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Adaptive Estimation of Periodic Regression Model in Short Panel Data

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This paper proposes the use of the adaptive estimation method for estimating the periodic regression parameters in short panel data. This will go through three phases. The first phase aims to show that the periodic regression model verifies the Uniform Local Asymptotic Normality (ULAN), the second phase focuses on constructing the local asymptotically Minimax (LAM) estimators, and the last phase deals with constructing the Adaptive Estimators (AE) of the periodic regression model using the results of phase one and phase two. The results obtained in the simulation show that the Adaptive Estimator is always better than the Least Squares Estimator. The AE is more efficient in the case of an asymmetric score function. Real data are used to compare the two methods and show that the periodic coefficient regression model outperforms both traditional regression and random regression models.

keywords: Periodic regression model, Panel data, Locally asymptotically minimax estimators, Uniform local asymptotic normality, Adaptive estimators.

1 Introduction

The adaptive estimation method has been treated and used by many authors. Kreiss (1987) has got adaptive estimators for the autoregressive moving average (ARMA) model. Linton (1993) has dealt with adaptive estimators for a regression model with the errors following the ARCH process. Ling (2003) constructed adaptive estimators for the

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ARFIMA model with GARCH errors. Allal and El Melhaoui (2006) proposed adaptive rank tests for linear regression models with autoregressive moving average errors. Bentarzi et al. (2009) gave adaptive estimators for the periodic autoregressive (PAR) model. The construction of the adaptive estimator relies on the derivation of the local asymptotically Minimax estimator. This was adopted by Fabian and Hannan (1982), who established a sufficient condition that Kreiss (1987) used in his derivation of LAM estimators in the ARMA model.
The application of periodic models in econometric modeling is also demonstrated by their excellent performance in a variety of applications. Environmental and meteorology studies are other application areas where periodic models show great promise, see Ghysels (1994); Franses and Paap (1994). In some real data, the periodic models may be suited better than random models which proposed by many authors such as, Beran et al. (1996); Akharif et al. (2020); Ou Larbi et al. (2021); Goto et al. (2023); Regui et al. (2024).
The present paper focuses on obtaining, using the ULAN property satisfied by a periodic regression model, an adaptive estimator of the parameters of a periodic regression model in short panel data.
This article is organized as follows. In the next section, we introduce notations, definitions, and assumptions, which we need for the following sections. In section 3, we show that the periodic regression model satisfies the Uniform Local Asymptotic Normality (ULAN) property, see Fihri et al. (2020); Lmakri et al. (2020); Regui et al. (2024). In section 4, we derive Locally Asymptotically Minimax (LAM) estimators. All of these results are used in section 5 for getting the Adaptive Estimator (AE) of parameters of the periodic regression model. Section 6 shows the performance of the adaptive estimation method and compare it with the Least Squares Estimation (LSE) method. The final section studies two real data sets to compare the adaptive estimation method with the least squares estimation method and to compare periodic coefficient regression with traditional regression and random regression models.

2 Notations, Definitions and main assumptions

2.1 Definitions and notations

We consider the following periodic regression model with period S in short panel data:

$$y_{i,t} = \mu + \beta_s x_{i,t} + \varepsilon_{i,t}, \qquad i = 1, ..., n, \qquad t = 1, ..., T, \qquad s = i - S\left[\frac{i}{S}\right], \tag{1}$$

where [x] denotes the largest integer less than or equal to x and $\{\varepsilon_{i,t},\ i = 1, ..., n, t = 1, ..., T\}$ are independent and identically distribution, f, with mean zero and variance ${\sigma_s}^2$. The regression parameters $\beta_i,\ i = 1, ..., n$ and the variance ${\sigma_i}^2$ are periodic in i, with period S i.e. $\beta_i = \beta_{i+[\frac{i}{S}]S}$ and ${\sigma_i}^2 = {\sigma^2}_{i+[\frac{i}{S}]S}$ for $i = 1, ..., n$.
Assume that n multiplications of S i.e. $n = Sm$. Let $i = s + Sr,\ s = 1, ..., S$ and

$r = 0, ..., m-1$. The model (1) takes the following form:

$$y^{(n)}_{s+Sr,t} = \mu + \beta_s x^{(n)}_{s+Sr,t} + \varepsilon^{(n)}_{s+Sr,t},\ s = 1, ..., S,\ r = 0, ..., m-1,\ t = 1, ..., T. \qquad (2)$$

We consider $\theta = \left(\mu, \sigma^{2'}, \beta'\right)'$ where $\sigma^2 = (\sigma^2_1, ..., \sigma^2_S)'$ and $\beta = (\beta_1, ..., \beta_S)'$ and let $\mathbb{P}^{(n)}_{\theta;f}$ the probability distribution of $\left(y^{(n)}_{1,1}, ..., y^{(n)}_{n,1}, ..., y^{(n)}_{1,T}, ..., y^{(n)}_{n,T}\right)$.

