1	1	0	1	0	0	1	0	1	1	0	0	1	1	0	1	0	0	1
23	bce	1	1	0	1	0	0	1	0	1	1	0	1	0	0	1	1	0	0	1	0	1	1	0	1	0	0	1	0	1	1	0
24	abce	1	1	0	1	0	0	1	0	1	1	0	1	0	0	1	0	1	1	0	1	0	0	1	0	1	1	0	1	0	0	1
25	de	1	0	1	1	0	1	0	0	1	0	1	1	0	1	0	0	1	0	1	1	0	1	0	0	1	0	1	1	0	1	0
26	ade	1	0	1	1	0	1	0	0	1	0	1	1	0	1	0	1	0	1	0	0	1	0	1	1	0	1	0	0	1	0	1
27	bde	1	0	1	1	0	1	0	1	0	1	0	0	1	0	1	1	0	1	0	0	1	0	1	0	1	0	1	1	0	1	0
28	abde	1	0	1	1	0	1	0	1	0	1	0	0	1	0	1	0	1	0	1	1	0	1	0	1	0	1	0	0	1	0	1
29	cde	1	0	1	0	1	0	1	1	0	1	0	1	0	1	0	0	1	0	1	0	1	0	1	1	0	1	0	1	0	1	0
30	acde	1	0	1	0	1	0	1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	0	1	0	1	0	1	0	1
31	bcde	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
32	abcde	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
CLNC		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
LTC		256	0	128	0	0	0	64	0	0	0	0	0	0	0	32	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	16
QTC		*	*	*	*	0	*	*	*	0	0	0	*	0	*	*	*	0	0	0	0	0	0	0	*	0	0	0	*	0	*	*

$I_1 = AB, I_2 = AC, I_3 = AD, I_4 = AE, I_5 = BC, I_6 = BD, I_7 = BE, I_8 = CD, I_9 = CE, I_{10} = DE, I_{11} = ABC, I_{12} = ABD, I_{13} = ABE, I_{14} = ACD, I_{15} = ACE,$
$I_{16} = ADE, I_{17} = BCD, I_{18} = BCE, I_{19} = BDE, I_{20} = CDE, I_{21} = ABCD, I_{22} = ABCE, I_{23} = ABDE, I_{24} = ACDE, I_{25} = BCDE, I26 = ABCDE.$
*Indicates Quadratic Time Count is not zero, i.e. that column effect is not quadratic time trend free.

The last two rows of Table (1) tell which of the 31 effect columns of the standard full 2^5 factorial experiment are orthogonal or resistant to the linear/quadratic time trend. Entries or values in these two rows are called by Draper and Stoneman (1968) the Linear and Quadratic Time Counts. These Time Counts are dot products of the runs order vector $(1, 2, \ldots, 32)$ in the first column of Table (1) and its squares $(1^2, 2^2, \ldots, 32^2)$ with each of the 31 effect columns of Table (1) after replacing factor level coding from $\{0,1\}$ into $\{-1,1\}$. Zero Linear Time Counts indicate columns' orthogonality to the linear time trend, while zero Quadratic Time Counts indicate orthogonality to the quadratic time trend. Therefore, the 26 interaction columns of the standard full 2^5 factorial experiment are linear trend-free, while the five main factor columns (A, B, C, D, E) are not trend-free. These non-trend-free columns (A, B, C, D, E) have level changes of $\{1, 3, 7, 15, 31\}$, respectively. Hence, the standard run order of the full 2^5 factorial experiment is not time trend-free and it is costly as well, having a large number of factor level changes.

We next apply the Interactions-Main Effects Assignment of Cheng and Jacroux (1988) to construct trend-free full 2^5 factorial experiments more time trend resistant than the standard run order. Each application of this assignment produces a different GFS run sequencing of the full 2^5 factorial experiment. Referring to Table (1), we select 5 independent columns from its 26 trend-free interaction columns and assign them as new main effect factors of a full 2^5 factorial experiment. We select trend-free columns (i.e., columns with zero Linear Time counts) with fewer level changes to minimize the cost of factor level changes and to avoid producing replicated rows (runs). Selecting the 5 trend-free interaction columns (DE, CD, CDE, BC, AB) and renaming them as 5 new main effect factors (A, B, C, D, E) produces the minimum cost trend-free full 2^5 factorial experiment in Table (2), whose effect columns (main and interaction effects) are listed in increasing level changes from 1 to 31 as follows: {BC, A, ABC, B, C, AB, AC, D, BCD, AD, ABCD, BD, CD, ABD, ACD, E, BCE, AE, ABCE, AE, ABCE, BE, CE, ABE, ACE, DE, BCDE, ADE, ABCDE, BDE, CDE, ABDE}. The 5 columns $1^{st}, 3^{rd}, 7^{th}, 15^{th}, 31^{st}$, which are the interaction columns (BC, ABC, AC, ACD, ABDE) of this trend-free full 2^5 factorial experiment, are not time trend-free since their Time Counts are non-zeros.

The total cost of level changes for these 5 factors (A, B, C, D, E) of this trend-free full 2^5 factorial experiment is $35 = 2+4+5+8+16$, which is more economical than the cost 57 of the standard full 2^5 factorial experiment in Table (1), costing only 4 above the minimal cost 31. The second column of Table (2) gives the GFS run sequencing of the 32 runs of this trend-free full 2^5 factorial experiment, where the $2^{nd}, 3^{rd}, 5^{th}, 9^{th}$, and 17^{th} runs of this column are the 5 independent GFS run generators $\{g_1 = e, g_2 = de, g_3 = bcd, g_4 = abc, g_5 = ac\}$.

It is worth noting that the column selection {1,2,4,8,16} from Table (1) for new 5 main effect factors of a full 2^5 factorial experiment yields a minimal cost full 2^5 factorial experiment with (cost $31 = 1 + 2 + 4 + 8 + 16$), but this experiment is not trend-free since its first factor (with 1 level change) is non-trend-free. Also, the selection of 5 effect columns (as new 5 main effects) from Table (1) with a total cost less than 31 yields duplicated runs (rows).

An unreplicated full 2^5 factorial experiment (standard, trend-free, or even non-trend-free) requires 32 homogeneous experimental units. However, if this is not fulfilled, then blocking this experiment into small incomplete blocks must be used. The blocking proposed here uses the effect columns of the full 2^5 factorial experiment with level changes $\{1, 3, 7, 15, 31\}$ for blocks where:

- (i) the column with 1 level change is for blocking into 2 blocks of 16 runs each,
- (ii) the column with 3 level changes is for 4 blocks of 8 runs each,
- (iii) the column with 7 level changes is for 8 blocks of 4 runs each, and
- (iv) the column with 15 level changes is for 16 blocks of 2 runs each.

