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By Alsarraf, Algamal

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A New Ridge – type in the Bell Regression Model

Israa Najeeb Saeed Alsarraf[*a] and Zakariya Yahya Algamal[†a]

[a]*Department of Statistics and Informatics, University of Mosul, Mosul, Iraq.*

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In scenario analysis, collinearity is a big issue in analyzing such relationship as between the response variable and several explanatory variables. As for these difficulties, the linear regression model, often traditionally, offers a range of shrinkage estimators. One such estimator is the ridge estimator. Thus, in order to fit count data with over-dispersion, for the bell regression model, this paper presents an improvement of the new Ridge-type estimator. Judging from the Monte Carlo simulation and the application of the Bell regression model, it was noted that the proposed estimate yields on average a smaller mean squared error than the other candidate estimators.

keywords: Collinearity; ridge-type estimator; Bell regression model; count data; Over-dispersion; Monte Carlo simulation.

1 Introduction

Since statistical modeling helps in explaining the gradient of the functionality between the response variable of interest and a number of explanatory variables, it is important in many scientific study areas. The dependent variable in the linear regression model is assumed to follow a normal distribution Further, it is assumed that observations in the dependent variable are independent and identically distributed. However, this assumption may not hold a lot of real-world applications particularly in day-to-day use of these technologies. For instance, the response variable in the medical sciences refers to an outcome that can be positive skewed. Consequently, using a linear regression model to some extent might not be reasonable. Linear regression models of the GLM

*Corresponding authors: Email: israa_alsarraf@uomosul.edu.iq
†Corresponding authors: E-mail: zakariya.algamal@uomosul.edu.iq

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are gradually finding their way into other models as a statistical modeling tool applicable for both continuous and discrete dependent variables(Algamal et al., 2023; Mahmood et al., 2020; Algamal, 2019).

In real applications, the design data matrix X has multicollinearity between explanatory variables, and, therefore, X^TX is singular or can be inflating the variance of the maximum likelihood estimator (MLE). Because of this, the methods of estimation such as the MLE often fail to generate good outcomes. In order to overcome multicollinearity problem in the linear regression model, one different alternative method to MLE is the ridge, Liu, Liu type and other estimators based on the other authors(Hoerl and Kennard, 1970; Kejian, 1993; Algamal, 2020; Algamal and Lee, 2017; Aladeitan et al., 2021; Algamal and Abonazel, 2022; Abonazel et al., 2022; Seifollahi et al., 2024). These estimators have been extended to the GLMs (Akram et al., 2022; Kibria, 2003; Kibria et al., 2012; Kurtoğlu and Özkale, 2016; Mackinnon and Puterman, 1989; Månsson and Shukur, 2011; Nyquist, 1991; Segerstedt, 1992; Shamany et al., 2019).

The main objective stated in this paper is to construct the new ridge type estimator to analyze the count data with over dispersion. It is very essential to know that the above proposed estimator will perform efficiently better than some of the other existing estimators in GLM. Existing comparative analyses within various simulated examples and a real data application prove the advantage of our proposed estimator.

2 New Ridge estimator in Bell regression model

Assume that (y_i, x_i), $i = 1, 2, ..., n$ is independent observed data with the predictor vector $x_i \in R^{p+1}$ and the response variable $y_i \in R$ which follows a distribution that belongs to the Bell distribution. Then, the density function of y_i can be expressed as

$$P(Y = y) = \frac{\theta^y e^{-e^\theta + 1} B_y}{y!}, \ \ y = 0, 1, 2, ..., \tag{1}$$

where $\theta > 0$ and $B_y = (1/e) \sum_{d=0}^{\infty} (d^y/d!)$ is the Bell numbers (Bell, 1934a,b; Castellares et al., 2018; Seifollahi and Bevrani, 2023; Erkoç et al., 2023; Abonazel and Taha, 2023). The mean and variance of the Bell distribution are respectively defined by

$$E(y) = \theta e^\theta, \tag{2}$$

$$Var(y) = \theta(1 + \theta) e^\theta. \tag{3}$$

Assuming $\psi = \theta e^\theta$ and $\theta = W_\circ(\psi)$ where $W_\circ(.)$ is the Lambert function. Then Eq(1) can be written in the new parameterization as

$$P(Y = y) = \exp\left(1 - e^{W_\circ(\psi)}\right) \frac{W_\circ(\psi)^y B_y}{y!}, \ \ y = 0, 1, 2, ..., \tag{4}$$