Let $K^{(n)}_s = \left(M^{(n)}_s\right)^{\frac{-1}{2}}$ for $s = 1, ..., S$ with $M^{(n)}_s = \frac{1}{mT}\sum\limits_{r=0}^{m-1}\sum\limits_{t=1}^{T} x^2_{s+Sr,t}$, $\theta^{(n)} = \left(\mu^{(n)}, {\sigma^{(n)}}^{2'}, {\beta^{(n)}}'\right)'$, $\tau^{(n)} = \left(\lambda^{(n)}, {\gamma^{(n)}}', {h^{(n)}}'\right)'$ where $\gamma^{(n)} = \left(\gamma^{(n)}_1, ..., \gamma^{(n)}_S\right)'$, $h^{(n)} = \left(h^{(n)}_1, ..., h^{(n)}_S\right)'$, ${\tau^{(n)}}'\tau^{(n)} < \infty$ and

$$\nu^{(n)} = \begin{bmatrix} 1 & \mathbf{0}_{1\times S} & \mathbf{0}_{1\times S} \\ \mathbf{0}_{S\times 1} & \mathbf{I}_{S\times S} & \mathbf{0}_{S\times S} \\ \mathbf{0}_{S\times 1} & \mathbf{0}_{S\times S} & \mathbf{K}^{(n)}_{S\times S} \end{bmatrix}$$

with

$$\mathbf{K}^{(n)}_{S\times S} = \begin{bmatrix} K^{(n)}_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & K^{(n)}_S \end{bmatrix}.$$

Note that the density of $\{\varepsilon_{s+Sr,t}\ s = 1, ..., S,\ r = 0, ..., m-1,\ t = 1, ..., T\}$ is $f(\varepsilon_{s+Sr,t}) = \frac{1}{\sigma_s} f_1(\frac{\varepsilon_{s+Sr,t}}{\sigma_s})$.

2.2 Main assumptions

Assumption(A)

A.a) $\phi_{f_1} = -\dfrac{f_1'}{f_1}$ with $f_1 \in \mathcal{C}^1$ and f_1' the first derivative of f_1.

A.b) Assume that $I_\phi(f_1) := \int\limits_{\mathbb{R}} \phi^2_{f_1}(x) f_1(x) dx < \infty$, $J_\phi(f_1) := \int\limits_{\mathbb{R}} x^2 \phi^2_{f_1}(x) f_1(x) dx < \infty$ and $K_\phi(f_1) := \int\limits_{\mathbb{R}} x \phi^2_{f_1}(x) f_1(x) dx < \infty$.

Assumption(B)

B.a) $\lim\limits_{n\longrightarrow+\infty} M^{(n)}_s = M_s$ for $s = 1, ..., S$, note that $\lim\limits_{n\longrightarrow+\infty} K^{(n)}_s = K_s = M_s^{\frac{-1}{2}}$.

B.b) For $s = 1, ..., S$, $\bar{x}_s = \frac{1}{mT}\sum\limits_{r=0}^{m-1}\sum\limits_{t=1}^{T} x_{s+Sr,t} = 0$.

3 Uniform Local Asymptotic Normality

Let $\mathcal{P}_{f_1}^{(n)} = \left\{\mathbb{P}^{(n)}_{\mu,\sigma^{2'},\beta';f_1} : (\mu,\beta') \in \mathbb{R}^{S+1} \text{ and } \sigma_s^2 > 0 \; s = 1,...,S\right\}$ be a sequence of hypotheses under which we suppose that $\left\{y^{(n)}_{s+Sr,t},\ t = 1,...,T, r = 0,...,m-1, s = 1,...,S\right\}$ verifies (2).

Let $\Lambda^{(n)}_{\theta^{(n)}+n^{\frac{-1}{2}}\nu^{(n)}\tau^{(n)}/\theta^{(n)};f}$ be the logarithm of the likelihood ratio for $d\mathbb{P}^{(n)}_{\theta^{(n)};f}$ against $d\mathbb{P}^{(n)}_{\theta^{(n)}+n^{\frac{-1}{2}}\nu^{(n)}\tau^{(n)};f}$. So we obtain that

$$\Lambda^{(n)}_{\theta^{(n)}+n^{\frac{-1}{2}}\nu^{(n)}\tau^{(n)}/\theta^{(n)};f} = \sum_{r=0}^{m-1}\sum_{t=1}^{T}\sum_{s=1}^{S}\log\left(f(\varepsilon_{s+Sr,t} - n^{\frac{-1}{2}}(\lambda^{(n)} + h_s^{(n)}K_s^{(n)}x_{s+Sr,t}))\right) - \sum_{r=0}^{m-1}\sum_{t=1}^{T}\sum_{s=1}^{S}\log\left(f(\varepsilon_{s+Sr,t})\right). \quad (3)$$

Putting Z the standardized residuals as $Z_{s+Sr,t} = \frac{y_{s+Sr,t}-\mu-\beta_s x_{s+Sr,t}}{\sigma_s}$ for all $s = 1,...,S, t = 1,...,T$ and $r = 0,...,m-1$.

Proposition 1. *Let **assumption(A)** and **assumption(B)** hold. Then, the family* $\mathcal{P}_{f_1}^{(n)}$ *is ULAN at any* $\theta = \left(\mu, \sigma^{2'}, \beta'\right)'$ *with*

$$\Delta^{(n)}_{f_1}(\theta) = \begin{bmatrix} \Delta^{(n)}_{1,f_1}(\theta) \\ \boldsymbol{\Delta}^{(n)}_{2,f_1}(\theta) \\ \boldsymbol{\Delta}^{(n)}_{3,f_1}(\theta) \end{bmatrix} \quad (4)$$

and

$$\Gamma^{(n)}_{f_1}(\theta) = \frac{T}{S}\begin{bmatrix} \Gamma_{11} & \boldsymbol{\Gamma}_{12} & \mathbf{0}_{1\times S} \\ \boldsymbol{\Gamma}'_{12} & (\boldsymbol{\Gamma}_{22})_{S\times S} & \mathbf{0}_{1\times S} \\ \mathbf{0}_{S\times 1} & \mathbf{0}_{S\times S} & (\boldsymbol{\Gamma}_{33})_{S\times S} \end{bmatrix}_{(2S+1)\times(2S+1)}, \quad (5)$$