To apply this proposed blocking on the trend-free full 2^5 factorial experiment in Table (2), we proceed as follows: Column $I_5 = BC$ of this table with 1 level change is the blocking factor ($B_1 = BC$) for 2 blocks, where the two blocks are: Block One = $\{(1), e, de, d, bcd, bcde, bce, bc, abc, abce, abcde, abcd, ad, ade, ae, a\}$ is the first 16 runs of the second column of Table (2), while Block Two is the last 16 runs of this column. The four GFS generators $\{g_1 = e, g_2 = de, g_3 = bcd, g_4 = abc\}$ in the $2^{nd}, 3^{rd}, 5^{th}$, and 9^{th} runs of this second column are the within-block GFS generators for sequencing the 16 runs of Block One, where $B_1 = BC = 0$ for this block. B_1 is the blocking factor. The fifth GFS generator $g_5 = ac$, which is the first run of Block Two, is the between-blocks generator for sequencing the 16 runs of this second block by multiplying all 16 runs of Block One by $g_5 = ac$, where $B_1 = BC = 1$ for this second block.

The time run order column $(1, 2, 3, \ldots, 32)$ in the first column of Table (2) is now changed into the sequence $(1, 2, 3, \ldots, 16)$ doubled twice (once for each block) to represent the linear time trend effect in each of the 2 blocks. The linear Time Counts are now the dot products of this doubled run order column with the 31 column effects of Table (2) in 32 entries each. Linear Time Counts of the 5 main effect columns (A, B, C, D, E) of this blocked trend-free full 2^5 factorial experiment are zeros due to factors' trend-freeness. The cost of level changes for these 5 main effect factors (A, B, C, D, E) within each of the 2 blocks is $(17 = 1 + 2 + 2 + 4 + 8)$, totaling 34 for the 2 blocks. That is, the blocked version of this economic trend-free full 2^5 experiment into 2 blocks is less costly than the unblocked version by only 1 level change.

Here, in this blocking, the blocking factor $B_1 = BC$ is confounded with the two-factor interaction BC having 1 level change. To ensure that blocking factors are confounded with higher-order interactions, we need to make a different 5-column selection from Table (1) as new factor main effects (A, B, C, D, E) for a second trend-free full 2^5 factorial experiment in a different GFS run order than the GFS run order of the first trend-free full 2^5 factorial experiment. That is, we need to find a trend-free full 2^5 factorial experiment by the Interactions-Main Effects Assignment where the blocking factor is a higher-order interaction. This will be done later in this section.

For blocking this first trend-free full 2^5 factorial experiment in Table (2) into 4 blocks of 8 runs each, the column ABC with 3 level changes is the blocking factor where the two independent blocking factors B_1 and B_2 are $(B_1 = I_5 = BC, B_2 = I_{11} = ABC)$, and the third blocking factor B_3 is their generalized interaction $(B_3 = B_1B_2 = A)$. That is, this blocking of the first trend-free full 2^5 factorial experiment into 4 blocks is not good since the main effect of factor A is confounded with blocks. Hence, we need to make a new 5-column selection from Table (1) for a second trend-free full 2^5 factorial experiment having better blocking into 4 blocks, where this will be done shortly. Table

Table 2: Minimum cost trend free full 2^5 factorial experiment with its columns in increasing level changes from 1 up 31.

Run Order	Trend free run order	The 5 factors (A, B, C, D, E) and their 26 interactions of orders (2, 3, 4 and 5)																													
		I_5	A	I_{11}	B	C	I_1	I_2	D	I_{17}	I_3	I_{21}	I_6	I_8	I_{12}	I_{14}	E	I_{18}	I_4	I_{22}	I_7	I_9	I_{13}	I_{15}	I_{10}	I_{25}	I_{16}	I_{26}	I_{19}	I_{20}	I_{23}	I_{24}
1	(1)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	e	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	de	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
4	d	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
5	bcd	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0
6	bcde	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1
7	bce	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0
8	bc	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
9	abc	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0
10	abce	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1
11	abcde	0	1	1	1	1	0	0	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	0	0	1	1	1	1	0	0
12	abcd	0	1	1	1	1	0	0	1	1	0	0	0	0	1	1	0	0	1	1	1	1	0	0	1	1	0	0	0	0	1	1
13	ad	0	1	1	0	0	1	1	1	1	0	0	1	1	0	0	0	0	1	1	0	0	1	1	1	1	0	0	1	1	0	0
14	ade	0	1	1	0	0	1	1	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	0	0	1	1	0	0	1	1
15	ae	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0
16	a	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
17	ac	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0
18	ace	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1
19	acde	1	1	0	0	1	1	0	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	0	1	1	0	0	1	1	0
20	acd	1	1	0	0	1	1	0	1	0	0	1	1	0	0	1	0	1	1	0	0	1	1	0	1	0	0	1	1	0	0	1
21	abd	1	1	0	1	0	0	1	1	0	0	1	0	1	1	0	0	1	1	0	1	0	0	1	1	0	0	1	0	1	1	0
22	abde	1	1	0	1	0	0	1	1	0	0	1	0	1	1	0	1	0	0	1	0	1	1	0	0	1	1	0	1	0	0	1
23	abe	1	1	0	1	0	0	1	0	1	1	0	1	0	0	1	1	0	0	1	0	1	1	0	1	0	0	1	0	1	1	0
24	ab	1	1	0	1	0	0	1	0	1	1	0	1	0	0	1	0	1	1	0	1	0	0	1	0	1	1	0	1	0	0	1
25	b	1	0	1	1	0	1	0	0	1	0	1	1	0	1	0	0	1	0	1	1	0	1	0	0	1	0	1	1	0	1	0
26	be	1	0	1	1	0	1	0	0	1	0	1	1	0	1	0	1	0	1	0	0	1	0	1	1	0	1	0	0	1	0	1
27	bde	1	0	1	1	0	1	0	1	0	1	0	0	1	0	1	1	0	1	0	0	1	0	1	0	1	0	1	1	0	1	0
28	bd	1	0	1	1	0	1	0	1	0	1	0	0	1	0	1	0	1	0	1	1	0	1	0	1	0	1	0	0	1	0	1
29	cd	1	0	1	0	1	0	1	1	0	1	0	1	0	1	0	0	1	0	1	0	1	0	1	1	0	1	0	1	0	1	0
30	cde	1	0	1	0	1	0	1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	0	1	0	1	0	1	0	1
31	ce	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
32	c	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
No. of column level changes		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
Linear Time Count		256	0	128	0	0	0	64	0	0	0	0	0	0	0	32	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	16

$I_1 = AB, I_2 = AC, I_3 = AD, I_4 = AE, I_5 = BC, I_6 = BD, I_7 = BE, I_8 = CD, I_9 = CE, I_{10} = DE, I_{11} = ABC, I_{12} = ABD, I_{13} = ABE, I_{14} = ACD, I_{15} = ACE,$ $I_{16} = ADE, I_{17} = BCD, I_{18} = BCE, I_{19} = BDE, I_{20} = CDE, I_{21} = ABCD, I_{22} = ABCE, I_{23} = ABDE, I_{24} = ACDE, I_{25} = BCDE, I_{26} = ABCDE.$

(2.3) provides a summary for blocking this first trend-free full 2^5 factorial experiment in Table (2) into $\{1, 2, 4, 8, 16\}$ blocks, where only blocking into 2 blocks is acceptable while blocking into more than 2 blocks is not good since they confound factor main effects with blocking factors. This table also gives the number of factor level changes within each block and the total number of factor level changes in all blocks. The table also gives the run order column representing the time trend within each blocking scheme.