In GLM, the mean of the response variable, $\mu_i = E(y_i)$, is conditionally related to a linear function of predictors through a link function. The linear function is stated as $\eta_i = \beta_0 + \sum_{j=1}^{p} x_{ij}\beta_j = x_i^T\beta$ with $x_i^T = (1, x_{i2}, x_{i3}, ..., x_{ip})$ and $\beta = (\beta_0, \beta_1, ..., \beta_p)^T$. The link function is providing the relation of the mean and the natural parameter as $\mu_i = g^{-1}(\eta_i) = g^{-1}(x_i^T\beta)$. The Bell regression model (BRM) can be modeled by assuming $\psi_i = \exp\left(x_i^T\beta\right)\ \exp\left(\exp\left(x_i^T\beta\right)\right)$ and $\log\psi_i = x_i^T\beta\ \exp\left(x_i^T\beta\right)$ as $y_i \sim \mathrm{B}ell\left(W_\circ\left(\psi_i\right)\right)$. The parameter estimation in the BRM is achieved through using the MLE based on the iteratively reweighted least-squares algorithm. The log-likelihood is defined

$$\ell(\beta,\psi) = \sum_{i=1}^n y_i \log\left(\exp\left(x_i^T\beta\right)\exp\left(e^{\left(x_i^T\beta\right)}\right)\right) + \sum_{i=1}^n \left(1 - e^{e^{\left(x_i^T\beta\right)}\ e^{e^{\left(x_i^T\beta\right)}}}\right) \\ + \log B_y - \log\left(\prod_{i=1}^n y_i!\right). \tag{5}$$

Then, the MLE is derived by equaling the first derivative of Eq(5) to zero. This derivative cannot be solved analytically because it is nonlinear in β. Fisher-scoring algorithm can be used to obtain the MLE where in each iteration, the parameter is updated by

$$\beta^{(r+1)} = \beta^{(r)} + I^{-1}(\beta^{(r)})S(\beta^{(r)}), \tag{6}$$

where $I^{-1}(\beta) = \left(-E\left(\partial^2\ell(\beta,\phi)/\partial\beta\partial\beta^T\right)\right)^{-1}$. After that, the estimated coefficients are defined as

$$\hat{\beta}_{MLE} = (X^T\hat{W}X)^{-1}X^T\hat{W}\hat{u}, \tag{7}$$

where $\hat{W} = \mathrm{d}iag\left[(\partial\mu_i/\partial\eta_i)^2/V(y_i)\right]$ and $\hat{u}$ is a vector where ith element equals to $\hat{u}_i = \log\hat{\psi}_i + [(y_i - \hat{\mu}_i)/\sqrt{var(\hat{\psi}_i)}$. The MLE is distributed asymptotically normal with a covariance matrix as

$$cov(\hat{\beta}_{MLE}) = \left[-E\left(\frac{\partial^2\ell(\beta,\phi)}{\partial\beta\,\partial\beta^T}\right)\right]^{-1} = (X^T\hat{W}X)^{-1}. \tag{8}$$

In the presence of multicollinearity, the $rank(X^T\hat{W}X) \leq rank(X)$, and, therefore, the near singularity of $X^T\hat{W}X$ makes the estimation unstable and enlarges the variance (Liu and Piantadosi, 2017). The ridge estimator (RE) (Hoerl and Kennard, 1970), Liu estimator (Kejian, 1993) have been consistently demonstrated to be an attractive and alternative to the MLE, when multicollinearity exists. In Bell regression model, the ridge estimator and Liu estimator have been proposed by Majid et al. (2022)and Akram et al. (2022), respectively. The Bell-Ridge estimator is defined as follows:

$$\widehat{\boldsymbol{\beta}}_{\mathrm{k-BRM}} = \left(\mathbf{I} + k\left(\mathbf{X}^T\widehat{\mathbf{W}}^T\mathbf{X}\right)^{-1}\right)^{-1}\widehat{\boldsymbol{\beta}}_{\mathrm{MLE}}, \tag{9}$$

where $k > 0$ is the shrinkage parameter. The Bell-Liu estimator is given as:

$$\widehat{\boldsymbol{\beta}}_{\mathrm{k-BRM}} = \left(\mathbf{I} + \left(\mathbf{X}^T\widehat{\mathbf{W}}^T\mathbf{X}\right)^{-1}\right)^{-1}\left(\mathbf{X}^T\widehat{\mathbf{W}}^{\boldsymbol{T}}\mathbf{X} + d\mathbf{I}\right)\mathbf{X}^T\widehat{\mathbf{W}}^T\mathbf{X}\widehat{\boldsymbol{\beta}}_{\mathrm{MLE}}, \tag{10}$$

where d $(0 < d < 1)$ is the shrinkage parameter.