where $\Delta^{(n)}_{1,f_1}(\theta) = n^{\frac{-1}{2}} \sum\limits_{t=1}^{T}\sum\limits_{s=1}^{S}\frac{1}{\sigma_s}\sum\limits_{r=0}^{m-1}\phi_{f_1}(Z_{s+Sr,t})$,

$$\boldsymbol{\Delta}^{(n)}_{2,f_1}(\theta) = \begin{pmatrix} \frac{n^{\frac{-1}{2}}}{2\sigma_1^2}\sum\limits_{t=1}^{T}\sum\limits_{r=0}^{m-1}(\phi_{f_1}(Z_{1+Sr,t})Z_{1+Sr,t} - 1) \\ \vdots \\ \frac{n^{\frac{-1}{2}}}{2\sigma_S^2}\sum\limits_{t=1}^{T}\sum\limits_{r=0}^{m-1}(\phi_{f_1}(Z_{S+Sr,t})Z_{S+Sr,t} - 1) \end{pmatrix},$$

$$\mathbf{\Delta}^{(n)}_{3,f_1}(\theta) = \begin{pmatrix} \frac{n^{\frac{-1}{2}}}{\sigma_1} \sum_{t=1}^{T} \sum_{r=0}^{m-1} \phi_{f_1}(Z_{1+Sr,t}) x_{1+Sr,t} K_1^{(n)} \\ \vdots \\ \frac{n^{\frac{-1}{2}}}{\sigma_S} \sum_{t=1}^{T} \sum_{r=0}^{m-1} \phi_{f_1}(Z_{S+Sr,t}) x_{S+Sr,t} K_S^{(n)} \end{pmatrix},$$

$\Gamma_{11} = I_\phi(f_1) \sum_{s=1}^{S} \frac{1}{\sigma_s^2}$, $\mathbf{\Gamma}_{12} = \frac{K_\phi(f_1)}{2} \left(\frac{1}{\sigma_1^3}, ..., \frac{1}{\sigma_S^3}\right)$,

$$\mathbf{\Gamma}_{22} = \frac{J_\phi(f_1) - 1}{4} diag\left(\frac{1}{\sigma_1^4}, ..., \frac{1}{\sigma_S^4}\right),$$

and

$$\mathbf{\Gamma}_{33} = I_\phi(f_1) diag\left(\frac{1}{\sigma_1^2}, ..., \frac{1}{\sigma_S^2}\right).$$

Under $\mathcal{P}^{(n)}_{\theta^{(n)};f_1}$, *for any* $\theta^{(n)}$ *such that* $\sqrt{n}\left(\theta^{(n)} - \theta\right) = O_{\mathbb{P}}(1)$ *and for any bounded sequence* $\tau^{(n)} \in \mathbb{R}^{2S+1}$, *we have*

$$\Lambda^{(n)}_{\theta^{(n)}+n^{\frac{-1}{2}}\nu^{(n)}\tau^{(n)}/\theta^{(n)};f_1} := \log\left(\frac{d\mathbb{P}^{(n)}_{\theta^{(n)}+n^{\frac{-1}{2}}\nu^{(n)}\tau^{(n)};f_1}}{d\mathbb{P}^{(n)}_{\theta^{(n)};f_1}}\right) =$$
$$\tau^{(n)'}\Delta^{(n)}_{f_1}(\theta^{(n)}) - \frac{1}{2}\tau^{(n)'}\Gamma_{f_1}(\theta)\tau^{(n)} + o_p(1), \quad (6)$$

$\Delta^{(n)}_{f_1}(\theta^{(n)})$ *converge to* $\mathcal{N}\left(\mathbf{0}_{(2S+1)\times 1}, \Gamma_{f_1}(\theta)\right)$ *when* $n \to \infty$ *and* $\Delta^{(n)}_{f_1}(\theta^{(n)})$ *converge to* $\mathcal{N}\left(\Gamma_{f_1}(\theta)\tau, \Gamma_{f_1}(\theta)\right)$ *when* $n \to \infty$ *under* $\mathcal{P}^{(n)}_{\theta^{(n)}+n^{\frac{-1}{2}}\nu^{(n)}\tau^{(n)};f_1}$.

Proof. See Appendix. □

4 Locally Asymptotically Minimax (LAM) Estimators

Definition 1 (**LAM estimator**)**.** *Using LAN property and under* ***Assumption(A)***, *the sequence estimator* $\{W_n\}$ *is LAM if*

$$\sqrt{n}(W_n - \theta) - \Gamma_f(\theta)^{-1}\Delta^{(n)}_f(\theta) = o_{\mathbb{P}_\theta}(1).$$

Definition 2. *A sequence of estimators* $\bar{\theta}_n$ *of* θ *is called :*

i) $\sqrt{n}$*-consistent if* $\sqrt{n}(\bar{\theta}_n - \theta) = O_{\mathbb{P}}(1)$.

ii) Discrete if there exists $k \in \mathbb{N}$ *such that* $\forall n \in \mathbb{N}$, $\bar{\theta}_n$ *takes at most* k *different values in*

$$Q_n = \left\{\delta \in \mathbb{R}^{2S+1} : \sqrt{n}||\delta - \theta|| \leq c\right\}, \ c > 0.$$

The $\sqrt{n}$-consistent condition is satisfied by least squares and maximum likelihood estimators, and the local discreteness condition goes back to LeCam (1960) and has become an important technical tool in the construction of efficient estimators.