For better blocking of a trend-free full 2^5 factorial experiment into $\{1, 2, 4, 8, 16\}$ blocks than the blocking in the summary Table (3), we refer to Table (1) and select the 5 trend-free effects columns: (DE, CDE, BC, AB, BE) with respective level changes $\{2, 5, 8, 16, 30\}$ as the new 5 main effect factors (A, B, C, D, E) of a second trend-free full 2^5 factorial experiment whose 32 runs are GFS sequenced as follows: $\{$*(1), de, cd, ce, bc, bcde, bd, be, ab, abde, abcd, abce, ac, acde, ad, ae,abe, abd, abcde, abc, ace, acd, ade, a, e, d, cde, c, bce, bcd, bde, b*$\}$ and whose 31 effects columns are sequenced in increasing column level changes from 1 up to 31 as in Table (2) as follows: {*ABCDE, A, BCDE, ACDE, B, CDE, AB, C, ABDE, AC, BDE, ADE, BC, DE, ABC, D, ABCE, AD, BCE, ACE, BD, CE, ABD, CD, ABE, ACD, BE, AE, BCD, E, ABCD*}.

All of these 31 effects columns of the second trend-free full 2^5 factorial experiment are linear trend-free (with zero Time Count) except the 5 interaction columns {*ABCDE, BCDE, AB, ABC, ABCD*} having nonzero Time Counts and with respective column level changes $\{1, 3, 7, 15, 31\}$. The $2^{nd}, 3^{rd}, 5^{th}, 9^{th}$, and 17^{th} runs of this new GFS runs sequencing of the full 2^5 factorial experiment are the 5 GFS run generators ($g_1 = de, g_2 = cd, g_3 = bc, g_4 = ab, g_5 = abe$), where the total cost of level changes for the 5 trend-free main effect factors (A, B, C, D, E) is $(61 = 2+5+8+16+30)$, which is a high cost but this second trend-free full 2^5 experiment provides better blocking as will be seen shortly.

Table 3: Minimum cost trend free full 2^5 factorial experiment of Table (2) blocked into 2, 4, 8, 16 blocks.

Full 2^5 unblocked						Full 2^5 in 2 blocks						Full 2^5 in 4 blocks						Full 2^5 in 8 blocks						Full 2^5 in 16 blocks					
Run order	A	B	C	D	E	Run order	A	B	C	D	E	Run order	A	B	C	D	E	Run order	A	B	C	D	E	Run order	A	B	C	D	E
1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0
2	0	0	0	0	1	2	0	0	0	0	1	2	0	0	0	0	1	2	0	0	0	0	1	2	0	0	0	0	1
3	0	0	0	1	1	3	0	0	0	1	1	3	0	0	0	1	1	3	0	0	0	1	1	1	0	0	0	1	1
4	0	0	0	1	0	4	0	0	0	1	0	4	0	0	0	1	0	4	0	0	0	1	0	2	0	0	0	1	0
5	0	1	1	1	0	5	0	1	1	1	0	5	0	1	1	1	0	1	0	1	1	1	0	1	0	1	1	1	0
6	0	1	1	1	1	6	0	1	1	1	1	6	0	1	1	1	1	2	0	1	1	1	1	2	0	1	1	1	1
7	0	1	1	0	1	7	0	1	1	0	1	7	0	1	1	0	1	3	0	1	1	0	1	1	0	1	1	0	1
8	0	1	1	0	0	8	0	1	1	0	0	8	0	1	1	0	0	4	0	1	1	0	0	2	0	1	1	0	0
9	1	1	1	0	0	9	1	1	1	0	0	1	1	1	1	0	0	1	1	1	1	0	0	1	1	1	1	0	0
10	1	1	1	0	1	10	1	1	1	0	1	2	1	1	1	0	1	2	1	1	1	0	1	2	1	1	1	0	1
11	1	1	1	1	1	11	1	1	1	1	1	3	1	1	1	1	1	3	1	1	1	1	1	1	1	1	1	1	1
12	1	1	1	1	0	12	1	1	1	1	0	4	1	1	1	1	0	4	1	1	1	1	0	2	1	1	1	1	0
13	1	0	0	1	0	13	1	0	0	1	0	5	1	0	0	1	0	1	1	0	0	1	0	1	1	0	0	1	0
14	1	0	0	1	1	14	1	0	0	1	1	6	1	0	0	1	1	2	1	0	0	1	1	2	1	0	0	1	1
15	1	0	0	0	1	15	1	0	0	0	1	7	1	0	0	0	1	3	1	0	0	0	1	1	1	0	0	0	1
16	1	0	0	0	0	16	1	0	0	0	0	8	1	0	0	0	0	4	1	0	0	0	0	2	1	0	0	0	0
17	1	0	1	0	0	1	1	0	1	0	0	1	1	0	1	0	0	1	1	0	1	0	0	1	1	0	1	0	0
18	1	0	1	0	1	2	1	0	1	0	1	2	1	0	1	0	1	2	1	0	1	0	1	2	1	0	1	0	1
19	1	0	1	1	1	3	1	0	1	1	1	3	1	0	1	1	1	3	1	0	1	1	1	1	1	0	1	1	1
20	1	0	1	1	0	4	1	0	1	1	0	4	1	0	1	1	0	4	1	0	1	1	0	2	1	0	1	1	0
21	1	1	0	1	0	5	1	1	0	1	0	5	1	1	0	1	0	1	1	1	0	1	0	1	1	1	0	1	0
22	1	1	0	1	1	6	1	1	0	1	1	6	1	1	0	1	1	2	1	1	0	1	1	2	1	1	0	1	1
23	1	1	0	0	1	7	1	1	0	0	1	7	1	1	0	0	1	3	1	1	0	0	1	1	1	1	0	0	1
24	1	1	0	0	0	8	1	1	0	0	0	8	1	1	0	0	0	4	1	1	0	0	0	2	1	1	0	0	0
25	0	1	0	0	0	9	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0
26	0	1	0	0	1	10	0	1	0	0	1	2	0	1	0	0	1	2	0	1	0	0	1	2	0	1	0	0	1
27	0	1	0	1	1	11	0	1	0	1	1	3	0	1	0	1	1	3	0	1	0	1	1	1	0	1	0	1	1
28	0	1	0	1	0	12	0	1	0	1	0	4	0	1	0	1	0	4	0	1	0	1	0	2	0	1	0	1	0
29	0	0	1	1	0	13	0	0	1	1	0	5	0	0	1	1	0	1	0	0	1	1	0	1	0	0	1	1	0
30	0	0	1	1	1	14	0	0	1	1	1	6	0	0	1	1	1	2	0	0	1	1	1	2	0	0	1	1	1
31	0	0	1	0	1	15	0	0	1	0	1	7	0	0	1	0	1	3	0	0	1	0	1	1	0	0	1	0	1
32	0	0	1	0	0	16	0	0	1	0	0	8	0	0	1	0	0	4	0	0	1	0	0	2	0	0	1	0	0
Level changes	2	4	5	8	16	Level changes	1	2	2	4	8	Level changes	0	1	1	2	4	Level changes	0	0	0	1	2	Level changes	0	0	0	0	1
Total cost = 35						Total cost per block= 17 Total cost for all 2 blocks = 34						Total cost per block = 8 Total cost for all 4 blocks = 32						Total cost per block = 3 Total cost for all 8 blocks = 24						Total cost per block= 1 Total cost for all 16 blocks = 16					
Independent blocking factors are:						$B_1 = BC$ for 2 blocks, $B_1 = BC$ and $B_2 = ABC$ for 4 blocks, $B_1 = BC, B_2 = ABC$ and $B_3 = AC$ for 8 blocks, $B_1 = BC, B_2 = ABC, B_3 = ACandB_4 = ACD$ for 16 blocks.																													