In this article, we propose a new one-parameter estimator in the class of ridge and Liu estimators, which will carry most of the characteristics from both ridge and Liu estimators.
The New One-Parameter Estimator. The proposed estimator is obtained by minimizing the following objective function:

$$(y - X\beta)'(y - X\beta) + k[(\beta + \widehat{\beta})'(\beta + \widehat{\beta}) - c], \tag{11}$$

with respect to β, will yield the normal equations

$$\left(X\acute{X} + kI_p\right)\beta = \acute{X}y - k\widehat{\beta}, \tag{12}$$

Where k is the nonnegative constant. The solution to (12) gives the new estimator as

$$\widehat{\beta}_{KL} = (S + kI_p)^{-1}(S - kI_p)\widehat{\beta} = W(k)M(k)\ \widehat{\beta}, \tag{13}$$

Where S= XX, $W(k) = [I_p + kS^{-1}]^{-1}$, and M(k) = $\left[I_p - kS^{-1}\right]$. The new proposed estimator will be called the Kibria-Lukman (KL) estimator and denoted by $\widehat{\beta}_{KL}$

3 Properties of the New Estimator

$$E\left(\widehat{\beta}_{KL}\right) = W(k)M(k)E\left(\widehat{\beta}\right) = W(k)M(k)\beta. \tag{14}$$

The proposed estimator is a biased estimator unless k=0.

$$B\left(\widehat{\beta}_{KL}\right) = [W(k)M(k)I_p]\beta, \tag{15}$$

$$D\left(\widehat{\beta}_{KL}\right) = \sigma^2 W(k)M(k)S^{-1}M'(k)W'(k), \tag{16}$$

And the mean square error matrix (MSEM) is defined as

$$MSEM\left(\widehat{\beta}_{KL}\right) = \sigma^2 W(k)M(k)S^{-1}M'(k)W'(k) + [W(k)M(k) - I_p]\beta\beta'[W(k)M(k) - I_p]'. \tag{17}$$

To compare the performance of the four estimators (OLS, RR, Liu, and KL), we rewrite (1) in the canonical form which gives

$$y = Z\alpha + \varepsilon, \tag{18}$$

Where Z = XQ and $\alpha = \acute{Q}\beta$. Here, Q is an orthogonal matrix such that $Z'Z = QX'XQ = \Lambda = diag\ (\lambda_1, \lambda_2, \ldots, \lambda_p)$. The OLS estimator of α is

$$\widehat{\alpha} = \Lambda^{-1}\acute{Z}y, \tag{19}$$

$$MSEM(\widehat{\alpha}) = \sigma^2\Lambda^{-1} \tag{20}$$

The ridge estimator (RE) of α is

$$\widehat{\alpha}\left(k\right)=W\left(k\right)\widehat{\alpha}, \tag{21}$$

Where W(k) = $\left[I_p+k\Lambda^{-1}\ \right]^{-1}$ and k is the biasing parameter.

$$MSEM\left(\widehat{\alpha}(K)\right)=\sigma^2W\left(k\right)\Lambda^{-1}W\left(k\right)+\left(W\left(k\right)-I_p\right)\alpha\alpha(W\left(k\right)-I_p)', \tag{22}$$

Where $(W\left(k\right)-I_p)=-k(\Lambda+kI_p)^{-1}$.The Liu estimator of α is

$$\widehat{\alpha}\left(d\right)=\left(\Lambda+I_p\right)^{-1}\left(\acute{Z}Y+d\widehat{\alpha}\right)=F\left(d\right)\widehat{\alpha}, \tag{23}$$

Where F(d) =$\left[\Lambda+I_p\right]^{-1}\left[\Lambda+dI_p\right]$.

$$MSEM(\widehat{\alpha}(d))=\sigma^2F_d\Lambda^{-1}F_d+\ (1-d)^2(1-d)^2(\Lambda+1)^{-1}.\alpha\alpha'(\Lambda+1)^{-1}, \tag{24}$$

Where $F_d=\left(\Lambda+I\right)^{-1}\left(\Lambda+dI\right)$.
The proposed one-parameter estimator of α is

$$\widehat{\alpha}_{KL}=\left(\Lambda+kI_p\right)^{-1}\left(\Lambda-kI_p\right)\widehat{\alpha}=W\left(k\right)M\left(k\right)\widehat{\alpha}, \tag{25}$$