For the model (1), we have $\bar{\theta}_n = \left(\bar{\mu}, \bar{\sigma}^{2'}, \bar{\beta}'\right)'$, where $\bar{\sigma}_s^2 = \frac{1}{(m-2)T} \sum_{r=0}^{m-1} \sum_{t=1}^{T} \hat{\varepsilon}_{s+Sr,t}^2$ for $s = 1, ..., S$, with

$$\hat{\varepsilon}_{s+Sr,t} = y_{s+Sr,t}^{(n)} - \bar{\mu} - \bar{\beta}_s x_{s+Sr,t}^{(n)} \tag{7}$$

and

$$\bar{\theta}_{\mu,\beta'} = \begin{bmatrix} \bar{\mu} \\ \bar{\beta}_1 \\ \vdots \\ \vdots \\ \bar{\beta}_S \end{bmatrix} = \left(X'X\right)^{-1} X'Y,$$

where

$$X = \begin{bmatrix} \mathbf{1}_{mT} & \mathbf{X}_1 & 0 & \dots & 0 \\ \mathbf{1}_{mT} & 0 & \mathbf{X}_2 & \dots & 0 \\ \vdots & 0 & 0 & \ddots & \vdots \\ \mathbf{1}_{mT} & 0 & 0 & 0 & \mathbf{X}_S \end{bmatrix},$$

$\mathbf{X}_s = \left(x_{s,1}, ..., x_{s+(m-1)S,1}; ...; x_{s,T}, ..., x_{s+(m-1)S,T}\right)^\top$, $Y = (\mathbf{Y}_1', ..., \mathbf{Y}_S')'$ and
$\mathbf{Y}_s = \left(y_{s,1}, ..., y_{(m-1)S+s,1}; ...; y_{s,T}, ..., y_{(m-1)S+s,T}\right)'$.
We have

$$E(\bar{\theta}_{\mu,\beta'}) = \left(X'X\right)^{-1} X'E(Y) = \theta_{\mu,\beta'}$$

and

$$Var(\bar{\theta}_{\mu,\beta'}) = \left(X'X\right)^{-1} X'E(\varepsilon\varepsilon')X \left(X'X\right)^{-1},$$

with $\varepsilon = \left(\boldsymbol{\varepsilon}_1', ..., \boldsymbol{\varepsilon}_S'\right)'$ where $\boldsymbol{\varepsilon}_s = \left(\varepsilon_{s,1}, .., \varepsilon_{s+(m-1)S,1}; ...; \varepsilon_{s,T}, .., \varepsilon_{s+(m-1)S,T}\right)'$ and

$$E(\varepsilon\varepsilon') = \begin{bmatrix} \sigma_1^2\mathbf{I}_m & 0 & 0 & \dots & 0 \\ 0 & \sigma_2^2\mathbf{I}_m & 0 & \dots & 0 \\ \vdots & 0 & \ddots & & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \dots & \sigma_S^2\mathbf{I}_m \end{bmatrix}.$$

Theorem 1. *Assume that $\left\{\bar{\theta}_n\right\}$ is discrete and $\sqrt{n}-$consistent of estimators θ, then*

$$\hat{\theta}_f = \bar{\theta}_n + \frac{1}{\sqrt{n}} \Gamma_f{}^{-1}(\bar{\theta}_n) \Delta_f^{(n)}(\bar{\theta}_n)$$

is LAM estimator.

Proof. See Appendix. □

5 Adaptive Estimation of Periodic Regression

The estimator $\hat{\theta}_f$ depends on the density f, which is unknown. In this section, we will estimate the density f using the Kernel method. We estimate the score function ϕ_f and the Fisher information $I_\phi(f_1)$ using the Gaussian Kernel function to obtain the adaptive estimator similar to $\hat{\theta}_f$, see Bentarzi et al. (2009); Ling (2003). We consider the following notations

1. $h(x; b_n) = \frac{1}{\sqrt{2\pi b_n^2}} \exp(-\frac{x^2}{2b_n^2}), \quad x \in \mathbb{R},$

2. $\widehat{f}_{b_n}(x) = \frac{1}{n\times T} \sum_{t=1}^{T} \sum_{r=0}^{m-1} \sum_{s=1}^{S} h(x - Z_{s+Sr,t}; b_n)$

3. $\widehat{f'}_{b_n}(x) = \frac{-1}{n\times T\times b_n^2} \sum_{t=1}^{T} \sum_{r=0}^{m-1} \sum_{s=1}^{S} (x - Z_{s+Sr,t})\, h(x - Z_{s+Sr,t}; b_n).$

Definition 3. *The estimation of ϕ_f is $\widehat{\phi}(x) = -\dfrac{\widehat{f'}_{b_n}(x)}{a_n + \widehat{f}_{b_n}(x)}$ where $a_n \to 0$ and $b_n \to 0$ when $n \to \infty$.*

Let $\hat{I}_n(\overline{\theta}_n) = \frac{1}{nT} \sum_{s=1}^{S} \sum_{r=0}^{m-1} \sum_{t=1}^{T} \hat{\phi}^2(\hat{\varepsilon}_{s+Sr,t}; \overline{\theta}_n)$, $\hat{K}_n(\overline{\theta}_n) = \frac{1}{nT} \sum_{s=1}^{S} \sum_{r=0}^{m-1} \sum_{t=1}^{T} \hat{\phi}^2(\hat{\varepsilon}_{s+Sr,t}; \overline{\theta}_n)\hat{\varepsilon}_{s+Sr,t}$ and $\hat{J}_n(\overline{\theta}_n) = \frac{1}{nT} \sum_{s=1}^{S} \sum_{r=0}^{m-1} \sum_{t=1}^{T} \hat{\phi}^2(\hat{\varepsilon}_{s+Sr,t}; \overline{\theta}_n)\hat{\varepsilon}^2_{s+Sr,t}$.