For blocking this second trend free full 2^5 factorial experiment = {*(1), de, cd, ce, bc,* *bcde, bd, be, ab, abde, abcd, abce, ac, acde, ad, ae, abe, abd, abcde, abc, ace, acd, ade, a, e, d, cde, c, bce, bcd, bde, b*} in 2 blocks, whose 31 effects columns are sequenced in increasing column level changes from 1 up to 31 as follows: {*ABCDE, A, BCDE, ACDE, B, CDE, AB, C, ABDE, AC, BDE, ADE, BC, DE, ABC, D, ABCE, AD, BCE, ACE, BD, CE, ABD, CD, ABE, ACD, BE, AE, BCD, E, ABCD*}, the interaction column ABCDE with 1 level change is the blocking factor (i.e. $B_1 = ABCDE$), where the two blocks are: Block One = {*(1), de, cd, ce, bc, bcde, bd, be, ab, abde, abcd, abce, ac, acde, ad, ae*} consisting of the first 16 runs of the GFS sequencing of this second trend free full 2^5 factorial experiment while the second block = $\{abe, abd, abcde, abc, ace, acd, ade, a, e, d, cde, c, bce, bcd, bde, b\}$ is the last 16 runs of this GFS sequencing. The four GFS generators $g_1 = de, g_2 = cd, g_3 = bc, g_4 = ab$ are the within block generators for the first block where ($B_1 = ABCDE = 0$), while the fifth GFS generator $g_5 = abe$ is the between blocks generator for this second block, where ($B_1 = ABCDE = 1$). The cost of level changes of the 5 main effect factors (A, B, C, D, E) of the second trend free full 2^5 factorial experiment within each of these two blocks is ($30 = 1+2+4+8+15$), totaling 60 for the 2 blocks. This cost is higher than the cost=34 of the blocking into 2 blocks of the first trend free full 2^5 factorial experiment but this blocking is better, since the blocking factor here is ($B_1 = ABCDE$) while the blocking factor of the first trend free full 2^5 factorial experiment into 2 blocks was ($B_1 = BC$).

For blocking this second trend free full 2^5 factorial experiment. {*(1), de, cd, ce, bc, bcde, bd, be, ab, abde, abcd, abce, ac, acde, ad, ae, abe, abd, abcde, abc, ace, acd, ade, a, e, d, cde, c, bce, bcd, bde, b*} in 4 blocks, whose 31 effects columns are sequenced in increasing column level changes from 1 up to 31 as follows: {*ABCDE, A, BCDE, ACDE, B, CDE, AB, C, ABDE, AC, BDE, ADE, BC, DE, ABC, D, ABCE, AD, BCE, ACE, BD, CE, ABD, CD, ABE, ACD, BE, AE, BCD, E, ABCD*}, we partition these 32 runs into four parts (blocks) by the two independent interaction columns with $\{1, 3\}$ level changes as the two independent blocking factors ($B_1 = ABCDE$, $B_2 = BCDE$), where their generalized interaction ($B_3 = B_1B_2 = A$) is the third blocking factor. Hence, this blocking of the second trend free full 2^5 factorial experiment into 4 blocks is not good (since a factor main effect is confounded with blocks) yet its blocking into 2 blocks was very good since ($B_1 = ABCDE$) is the highest order interaction.

Therefore, and for better blocking into 4 blocks, we refer to Table (1) and select 5 new trend free columns as 5 main effect factors for a third trend free full 2^5 factorial experiment in a different GFS run sequence. Hence, we select the 5 trend free columns (CDE, BCD, $ABCD$, BC, AB) with respective level changes $\{5, 11, 20, 8, 16\}$ as the new 5 main effect factors (A, B, C, D, E) of the following third trend free full 2^5 factorial experiment whose GFS runs sequencing is as follows: {*(1), ce, bcde, bd, abcd, abde, ae, ac, abc, abe, ade, acd, d, cde, bce, b, a, ace, abcde, abd, bcd, bde, e, c, bc, be, de, cd, ad, acde, abce, ab*} and whose 31 effects columns are sequenced in increasing column level changes from 1 up to 31 as follows: {*ACE, ABCDE, BD, CE, A, ABD, BCDE, D, ACDE, ABCE, B, CDE, AD, AB, BCE, E, AC, ABCD, BDE, C, AE, ABDE, BCD, DE, ACD, ABC, BE, CD, ADE, ABE, BC*}.

All of these 31 effects columns are linear trend free except the 5 interaction columns {ACE, BD, BCDE, BCE, BC} with respective level changes {1, 3, 7, 15, 31}. The

$2^{nd}, 3^{rd}, 5^{th}, 9^{th}$, and 17^{th} runs of this GFS runs sequencing are the 5 GFS run generators $g_1 = ce, g_2 = bcde, g_3 = abcd, g_4 = abc, g_5 = a$, where the cost of level changes for the 5 trend free main effect factors (A, B, C, D, E) equals $60 = 5 + 11 + 20 + 8 + 16$.

For blocking this third trend free full 2^5 factorial experiment {*(1), ce, bcde, bd, abcd, abde, ae, ac, abc, abe, ade, acd, d, cde, bce, b, a, ace, abcde, abd, bcd, bde, e, c, bc, be, de, cd, ad, acde, abce, ab*} into 4 blocks whose 31 effects columns are sequenced in increasing column level changes from 1 up to 31 as follows: {*ACE, ABCDE, BD,CE, A, ABD, BCDE, D, ACDE, ABCE, B, CDE, AD, AB, BCE, E, AC, ABCD, BDE, C, AE, ABDE, BCD, DE, ACD, ABC, BE, CD, ADE, ABE, BC*}, the two effects columns with level changes {1, 3} are the independent blocking factors $B_1 = ACE, B_2 = BD$ and their generalized interaction is the third blocking factor $B_3 = B_1B_2 = ABCDE$, which is the highest order interaction. Hence, this blocking of the third trend free full 2^5 factorial experiment into 4 blocks is better than the blocking into 4 blocks of the second trend free full 2^5 factorial experiment.