Where W(k) = $\left[I_p+k\Lambda^{-1}\right]^{-1}$and M(k) = $\left[I_p-k\Lambda^{-1}\right]$.
The following notations and lemmas are needful to prove the statistical property of $\widehat{\alpha}_{KL}$:
Lemma 1. Let n$\times n$ matrices M>0 and N>0 (or N$\geq$ 0); then, M>N if and only if $\lambda_1\left(NM^{-1}\right)<1$, where $\lambda_1(NM^{-1})$ is the largest eigenvalue of matrix NM^{-1} [28].
Lemma 2. Let M be an n$\times n$ positive definite matrix, that is M>0 and α be some vector; then, M-$\alpha\alpha'\geq 0$ if and only if $\alpha' M^{-1}\alpha\leq 1$ [29].
Lemma 3. Let $\widehat{\alpha}_i=\ A_iy$, i=1,2, be two linear estimators of α. Suppose that D=Cov($\widehat{\alpha}_1$) $-$ Cov$\left(\widehat{\alpha}_2\right)>0$, where Cov($\widehat{\alpha}_i$) i=1,2 denotes the covariance matrix of $\widehat{\alpha}_i$ and $b_i=Bias\ \left(\widehat{\alpha}_i\right)=\left(A_iX-I\right)\alpha$, i=1,2. Consequently,

$$\Delta\left(\widehat{\alpha}_1-\widehat{\alpha}_2\right)=MSEM\left(\widehat{\alpha}_1\right)-MSEM\left(\widehat{\alpha}_2\right)=\sigma^2D+b_1b_2'-b_2b_2'>0 \tag{26}$$

If and only if $b_2'\left[\sigma^2D+b_1b_1\right]^{-1}b_2<1$, where MSEM($\widehat{\alpha}_i$) = Cov($\widehat{\alpha}_i$) + b_ib_i' [30].

The other parts of this article are as follows. The theoretical comparison among the estimators and estimation of the biasing parameters are given in Section 3. We conducted two numerical examples in Section 4. This paper ends up with concluding remarks in Section 5.

4 Simulation Study

In this section, we simulate explanatory variables that are collinear and a response variable y that follows a bell distribution. The explanatory variables are obtained in line with the study of Lukman et al. (2019, 2022) as follows:

$$x_{ij}=\sqrt{(1-\rho^2)}m_{ij}+\rho m_{i(j+1)},\ i=1,\ldots,n;j=1,\ldots p \tag{27}$$

where m_{ij} are independent standard normal pseudo-random numbers and ρ^2 denotes the correlation between the explanatory variables such that $\rho = 0.7,\ 0.8,\ 0.9$, and 0.999. We assumed that $y_i \sim bell\left(W_o\left(\mu_i\right)\right)$, where

$$\log(\mu_i) = \eta_i = \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ip} \tag{28}$$

The sample sizes are varied such that n=30, 50, and 100 while p is taken to be 4, 8,12 and 16. The true values of the regression parameter β are chosen such that $\sum_{i=1}^{p} \hat{\beta}_i^2 = 1$ Alkhateeb and Algamal (2022); Kibria and Lukman (2020). The simulation study is conducted by adopting the RStudio programming language with the help of bellreg-package. The experiment was replicated 1000 times and the mean squared error (MSE) was employed to evaluate the estimators' performance.

$$MSE(\beta^*) = \frac{1}{1000}\sum_{j=1}^{100}(\beta^*_{ij} - \beta_i)'(\beta^*_{ij} - \beta_i) \tag{29}$$

where β^*_{ij} is the estimator and β_i is the parameter.

The MSE of the simulated data is provided in Tables 1-4 under different simulation conditions. MLE performance is not satisfactory due to the presence of multicollinearity. For instance, from Table 3 at sample size 30, ρ=0.9 and p=12, the MSE for MLE is 50.097. The MSEs for the other estimators are as follows: 14.221, 29.465, and 11.964. This agrees with the literature that MLE suffers setback when the regressors are collinear.