Definition 4. *The estimator of $I_\phi(f_1)$, $K_\phi(f_1)$ and $J_\phi(f_1)$ are respectively $\hat{I}_n(\overline{\theta}_n)$, $\hat{K}_n(\overline{\theta}_n)$ and $\hat{J}_n(\overline{\theta}_n)$.*

Definition 5. *The estimators of $\Delta^{(n)}_{f_1}(\theta)$ and $\Gamma^{(n)}_{f_1}(\theta)$ are respectively $\hat{\Delta}^{(n)}(\overline{\theta}_n)$ and*

$\hat{\Gamma}^{(n)}(\overline{\theta}_n)$, *where*

$$\hat{\Delta}^{(n)}(\overline{\theta}_n)=\begin{bmatrix}\hat{\Delta}_1(\overline{\theta}_n)\\ \hat{\mathbf{\Delta}}_2(\overline{\theta}_n)\\ \hat{\mathbf{\Delta}}_3(\overline{\theta}_n)\end{bmatrix}=\begin{bmatrix} n^{\frac{-1}{2}}\sum_{t=1}^{T}\sum_{s=1}^{S}\frac{1}{\overline{\sigma}_s}\sum_{r=0}^{m-1}\hat{\phi}(Z_{s+Sr,t};\overline{\theta}_n)\\ \frac{n^{\frac{-1}{2}}}{2\overline{\sigma}_1^2}\sum_{t=1}^{T}\sum_{r=0}^{m-1}\left(\hat{\phi}(Z_{1+Sr,t};\overline{\theta}_n)Z_{1+Sr,t}-1\right)\\ \vdots\\ \frac{n^{\frac{-1}{2}}}{2\overline{\sigma}_S^2}\sum_{t=1}^{T}\sum_{r=0}^{m-1}\left(\hat{\phi}(Z_{S+Sr,t};\overline{\theta}_n)Z_{S+Sr,t}-1\right)\\ \frac{n^{\frac{-1}{2}}}{\overline{\sigma}_1}\sum_{t=1}^{T}\sum_{r=0}^{m-1}\hat{\phi}(Z_{1+Sr,t};\overline{\theta}_n)x_{1+Sr,t}K_1^{(n)}\\ \vdots\\ \frac{n^{\frac{-1}{2}}}{\overline{\sigma}_S}\sum_{t=1}^{T}\sum_{r=0}^{m-1}\hat{\phi}(Z_{S+Sr,t};\overline{\theta}_n)x_{S+Sr,t}K_S^{(n)}\end{bmatrix}, \quad (8)$$

$$\hat{\Gamma}^{(n)}(\overline{\theta}_n)=\frac{T}{S}\begin{bmatrix}\hat{\Gamma}_{11} & \hat{\mathbf{\Gamma}}_{12} & \mathbf{0}_{1\times S}\\ \hat{\mathbf{\Gamma}}_{12}^{'} & \left(\hat{\mathbf{\Gamma}}_{22}\right)_{S\times S} & \mathbf{0}_{1\times S}\\ \mathbf{0}_{S\times 1} & \mathbf{0}_{S\times S} & \left(\hat{\mathbf{\Gamma}}_{33}\right)_{S\times S}\end{bmatrix}, \quad (9)$$

$\hat{\Gamma}_{11}=\hat{I}_n(\overline{\theta}_n)\sum_{s=1}^{S}\frac{1}{\overline{\sigma}_s^2}$, $\hat{\mathbf{\Gamma}}_{12}=\frac{\hat{K}_n(\overline{\theta}_n)}{2}(\frac{1}{\overline{\sigma}_1^3},...,\frac{1}{\overline{\sigma}_S^3})$, $\hat{\mathbf{\Gamma}}_{22}=\frac{\hat{J}_n(\overline{\theta}_n)-1}{4}diag(\frac{1}{\overline{\sigma}_1^4},...,\frac{1}{\overline{\sigma}_S^4})$, *and*

$$\hat{\mathbf{\Gamma}}_{33}=\hat{I}_n(\overline{\theta}_n)diag(\frac{1}{\overline{\sigma}_1^2},...,\frac{1}{\overline{\sigma}_S^2}).$$

Theorem 2. *Under regularity conditions* ***Assumption(A)****, the estimator* $\widetilde{\theta}_n=\bar{\theta}_n+\frac{1}{\sqrt{n}}\hat{\Gamma}(\bar{\theta}_n)^{-1}\hat{\Delta}^{(n)}(\bar{\theta}_n)$ *is LAM and we have*

$$L\left(\sqrt{n}\left(\widetilde{\theta}_n-\theta\right)/\mathbb{P}^{(n)}_{\theta^{(n)};f_1}\right)\Rightarrow\mathcal{N}\left(\mathbf{0},\Gamma_{f_1}(\theta)^{-1}\right).$$

Proof. See Appendix. □

6 Simulation

We estimate the parameters of the model (2) for showing the performance of the adaptive estimators (AE) and compare these estimators with Least square estimators (LSE), for a different size n, period S, and the white noise density f. We have got the following results:

The densities of the white noise are estimated using the Kernel method. The first simulation (table 1) shows that the Adaptive Estimator(AE) is more efficient than the

Table 1: $S = 2$, $T = 10$, $\mu = 3$, $\beta = (2, 4)$, $x \sim 10\mathcal{N}(0, 1)$, $\sigma^2 = (1, 0.9)$, $\varepsilon^1_{S\tau+s} \sim \mathcal{N}(0, \sigma_s^2)$, $\varepsilon^2_{S\tau+s} \sim \sqrt{2}Laplace(0, \sigma_s^2)$, $\varepsilon^3_{S\tau+s} \sim skew\ Normal(0, \sigma_s^2, \delta = 10)$ and $\varepsilon^4_{S\tau+s} \sim skew\ Student_5(0, \sigma_s^2, \delta = 10)\ for\ s = 1, 2$.

				n=20		
$f/\hat{\theta}$		$\hat{\mu}$	$\hat{\sigma}_1^2$	$\hat{\sigma}_2^2$	$\hat{\beta}_1$	$\hat{\beta}_2$
f_1	LSE	3.050919	1.203777	0.9675306	1.985747	4.005323
	AE	3.073068	1.184051	0.9559971	1.870641	3.976649
f_2	LSE	2.988219	1.231902	0.9961245	2.020256	3.994478
	AE	3.128327	1.345157	1.028807	2.082048	3.966505
f_3	LSE	2.965729	1.369362	0.7514024	1.982971	4.004441
	AE	3.02707	1.311004	0.745587	2.001936	3.994973
f_4	LSE	2.781497	0.9775358	0.8269645	2.009052	3.998216
	AE	2.904733	0.9643236	0.8199164	1.99059	3.978953
				n=30		
$f/\hat{\theta}$		$\hat{\mu}$	$\hat{\sigma}_1^2$	$\hat{\sigma}_2^2$	$\hat{\beta}_1$	$\hat{\beta}_2$
f_1	LSE	2.995833	1.092605	0.9065589	2.007034	3.997423
	AE	3.026603	1.075258	0.9006608	2.042606	3.962786
f_2	LSE	3.098742	1.075586	0.9610971	2.032165	3.990511
	AE	3.14614	1.07574	0.9969977	2.013062	3.908403
f_3	LSE	2.900636	1.166994	0.9080824	1.993763	4.004968
	AE	2.971603	1.151715	0.9062038	1.986187	4.029125
f_4	LSE	2.945182	1.000806	0.9272752	1.997739	3.996383
	AE	2.956128	0.9837186	0.9124315	1.981714	4.079678

Least Square Estimator (LSE) for the densities f_3 (skew normal) and f_4 (skew student). Even though the size n decreases, the AE is still more suitable. In addition, we generated 100 replications independent based on many samples of the same size, as shown in table 2. Table 2 confirms that, for the densities f_2 (skew normal) and f_3 (skew student), the adaptive estimation method gives the efficient estimators of θ, as the Root Mean Square (RMS) errors show. Moreover, simulation results (table 1 and table 2) show that even when the size of samples decreases, the AE remains the most appropriate option.
Many replications are used for each simulation to get the adaptive estimator $\widetilde{\theta}_n$ and the least square estimator $\bar{\theta}_n$. We have used the Root Mean Square errors, which are defined by: $RMS\ errors = \sqrt{Variance + Bias^2}$ (table 2), to compare the AE and the LSE.

Table 2: $S = 2$, $T = 10$ $\mu = 4$, $\beta = (2, 1.5)$, $x \sim 10\mathcal{N}(0,1)$, $\sigma^2 = (1, 0.9)$, $\varepsilon^1_{S\tau+s} \sim Normal(0, \sigma_s^2)$, $\varepsilon^2_{S\tau+s} \sim skew\ Normal(0, \sigma_s^2, \delta = 10)$ and $\varepsilon^3_{S\tau+s} \sim skew\ Student_7(0, \sigma_s^2, \delta = 10)$ $for\ s = 1, 2.$

			n=20				
$f/\hat{\theta}$		$\hat{\mu}$	$\hat{\sigma}_1^2$	$\hat{\sigma}_2^2$	$\hat{\beta}_1$	$\hat{\beta}_2$	RMS
f_1	LSE Mean	3.998957	1.230349	1.101603	1.999469	1.499308	0.1557777
	AE Mean	4.000183	1.19524	1.075913	2.00546	1.489938	0.1359267
f_2	LSE Mean	4.011263	1.133402	1.097992	2.000329	1.49992	0.5618598
	AE Mean	3.996893	1.089179	1.052315	2.000244	1.504165	0.4681877
f_3	LSE Mean	3.89535	1.329362	1.056403	2.000967	1.500932	0.7195472
	AE Mean	3.803372	1.284293	1.032338	1.99637	1.487846	0.5595058
			n=30				
$f/\hat{\theta}$		$\hat{\mu}$	$\hat{\sigma}_1^2$	$\hat{\sigma}_2^2$	$\hat{\beta}_1$	$\hat{\beta}_2$	RMS
f_1	LSE Mean	3.995845	1.128235	1.033385	2.000869	1.50053	0.06875505
	AE Mean	4.006002	1.109741	1.019394	2.001264	1.498373	0.0687399
f_2	LSE Mean	3.998353	1.016296	0.9603092	1.999799	1.500601	0.4969943
	AE Mean	4.003026	0.9924481	0.934455	2.003751	1.505784	0.4483293
f_3	LSE Mean	3.955123	1.413475	1.090372	1.999801	1.500141	0.4783673
	AE Mean	3.883805	1.380217	1.074936	1.996667	1.507942	0.4559947

7 Real data

In examples 1 and 2, the adaptive estimation method will be compared to the least squares estimation method. The periodic coefficient regression model will also be considered in comparison with the traditional regression and random regression models.