The four blocks of this third trend free full 2^5 factorial experiment are constructed by partitioning the 32 runs of its GFS sequencing into four parts (blocks), 8 runs per block. The three GFS generators $g_1 = ce, g_2 = bcde, g_3 = abcd$ are the within block generators where $(B_1 = 0, B_2 = 0, B_3 = 0)$ for all 8 runs of the first block. The fourth GFS generator $g_4 = abc$ is the between block generator for the second block, where $(B_1 = 0, B_2 = 1, B_3 = 1)$. The fifth GFS generator $g_5 = a$ is the between blocks generator for the third block, where $(B_1 = 1, B_2 = 0, B_3 = 1)$. Finally, the 8 runs of the fourth block can be generated by multiplying each of the 8 runs of Block One by $g_4g_5 = bc$, where $(B_1 = 1, B_2 = 1, B_3 = 0)$ for all 8 runs of Block Four. The cost of level changes for the 5 factors (A, B, C, D, E) of this third trend free full 2^5 factorial experiment within each block is $14 = 1 + 2 + 5 + 2 + 4$, with the total cost for all four blocks being 56.

The run order column representing the linear time trend effect in each of these four blocks of the third trend free full 2^5 factorial experiment is now the sequence $(1, 2, \ldots, 8)$ replicated 4 times, once for each block. Linear Time Counts are now the dot products of this quadrupled run order column with the 31 column effects of 32 entries each.

For blocking this third trend free full 2^5 factorial experiment ={*(1), ce, bcde, bd, abcd, abde, ae, ac, abc, abe, ade, acd, d, cde, bce, b, a, ace, abcde, abd, bcd, bde, e, c, bc, be, de, cd, ad, acde, abce, ab*} into 8 blocks whose 31 effects columns are sequenced in increasing column level changes from 1 up to 31 as follows: {*ACE, ABCDE, BD, CE, A, ABD, BCDE, D, ACDE, ABCE, B, CDE, AD, AB, BCE, E, AC, ABCD, BDE, C, AE, ABDE, BCD, DE, ACD, ABC, BE, CD, ADE, ABE, BC*}, the three effects columns with respective level changes {1, 3, 7} are the three independent blocking factors ($B_1 = ACE$, $B_2 = BD$, $B_3 = BCDE$), where their four generalized interactions are the other single-degree-of-freedom blocking factors: ($B_4 = B_1B_2 = ABCDE$, $B_5 = B_1B_3 = ABD$, $B_6 = B_2B_3 = CE$, $B_7 = B_1B_2B_3 = A$). Hence, this blocking of the third trend free full 2^5 factorial experiment into 8 blocks is not good, since factor main effects are confounded with blocks. So we need to go back to Table (1) and construct a fourth trend free full 2^5 factorial experiment by the Interactions-Main effects Assignment for better blocking into 8 blocks. That is, we need to look for another trend free GFS run order for the full

2^5 factorial experiment.

Referring to Table (1) and selecting the 5 trend-free columns (BC, AB, BCE, BCDE, BDE) with respective level changes {8, 16, 9, 10, 13} as the new 5 factors (A, B, C, D, E) for the fourth trend-free full 2^5 factorial experiment, whose 32 runs are GFS sequenced as follows: {*(1), b, abcde, acde, acd, abcd, be, e, de, bde, abc, ac, ace, abce, bd, d, cde, bcde, ab, a, ae, abe, bcd, cd, c, bc, abde, ade, ad, abd, bce, ce*}.

The 31 effects columns are sequenced in increasing column level changes from 1 up to 31 as follows: {*AC, AD, CD, CE, AE, ACDE, DE, A, C, D, ACD, ACE, E, CDE, ADE, B, ABC, ABD, BCD, BCE, ABE, ABCDE, BDE, AB, BC, BD, ABCD, ABCE, BE, BCDE, ABDE*}. All these 31 effects columns except the 5 interaction columns {AC, CD, DE, ADE, ABDE} are linear trend-free. These 5 non-trend-free effect columns are with respective level changes {1, 3, 7, 15, 31}. The 2^{nd}, 3^{rd}, ^{5th}, 9^{th}, and 17^{th} runs of this fourth GFS runs sequencing are the 5 GFS run generators ($g_1 = b$, $g_2 = abcde$, $g_3 = acd$, $g_4 = de$, $g_5 = cde$), where the cost of level changes for the 5 trend-free main effect factors (A, B, C, D, E) equals (56=8+16+9+10+13).

For blocking this fourth trend-free full 2^5 factorial experiment into 8 blocks, the three effects columns {AC, CD, DE} with level changes {1, 3, 7} are the independent blocking factors ($B_1 = AC$, $B_2 = CD$, $B_3 = DE$), where their generalized interactions ($B_4 = B_1B_2 = AD$, $B_5 = B_1B_3 = ACDE$, $B_6 = B_2B_3 = CE$, $B_7 = B_1B_2B_3 = AE$) are the other blocking factors. This blocking structure for the fourth trend-free full 2^5 factorial experiment is reasonable as it involves two-factor interactions and not factor main effects, while blocking of the third trend-free full 2^5 factorial experiment into 8 blocks was not reasonable.

The eight blocks of this fourth trend-free full 2^5 factorial experiment are constructed by partitioning the 32 run GFS sequencing into 8 blocks with 4 runs each. The two GFS generators ($g_1 = b$, $g_2 = abcde$) are the within block generators for Block One, where ($B_1 = 0$, $B_2 = 0$, $B_3 = 0$). The third GFS generator ($g_3 = acd$) is the between blocks generator for Block Two, where ($B_1 = 0$, $B_2 = 0$, $B_3 = 1$). The fourth GFS generator ($g_4 = de$) is the between blocks generator for Block Three, where ($B_1 = 0$, $B_2 = 1$, $B_3 = 0$). The four runs of Block Four are generated by $g_3g_4 = acd.de = ace$, where ($B_1 = 0$, $B_2 = 1$, $B_3 = 1$). The fifth GFS generator ($g_5 = cde$) is the between blocks generator for Block Five, where ($B_1 = 1$, $B_2 = 0$, $B_3 = 0$). The four runs of Block Six are generated by $g_3g_5 = acd.cde = ae$, where ($B_1 = 1$, $B_2 = 0$, $B_3 = 1$). The four runs of Block Seven are generated by $g_4g_5 = de.cde = c$, where ($B_1 = 1$, $B_2 = 1$, $B_3 = 0$). Finally, the four runs of Block Eight are generated by $g_3g_4g_5 = acd.de.cde = ad$, where ($B_1 = 1$, $B_2 = 1$, $B_3 = 1$). The cost of level changes for the 5 factors (A, B, C, D, E) within each block is $(6 = 1 + 2 + 1 + 1 + 1)$, with a total cost of 48 for the eight blocks.

The run order column representing the linear time trend effect in each of the 8 blocks of the fourth trend-free full 2^5 factorial experiment is the sequence $(1, 2, 3, 4)$ replicated eight times, once for each block. Linear Time Counts are now the dot products of this run order column (replicated eight times) with the 31 effects columns of 32 runs each, where the 5 factor main effects (A, B, C, D, E) are time trend-free with zero linear Time Counts.

We next generalize all these illustrative blocking results into blocking trend-free full

2^k factorial experiments into $\{1, 2, 4, 8, \ldots, 2^{k-1}\}$ blocks. This generalization is found inductively by considering various blockings of trend-free full 2^k factorial experiments in $k = 4, 5, \ldots, 15$ factors.