We also observed that the MSE of each of the estimators increase when the level of multicollinearity increases at a particular sample size. For instance, from Table 2 when n=50 for p=8, the MSE values for the proposed estimator are 3.384, 4.39, 5.402 and 10.662, respectively. Also, the MSE of each of the estimators decreases as the sample sizes increases when other factors are kept constant. From Table 1, the MSE for proposed estimator for p=3 and ρ=0.99 are as follows: 1.961 (n=30), 1.571 (n=50), and 1.480 (n=100). It is very obvious that the MSE rise as the number of explanatory variables (p) increase. The performances of the biased estimators and the proposed estimator are competitive especially when the level of multicollinearity is moderate- say ρ=0.7. However, the proposed estimator shows superiority when the level of multicollinearity becomes high.

Table 1: Mean squared error of simulated data when ($p = 3$)

N	Estimator				
		r=0.70	r=0.80	r=0.90	r=0.99
30	MLE	5.1548	5.3434	5.7976	13.7301
	RIDGE	1.7183	1.7483	1.8673	2.0603
	Liu	1.7607	1.7633	1.8826	9.5983
	Proposed	1.5591	1.7211	1.8019	1.9612
50	MLE	1.9142	2.174	2.3065	7.4765
	RIDGE	1.4788	1.5557	1.5769	1.5796
	Liu	1.644	1.6796	1.7125	7.3127
	Proposed	1.4777	1.5516	1.5621	1.5713
100	MLE	1.6162	1.6763	1.7735	3.4816
	RIDGE	1.4033	1.4606	1.4728	1.4812
	Liu	1.5873	1.5917	1.5954	1.6526
	Proposed	1.3807	1.4576	1.4724	1.4804

Table 2: Mean squared error of simulated data when ($p = 8$)

n	Estimator				
		r=0.70	r=0.80	r=0.90	r=0.99
30	MLE	11.732	16.9209	25.212	57.352
	RIDGE	5.0912	6.5025	8.046375	24.148
	Liu	5.6934	12.9963	16.7967	27.159
	Proposed	5.0609	5.499	6.792	22.0609
50	MLE	5.8046	9.0094	15.1376	36.2925
	RIDGE	3.3884	5.4176	7.4287	12.6707
	Liu	3.5951	6.3373	13.4532	28.3894
	Proposed	3.3849	4.3937	5.4022	10.662
100	MLE	4.0996	5.0711	6.2723	7.5902
	RIDGE	1.4688	2.4586	4.4377	6.3519
	Liu	1.799	2.485	4.4723	6.5441
	Proposed	1.4678	2.4408	3.4377	6.2921

Table 3: Mean squared error of simulated data when ($p = 12$)

n	Estimator				
		r=0.70	r=0.80	r=0.90	r=0.99
30	MLE	19.9573	33.5149	50.097	91.567
	RIDGE	6.9245	12.678	14.22188	33.6764
	Liu	15.3437	16.1636	29.4657	43.2582
	Proposed	6.5874	10.6709	11.964	32.1194
50	MLE	11.2081	19.428	24.8753	72.258
	RIDGE	5.4148	7.584	11.5934	25.0145
	Liu	5.4286	8.5835	12.2697	35.4257
	Proposed	4.4143	6.5799	11.5459	20.6723
100	MLE	7.8722	12.9014	17.5684	36.6432
	RIDGE	2.6106	4.5886	9.7817	13.2375
	Liu	3.2709	4.409	9.8613	12.1961
	Proposed	2.3804	4.3108	7.7927	9.7007

Table 4: Mean squared error of simulated data when ($p = 16$)

n	Estimator				
		r=0.70	r=0.80	r=0.90	r=0.99
30	MLE	37.627	58.537	126.757	283.357
	RIDGE	12.867	21.987	37.067	103.777
	Liu	28.857	28.107	75.177	133.497
	Proposed	12.217	18.467	31.417	98.947
50	MLE	20.997	33.827	63.697	223.457
	RIDGE	9.997	13.057	30.497	76.907
	Liu	10.017	14.807	32.187	109.207
	Proposed	8.097	11.297	30.377	63.437
100	MLE	14.667	22.387	45.427	112.977
	RIDGE	4.667	7.797	25.967	40.377
	Liu	5.917	7.487	26.167	37.147
	Proposed	4.227	7.317	20.987	29.407

5 Numerical Result

In this section, we will adopt two real-life data to evaluate the performance of the existing estimators and the proposed. The Aircraft Data dataset is originally assumed to follow the Poisson regression model (see Myers et al., 2012; Asar & Genc, 2017; Amin et al. 2020; Lukman et al., 2021a,b), among others. The response variable y represent the number of locations with damage on the aircraft and it follows a Poisson distribution Myers et al. (2012); Asar and Genç (2017); Lukman et al. (2022) . The explanatory variables are described as follows: x_1 denotes aircraft type (A-4 coded as 0 and A-6 coded as 1), x_2 and x_3 denote bomb load in tons and total months of aircrew experience, respectively. Lukman et al. (2021a,b) diagnosed the model and conclude that the model suffers from multicollinearity because the condition number is 219.3654. The output of the Poisson regression model using the maximum likelihood method is presented in Table 5.