7.1 Example 1

The aim of this Subsection is to show that the regression model with periodic coefficients is more efficient than both the traditional regression and random regression models for predicting monthly water temperature (T_{Water}) from air temperature (T_{Air}) for the Missouri River studied by Zhu et al. (2018). The estimated model is:

- For traditional regression:

$$T_{\text{Water}} = 2.8364 + 0.2036 T_{\text{Air}} + \widehat{\varepsilon},$$

  with $\widehat{\varepsilon}$ is estimated residual.

- For periodic coefficient regression using the LSE method:

$$T_{\text{Water},s+12r} = \mu + \beta_s T_{\text{Air},s+12r} + \varepsilon_{s+12r} \quad \text{with } r = 0, ..., 6,$$

where $\widehat{\mu} = 3.1664$, $\widehat{\beta}_1 = 0.8151$, $\widehat{\beta}_2 = 0.2589$, $\widehat{\beta}_3 = 0.5832$, $\widehat{\beta}_4 = 0.7565$, $\widehat{\beta}_5 = 0.8318$, $\widehat{\beta}_6 = 0.8396$, $\widehat{\beta}_7 = 0.9124$, $\widehat{\beta}_8 = 0.9354$, $\widehat{\beta}_9 = 0.9655$, $\widehat{\beta}_{10} = 0.9103$, $\widehat{\beta}_{11} = 0.7423$, and $\widehat{\beta}_{12} = 0.4668$.

- For periodic coefficient regression using the AE method:

$$T_{\text{Water},s+12r} = \mu + \beta_s T_{\text{Air},s+12r} + \varepsilon_{s+12r} \quad \text{with } r = 0, ..., 6,$$

where $\widehat{\mu} = 3.1681$, $\widehat{\beta}_1 = 0.7706$, $\widehat{\beta}_2 = 0.2587$, $\widehat{\beta}_3 = 0.5796$, $\widehat{\beta}_4 = 0.7394$, $\widehat{\beta}_5 = 0.8249$, $\widehat{\beta}_6 = 0.8343$, $\widehat{\beta}_7 = 0.8994$, $\widehat{\beta}_8 = 0.9301$, $\widehat{\beta}_9 = 0.9632$, $\widehat{\beta}_{10} = 0.9073$, $\widehat{\beta}_{11} = 0.7381$, and $\widehat{\beta}_{12} = 0.4664$.

Table 3: Root mean square errors (RMSE) for traditional regression, periodic regression models using LSE and adaptive estimation methods, and random regression.

Model	Traditional regression	Periodic regression ($S = 12$)		Random regression
		LSE Method	AE Method	
RMSE	1.6702	0.7956	0.7766	1.4343

Table 3 shows that the value of RMSE for the periodic coefficient regression using the adaptive estimation method is the smallest compared to traditional regression, random regression, and periodic regression using the LSE method, and the periodic coefficient regression model is more appropriate than the traditional regression and random regression models.

7.2 Example 2

We consider the data set, which exists in Newbold and Bos(1985), with variables **Rate of inflation** ($R_{\text{inflation}}$) and **Rate on 3-month T-bills** (R_{bills}), and 110 observations. The estimated model is:

- For traditional regression:

$$R_{\text{inflation}} = -2.067 + 1.388 R_{\text{bills}} + \widehat{\varepsilon},$$

- For periodic coefficient regression using the LSE method:

$$R_{\text{inflation},s+4r} = \mu + \beta_s R_{\text{bills},s+4r} + \varepsilon_{s+4r} \quad \text{with } r = 0, ..., 24,$$

where $\widehat{\mu} = -2.4917$, $\widehat{\beta}_1 = 1.4423$, $\widehat{\beta}_2 = 1.5341$, $\widehat{\beta}_3 = 1.3619$, and $\widehat{\beta}_4 = 1.3111$.

- For periodic coefficient regression using the AE method:

$$R_{\text{inflation},s+4r} = \mu + \beta_s R_{\text{bills},s+4r} + \varepsilon_{s+4r} \quad \text{with } r = 0, ..., 24,$$

where $\widehat{\mu} = -2.4715$, $\widehat{\beta}_1 = 1.6234$, $\widehat{\beta}_2 = 1.6213$, $\widehat{\beta}_3 = 1.3821$, and $\widehat{\beta}_4 = 1.3625$.

Table 4: Root mean square errors (RMSE) for traditional regression, periodic regression models using LSE and adaptive estimation methods, and random regression.

Model	Traditional regression	Periodic regression ($S = 4$)		Random regression
		LSE Method	AE Method	
RMSE	1.6978	0.4523	0.4521	0.4567

Table 4 shows that the adaptive estimation method outperforms the least squares estimation method, and the periodic coefficient regression is the best model compared to the traditional regression and random regression models.

Conclusion

The paper shows that the adaptive estimation (AE) method is better than the least squares estimation (LSE) method. In addition, the periodic coefficient regression model outperforms both the traditional regression and random coefficient regression models for the real data studied.