First, we note that the full 2^k factorial experiment in k two-level factors $(A_1, A_2, \ldots, A_k)$ has 2^k treatment combinations (runs), which can be sequenced run-after-run in $2^k!$ possible run orders, where symbol (!) stands for permutations. A subset of these $2^{k!}$ run orders can be found by the GFS runs sequencing scheme of Coster and Cheng (1988) using k independent generator runs $\{g_1, g_2, \ldots, g_k\}$, where some of the GFS run orders are minimal cost-wise and/or time trend free. One of these GFS run orders of the full 2^k factorial experiment is the standard run order: (1), a_1, a_2, a_1a_2, a_3, a_1a_3, a_2a_3, $a_1a_2a_3$, a_4, $\ldots$, $a_1a_2\ldots a_k$, (where $a_i = 0, 1$, and $i = 1, 2, 3, \ldots, k$). whose k GFS run generators are: $\{g_1 = a_1, g_2 = a_2, g_3 = a_3, \ldots, g_k = a_k\}$ located in the 2^{nd}, 3^{rd}, 5^{th}, 9^{th}, $\ldots$, $(2^{k-1}+1)^{\text{th}}$ runs of this standard order of the sequence of 2^k runs. This standard run order makes a total cost of factor level changes between the 2^k successive runs equal to:

$$(2^{k+1} - k - 2) = [(2^k - 1) + (2^{k-1} - 1) + (2^{k-2} - 1) + (2^{k-3} - 1) + \ldots + (2^{k-(k-1)} - 1)],$$

which is above the minimal cost $(2^k - 1)$ by $(2^{k+1} - 2^k - k - 1)$, i.e., it is a costly run order. When $k = 5$, this standard run order is given in Table (2.1).

The $(2^k - 1)$ effects columns of the standard full 2^k factorial experiment can be arranged in increasing level changes from 1 up to $(2^k - 1)$, where its k main effect columns $(A_1, A_2, \ldots, A_k)$ are in respective level changes $\{1, 3, 7, 15, \ldots, 2^k - 1\}$, but these k factor main effects are not time trend free, since their linear Time Counts of Draper and Stoneman (1968) are nonzero. The remaining $(2^k - 1 - k)$ interaction columns of orders $(2, 3, \ldots, k)$ are, however, time trend free, having zero linear Time Counts. See Time Count illustration above when $k = 5$.

Trend-free full 2^k factorial experiments can be constructed from the standard full 2^k factorial experiment by arranging the columns in increasing level changes from 1 up to $(2^k - 1)$ using Cheng and Jacroux's (1988) assignment approach. This involves selecting k columns from the $(2^k - 1 - k)$ trend-free columns of the standard full 2^k factorial experiment and assigning them as new k main effect factors $(A_1, A_2, \ldots, A_k)$. To minimize the cost of factor level changes, trend-free columns with small level changes must be selected as new factor main effects. There are many possibilities for these k main effect columns, hence there are many trend-free full 2^k factorial experiments which can be generated by the Interactions-Main effect assignment. Some generated full 2^k factorial experiments are economic with minimal or near-minimal cost of factor level changes, while others are not economical nor trend-free.

The k independent GFS rows (runs) for each trend-free full 2^k factorial experiment change as we shift from the standard order to a trend-free run order, and they also change when we shift from one trend-free run order to another. Additionally, each trend-free full 2^k factorial experiment can be laid out in main effect and interaction effects columns in increasing level changes from 1 up to $(2^k - 1)$. Effect columns with level changes $\{1, 3, 7, 15, 31, \ldots, 2^k - 1\}$ are not time trend-free; they have nonzero Time Counts.

After the detailed blocking illustration of the full 2^5 factorial experiment and after extensive inductive blocking of the full 2^k factorial experiment (for $k = 5, 6, 7, 8, 9, \ldots, 15$) along the lines of these illustrations, the following are the general blocking results for trend-free full 2^k factorial experiments produced from the standard full 2^k factorial experiment in increasing column level changes from 1 up to (2^k-1) by the Interaction-Main Effect assignment of Cheng and Jacroux (1988). Here, k trend-free main effect columns $(A_1, A_2, \ldots, A_k)$ are selected from the $(2^k - k - 1)$ trend-free effect columns as the k factor main effects of an economic trend-free full 2^k factorial experiment having good blocking in 2^k blocks $(r = 0, 1, 2, 4, \ldots, 2^{k-1})$. The k non-trend-free effect columns of the generated trend-free full 2^k factorial experiments having level changes $\{1, 3, 7, \ldots, 2^k-1\}$ are the blocking factors.

2.1 Unblocked minimum cost trend free full 2^k factorial experiment in 2^r blocks, $r = 0$

The k effect columns to be selected from the $(2^k - k - 1)$ trend-free columns of the standard full 2^k factorial experiment in increasing column level changes from 1 up to (2^k-1) as k main factors $A_1, A_2, \ldots, A_k$ $(k > 3)$ of an unblocked minimum cost trend-free full 2^k factorial experiment and their column level changes are:

$$\mathrm{A}_i = \begin{cases} 2^i, & if\ i = 1, 2 \\ 5, & if\ i = 3 \\ 2^{i-1}, & if\ i = 4, 5, \ldots k \end{cases}$$

The total cost of level changes for these k main effect factors equals $\sum_{i=1}^{k} \mathrm{A}_i = 2^k + 3$, which is above the minimal cost (2^k-1) by only 4. The k GFS run generators $g_1, g_2, \ldots g_k$ of this trend free full 2^k factorial experiment are:

$$g_i = \begin{cases} a_k, & if\ i = 1; \\ a_{k-i+1}a_{k-i+2}, & if i = 2, 3, .., k-3, (k \geq 5); \\ a_{k-i}a_{k-i+1}a_{k-i+2},, & if i = k-2, k-1; \\ a_1a_3, & if i = k. \end{cases}$$

which are the $2^{nd}, 3^{rd}, 5^{th}$, ..., $(2^{k-1}+1)^{th}$ runs in this systematic GFS run order of 2^k successive runs. Putting $k = 5$ produces the minimum cost trend-free full 2^5 factorial experiment in Table (2).

2.2 Blocked minimum cost trend free full 2^k factorial experiment into 2^r blocks $(r = 1)$

The k effect columns to be selected from the $(2^k - k - 1)$ trend-free columns of the standard full 2^k factorial experiment in increasing column level changes from 1 up to $(2^k - 1)$ as k main factors $A_1, A_2, \ldots, A_k$ $(k > 3)$ of a blocked minimum cost trend-free full 2^k factorial experiment into 2 blocks, and their column level changes are:

$$\mathrm{A}_i = \begin{cases} 2^i, & if\ i = 1, 3, \ldots k-1 \\ 5, & if\ i = 2 \\ 2^k - 2, & if\ i = k \end{cases}$$