Table 5: Poisson regression estimates using MLE

Coef.	Estimate	Std. Error	z value	Pr(>|z|)
(Intercept)	-0.4060	0.8775	-0.463	0.6436
x1	0.5688	0.5044	1.128	0.2595
x2	0.1654	0.0675	2.449	0.0143
x3	-0.0135	0.0083	-1.633	0.1025

However, the variance of the number of locations with damage on the aircraft is more than twice the mean (2.0569). With this, it is evident that the data exhibit over-dispersion. Bell Regression models account for over-dispersion in count data (Castellares et al., 2018). Recently, This data was employed the bell regression model to model the same dataset. Table 6 provides the regression estimates and the mean squared error of each of the adopted estimators in this study. The biasing parameter k proposed by Hoerl et al. (1975) was adopted as the biasing parameter for the Bell ridge and the Bell KL estimators.

$$\hat{k} = \frac{p}{\sum_{j=1}^{p} \widehat{\boldsymbol{v}}_j^2} \tag{30}$$

The Scalar mean squared error (MSE) for the other adopted method of estimation in this study are as follows:

$$MSE\left(\widehat{\boldsymbol{\beta}}_{MLE}\right) = \sum_{j=1}^{p} \frac{1}{\lambda_j} \tag{31}$$

where λ_j is the eigenvalue of $\boldsymbol{X}^T\widehat{\boldsymbol{W}}\boldsymbol{X}$.

$$MSE\left(\widehat{\boldsymbol{\beta}}_{\mathrm{k-BRM}}\right) = \sum_{j=1}^{p} \frac{\lambda_j}{\left(\lambda_j + k\right)^2} + k^2 \sum_{j=1}^{p} \frac{\widehat{\boldsymbol{v}}_j^2}{\left(\lambda_j + k\right)^2} \tag{32}$$

where $\widehat{\alpha}_j^2$ is the jth squared of the maximum likelihood estimate.

$$MSE\left(\widehat{\boldsymbol{\beta}}_{\text{d}-\text{BRM}}\right) = \sum_{j=1}^{p} \frac{(\lambda_j + d)^2}{\lambda_j (\lambda_j + 1)^2} + (1-d)^2 \sum_{j=1}^{p} \frac{\widehat{\boldsymbol{\upsilon}}_j^2}{(\lambda_j + 1)^2} \tag{33}$$

Table 6: Bell regression estimates for Aircraft Data

Coef.	$\widehat{\boldsymbol{\beta}}_{\text{MLE}}$	$\widehat{\boldsymbol{\beta}}_{\text{k}-\text{BRM}}$	$\widehat{\boldsymbol{\beta}}_{\text{d}-\text{BRM}}$	$\widehat{\boldsymbol{\beta}}_{\text{proposed}}$
Intercept	-0.5422	-0.1509	-0.3006	-0.0211
x1	0.5990	0.3433	0.0034	0.3176
x2	0.1630	0.1665	0.0119	0.1513
x3	-0.0117	-0.0146	-0.0023	-0.0128
MSE	1.7447	0.1609	0.5327	**0.1088**

6 Conclusion

Count data are modelled by such GLMs as Poisson regression mentioned before or negative binomial regression. Nevertheless, it can be seen that utilization of the Poisson regression model results into a strapping fit for count data which is over-dispersed. Some models that have been put forward to handle over-dispersion in the context of count data regression analysis are developed by McElreath and Peble named the Bell regression model. When applying the frequentist approach, it is possible to estimate parameters of the Bell regression model by the maximum likelihood method; the Fisher information is calculated. Thus, in this work, it was proposed to apply the new estimation method of parameters called the ridge-type estimator. To elaborate the proposed methodology, the results of the simulation study conducted for this purpose, and the application of the developed methodology to two empirical datasets are presented. Therefore, it can be concluded that the employment of the Bell regression model is more appropriate than the other models for count data displaying over-dispersion information. Moreover, it is important when the model is not free from multicollinearity problem according to the proposed estimator.

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