The total cost of level changes for the k main effect factors $(A_1, A_2, \ldots, A_k)$ equals $\sum_{i=1}^{k} \mathrm{A}_i \ = 2^{k+1}-3$. The blocking factor is the highest order interaction $(B_1 = A_1 A_2 \ldots A_k)$, where the $2^k - 1$ runs of Block One have $(B_1 = 0)$ and the $2^k - 1$ runs of Block Two have $(B_1 = 1)$. The k GFS within and between generators $\{g_1, g_2, \ldots, g_k\}$ to sequence all 2^k runs of this trend-free full 2^k factorial experiment in 2 blocks are:

$$g_i = \begin{cases} a_{k-i} a_{k-i+1}, & if\ i = 1, 2, \ldots k-1 \quad (within) \\ a_1 a_2 a_k, & if\ i = k \quad (between) \end{cases}$$

which are the $2^{nd}, 3^{rd}, 5^{th}$, ..., $(2^{k-1}+1)^{th}$ runs in this trend-free full 2^k factorial experiment? The number of level changes $n(A_i)_j$ for the ith main effect column A_i in block j $(j = 1, 2)$ is:

$$n(\mathrm{A}_i)_j = \begin{cases} 2^{i-1}, & if\ i = 1, 2, \ldots, k-1 \\ 2^{k-1} - 1, & if\ i = k \end{cases}, \text{ where } n(\mathrm{A}_i)_1 = n(\mathrm{A}_i)_2, \forall \mathrm{A}_i.$$

while the number of level changes for the ith main effect column factor A_i in all 2 blocks is:

$$n(\mathrm{A}_i) = \sum_{j=1}^{2} n(\mathrm{A}_i)_j = \begin{cases} 2^i, & if\ i = 1, 2, \ldots k-1 \\ 2^k - 2, & if\ i = k \end{cases}$$

The total cost for all k main effect factors $(A_1, A_2, \ldots, A_k)$ in block j, $(j = 1, 2)$, is:

$$\mathrm{TC}(\text{block}\, j) = \sum_{i=1}^{k} n(\mathrm{A}_i)_j \ = 2^k - 2, \text{ where } \mathrm{TC}(\text{block}\, 1) = \mathrm{TC}(\text{block}\, 2)$$

while the total cost for all k main effect factors $(A_1, A_2, \ldots, A_k)$ in all two blocks is:

$$\mathrm{TC}(\text{all blocks}) = \sum_{j=1}^{2} \mathrm{TC}\,(\text{bolck}\, j) = 2^{k+1} - 4$$

Putting $k = 5$ produces the second trend-free full 2^5 factorial experiment in the above illustration.

2.3 Blocked minimum cost trend free full 2^k factorial experiment into 2^r blocks $(r > 1)$

The k effect columns to be selected from the $(2^k - k - 1)$ trend-free columns of the standard full 2^k factorial experiment in increasing column level changes from 1 up to

$(2^k - 1)$ as k main factors $A_1, A_2, \ldots, A_k$ $(k > 3)$ of a blocked minimum cost trend-free full 2^k factorial experiment into 2^r blocks, and their column level changes are:

$$\mathrm{A}_i = \begin{cases} \left.\begin{array}{ll} 2^r + \frac{1}{3}(2^i - 2^{i \bmod 2}), & if\ i = 1, 2, \ldots r+1 \\ 2^{i-1}, & if\ i = r+2, \ldots k, (k > r+1) \end{array}\right\}, r+1 \leq k < 2r, (r \geq 3) \\ \left.\begin{array}{ll} 2^{r-1+i} + 2^i - 1, & if\ i = 1, 2, \ldots r \\ 2^{r-1+i} + 2^{i-1}, & if\ i = r+1, \ldots k-r, (k \geq 2r+1) \\ 2^{i-1}, & if\ i = k-r+1, \ldots k \end{array}\right\}, k \geq 2r \end{cases}$$

The total cost of level changes for all its k main effect factors $(A_1, A_2, \ldots, A_k)$ equals

$$\sum_{i=1}^{k} \mathrm{A}_i = \begin{cases} 2^k + r2^r - 2^r + \sum\limits_{j=1}^{r} [\frac{r+2-j}{2}] 2^{j-1}, & if\ r+1 \leqslant k < 2r, (r \geqslant 3) \\ 2^{k+1} - r - 2, & if\ k \geqslant 2r \end{cases}$$

The k GFS within and between generators $\{g_1, g_2, \ldots, g_k\}$ to sequence all its 2^k runs in the 2^r blocks $(r = 2, 3, \ldots, k-1)$ are:

$$g_i = \begin{cases} \left.\begin{array}{l} \left.\begin{array}{ll} \left\{\begin{array}{ll} a_1 a_2 \ldots a_k, & k = r+1 \\ a_k, & k > r+1 \end{array}\right\}, & if\ i = 1, \\ a_{k+1-i} a_{k+2-i}, & if\ i = 2, \ldots k-r-1, (k > r+2) \\ a_1 a_2 \ldots a_{r+2}, & if\ i = k-r > 1 \end{array}\right\} within \\ \left.\begin{array}{ll} a_1 a_2 \ldots a_r, & if\ i = k-r+1 \\ a_{k+2-i} \ldots a_{r+1}, & if\ i = k-r+2, \ldots k \end{array}\right\} between \end{array}\right\}, r+1 \leqslant k < 2r, (r \geqslant 3) \\ \left.\begin{array}{l} \left.\begin{array}{ll} a_{k-r} a_k, & if\ i = 1 \\ a_{k-r+1-i} a_{k-r+2-i} a_{k+1-i} a_{k+2-i}, & if\ i = 2, 3, \ldots k-r \end{array}\right\} within \\ \left.\begin{array}{ll} a_1 a_r a_{r+1}, & if\ i = k-r+1 \\ a_{k+1-i}, & if\ i = k-r+2, \ldots k \end{array}\right\} between \end{array}\right\}, k \geqslant 2r \end{cases}$$

which are the $2^{nd}, 3^{rd}, 5^{th}, \ldots, (2^{k-1}+1)^{th}$ runs of this systematic order of 2^k successive runs blocked in 2^r blocks. The independent blocking factors $(B_1, B_2, \ldots, B_r)$ for blocking this trend-free full 2^k factorial experiment $(k > 3)$ are:

$$\mathrm{B}_i = \left\{\begin{array}{ll} \mathrm{A}_i \mathrm{A}_{i+1}, & if\ r+1 \leqslant k < 2r,\ r \geqslant 3 \\ \mathrm{A}_i \mathrm{A}_{i+r} \mathrm{A}_{i+2r} \ldots \mathrm{A}_{i+mr}, & if\ m = [\frac{k-i}{r}],\ k \geqslant 2r \end{array}\right\}, i = 1, 2, \ldots r,$$

where $\lfloor x \rfloor$ is the floor function and where none of these blocking factors (or their generalized interactions) is a factor main effect. The number of level changes for the ith main effect column $n(A_i)_j$ in block j, $j = 1, 2, \ldots, 2^r$ is given by:

$$n(\mathrm{A}_i)_j = \begin{cases} \left.\begin{array}{ll} 1, & if\ i = 1, 2, \ldots r+1 \\ 2^{i-1-r}, & if\ i = r+2, \ldots k \end{array}\right\}, r+1 \leqslant k < 2r, (r \geqslant 3) \\ \left.\begin{array}{ll} 2^{i-1}, & if\ i = 1, 2, \ldots r \\ 2^{i-1} + 2^{i-1-r}, & if\ i = r+1, \ldots k-r, (k \geqslant 2r+1) \\ 2^{i-1-r}, & if\ i = k-r+1, \ldots k \end{array}\right\}, \ k \geqslant 2r \end{cases},$$

$$\text{where } n(\mathrm{A}_i)_1 = n(\mathrm{A}_i)_2 = \ldots = n(\mathrm{A}_i)_{2^r}, \forall \mathrm{A}_i$$

while the number of level changes for the i^{th} main effect column factor A_i in all 2^r blocks is:

$$n(\mathrm{A}_i) = \sum_{j=1}^{2^r} n(\mathrm{A}_i)_j = \begin{cases} \left.\begin{array}{ll} 2^r, & if\ i = 1, 2, \ldots r+1 \\ 2^{i-1}, & if\ i = r+2, \ldots k \end{array}\right\}, r+1 \leqslant k < 2r, (r \geqslant 3) \\ \left.\begin{array}{ll} 2^{r+i-1}, & if\ i = 1, 2, \ldots r \\ 2^{r+i-1} + 2^{i-1}, & if\ i = r+1, \ldots k-r, (k \geqslant 2r+1) \\ 2^{i-1}, & if\ i = k-r+1, \ldots k \end{array}\right\}, k \geqslant 2r. \end{cases}$$

The total cost for all k main effect factors $(A_1, A_2, \ldots, A_k)$ in block j, $(j = 1, 2, 3, \ldots, 2^r)$, is:

$$\mathrm{TC}(\text{block } j) = \sum_{i=1}^{k} n(\mathrm{A}_i)_j = \begin{cases} 2^{k-r} + r - 1, & if\ r+1 \leqslant k < 2r,\ (r \geqslant 3) \\ 2^{k-r+1} - 2, & if\ k \geqslant 2r \end{cases}$$

where

$$\mathrm{TC}(block\ 1) = TC(block\ 2) = \ldots = TC(block\ 2^r).$$

while the total cost for all k main effect factors $(A_1, A_2, \ldots A_k)$ in all 2^r blocks equals:

$$\mathrm{TC}(\text{all } blocks) = \sum_{j=1}^{2^r} \mathrm{TC}(\text{block } j) = \begin{cases} 2^k + (r-1)2^r, & if\ r+1 \leqslant k < 2r,\ (r \geqslant 3) \\ 2^{k+1} - 2^{r+1}, & if\ k \geqslant 2r \end{cases}$$

Putting $k = 5$ produces the fourth trend-free full 2^5 factorial experiment in the above illustration blocked into $4, 8$ blocks (i.e., $r = 2, 3$). The next four tables, namely Tables (4) - (7), contain some blocking results of the full 2^5 factorial experiment, which are identical to the blocking results in the illustrations portion of this section.

Table 4: Minimum cost trend free full 2^5 factorial experiment in one block (i.e. no blocking)

Columns selected from Table (1)	Factors and their column level changes					GFS generators
	A_1	A_2	A_3	A_4	A_5	
	2	4	5	8	16	
	0	0	0	0	1	$g_1 = a_5$
	0	0	0	1	1	$g_2 = a_4a_5$
	0	1	1	1	0	$g_3 = a_2a_3a_4$
	1	1	1	0	0	$g_4 = a_1a_2a_3$
	1	0	1	0	0	$g_5 = a_1a_2$
Cost of level changes for each factor	2	4	5	8	16	Total cost=35

Table 5: Trend free full 2^5 factorial experiment in 2 blocks with blocking factor ($B_1 = A_1A_2A_3A_4A_5$)

Columns selected from Table (1)	Factors and their column level changes					GFS generators	
	A_1	A_2	A_3	A_4	A_5		
	2	5	8	16	30		
	0	0	0	1	1	$g_1 = a_4a_5$	within
	0	0	1	1	0	$g_2 = a_3a_4$	within
	0	1	1	0	0	$g_3 = a_2a_3$	within
	1	1	0	0	0	$g_4 = a_1a_2$	within
	1	1	0	0	1	$g_5 = a_1a_2a_5$	between
Cost of level changes for each factor per block	1	2	4	8	15	Total cost per block=30	
Cost of level changes for each factor in all blocks	2	4	8	16	30	Total cost for all blocks=60 Total cost of full 2^5 unblocked=61	

Table 6: Trend free 2^5 factorial experiment in 4 blocks with Blocking Factors:($B_1 = A_1A_3A_5, B_2 = A_2A_4, B_3 = B_1B_2 = A_1A_2A_3A_4A_5$)

Columns selected from Table (1)	Factors and their column level changes					GFS generators	
	A_1	A_2	A_3	A_4	A_5		
	5	11	20	8	16		
	0	0	1	0	1	$g_1 = a_3a_5$	within
	0	1	1	1	1	$g_2 = a_2a_3a_4a_5$	
	1	1	1	1	0	$g_3 = a_1a_2a_3a_4$	
	1	1	1	0	0	$g_4 = a_1a_2a_3$	between
	1	0	0	0	0	$g_5 = a_1$	
Cost of level changes for each factor per block	1	2	5	2	4	Total cost per block=14	
Cost of level changes for each factor in all blocks	4	8	20	8	16	Total cost for all blocks=56 Total cost of full 2^5 unblocked=60	

Table 7: Trend free full 2^5 factorial experiment in 8 blocks with Blocking Factors: ($B_1 = A_1A_2, B_2 = A_2A_3, B_3 = A_3A_4, B_4 = B_1B_2 = A_1A_3, B_5 = B_1B_3 = A_1A_2A_3A_4, B_6 = B_2B_3 = A_2A_4, B_7 = B_1B_2B_3 = A_1A_4$)

Columns selected from Table (1)	Factors and their column level changes					GFS generators	
	A_1	A_2	A_3	A_4	A_5		
	8	9	10	13	16		
	0	0	0	0	1	$g_1 = a_5$	within
	1	1	1	1	1	$g_2 = a_1a_2a_3a_4a_5$	
	1	1	1	0	0	$g_3 = a_1a_2a_3$	
	0	0	1	1	0	$g_4 = a_3a_4$	between
	0	1	1	1	0	$g_5 = a_2a_3a_4$	
Cost of level changes for each factor per block	1	1	1	1	2	Total cost per block=6	
Cost of level changes for each factor in all blocks	8	8	8	8	16	Total cost for all blocks=48 Total cost of full 2^5 unblocked=56	

3 Conclusion and Recommendation

Having proposed this blocking scheme for trend-free full 2^k factorial experiments into blocks of sizes $\{1, 2, 4, \ldots, 2^{k-1}\}$, experimenters now have a handy catalog of systematically blocked trend-free full 2^k factorial experiments to select from. In these designs, factor main effects are time trend-free and uncompounded with blocks, and the number of factor level changes (i.e., cost) between successive runs within blocks is minimized. Additionally, blocking factors are confounded with negligible high-order interactions, not with main effects or low-order interactions.

All this has been achieved without limiting the number of factors in the experiment and without limiting the number of blocks. The proposed blocking scheme can be extended to blocking trend-free 2^{k-p} fractional factorial experiments, which will be the focus of upcoming research. Resistance to higher-degree time trends is another possible extension, as well as blocking with different polynomial time trends within blocks.

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