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Appendix

Proof of Proposition 1. The main point is to show that $q^{\frac{1}{2}}_{\mu,\sigma_s^2,\beta_s;f_1}$ is quadratic mean differentiability at any $\left(\mu,\sigma_s^2,\beta_s\right)$ for s fixed. We have for $s = 1,...,S$:

$$q^{\frac{1}{2}}_{\mu,\sigma_s^2,\beta_s;f_1}(y) = \left(\frac{1}{\sigma_s} f_1\left(\frac{y-\mu-\beta_s x}{\sigma_s}\right)\right)^{1/2}.$$

$q^{\frac{1}{2}}_{\mu,\sigma_s^2,\beta_s;f_1}$ is quadratic mean differentiability at $\left(\mu,\sigma_s^2,\beta_s\right)$ for all $s = 1,...,S$. i.e. for w, ν and $t \to 0$

$$\int_{\mathbb{R}} \left[q^{\frac{1}{2}}_{\mu+w,\sigma_s^2+\nu,\beta_s+t;f_1}(y) - q^{\frac{1}{2}}_{\mu,\sigma_s^2,\beta_s;f_1}(y) - (r,\nu,t) \begin{pmatrix} \frac{\partial}{\partial \mu} q^{\frac{1}{2}}_{\mu,\sigma_s^2,\beta_s;f_1}(y) \\ \frac{\partial}{\partial \sigma_s^2} q^{\frac{1}{2}}_{\mu,\sigma_s^2,\beta_s;f_1}(y) \\ \frac{\partial}{\partial \beta_s} q^{\frac{1}{2}}_{\mu,\sigma_s^2,\beta_s;f_1}(y) \end{pmatrix} \right]^2 dy = O\left(\left\| \begin{pmatrix} w \\ \nu \\ t \end{pmatrix} \right\|^2 \right),$$

with

$$\frac{\partial}{\partial \mu} q^{\frac{1}{2}}_{\mu,\sigma_s^2,\beta_s;f_1}(y) = \frac{1}{2\sigma_s} q^{\frac{1}{2}}_{\mu,\sigma_s^2,\beta_s;f_1}(y)\phi_{f_1}\left(\frac{y-\mu-\beta_s x}{\sigma_s} \right),$$

$$\frac{\partial}{\partial \sigma_s^2} q^{\frac{1}{2}}_{\mu,\sigma_s^2,\beta_s;f_1}(y) = \frac{1}{2\sigma_s} q^{\frac{1}{2}}_{\mu,\sigma_s^2,\beta_s;f_1}(y)\phi_{f_1}\left(\frac{y-\mu-\beta_s x}{\sigma_s} \right) K_s^{(n)} x,$$

and

$$\frac{\partial}{\partial \beta_s} q^{\frac{1}{2}}_{\mu,\sigma_s^2,\beta_s;f_1}(y) = \frac{1}{4\sigma_s^2} q^{\frac{1}{2}}_{\mu,\sigma_s^2,\beta_s;f_1}(y) \left(\frac{y-\mu-\beta_s x}{\sigma_s} \phi_{f_1}\left(\frac{y-\mu-\beta_s x}{\sigma_s} \right) - 1 \right).$$

The problem here is reduced to the one discussed by Lmakri et al. (2020) with $b = 0$. The proof of Proposition 1 is therefore complete. □

Proof of Theorem 1. We have

$$\begin{aligned}
\sqrt{n}\left(\hat{\theta}_f - \theta\right) - \Gamma_f(\theta)^{-1}\Delta_f^{(n)}(\theta) &= \Gamma_f(\overline{\theta}_n)^{-1}\Delta_f^{(n)}(\overline{\theta}_n) - \Gamma_f(\theta)^{-1}\Delta_f^{(n)}(\theta) + \sqrt{n}\left(\overline{\theta}_n - \theta\right) \\
&= \Gamma_f(\overline{\theta}_n)^{-1}(\Delta_f^{(n)}(\overline{\theta}_n) - \Gamma_f(\overline{\theta}_n)\Gamma_f(\theta)^{-1}\Delta_f^{(n)}(\theta) \\
&+ \sqrt{n}\Gamma_f(\overline{\theta}_n)\left(\overline{\theta}_n - \theta\right)) \\
&= \Gamma_f(\overline{\theta}_n)^{-1}\Big[\Delta_f^{(n)}(\overline{\theta}_n) - \Delta_f^{(n)}(\theta) \\
&+ \left(1 - \Gamma_f(\overline{\theta}_n)\Gamma_f(\theta)^{-1}\right)\Delta_f^{(n)}(\theta) + \sqrt{n}\Gamma_f(\theta)\left(\overline{\theta}_n - \theta\right) \\
&+ \sqrt{n}\left(\Gamma_f(\overline{\theta}_n) - \Gamma_f(\theta)\right)\left(\overline{\theta}_n - \theta\right)\Big] \\
&= o_{\mathbb{P}_\theta}(1).
\end{aligned}$$

Because $\Delta_f^{(n)}(\overline{\theta}_n) - \Delta_f^{(n)}(\theta) + \sqrt{n}\Gamma_f(\theta)\left(\overline{\theta}_n - \theta\right) = o_{\mathbb{P}_\theta}(1)$ (Asymptotic linearity) and $\Gamma_f(\overline{\theta}_n) - \Gamma_f(\theta) = o_{\mathbb{P}_\theta}(1)$. □

Proof of Theorem 2. We have

$$\begin{aligned}\sqrt{n}\left(\widetilde{\theta}_n-\theta\right)-\Gamma_f(\theta)^{-1}\Delta_f^{(n)}(\theta) &= \sqrt{n}\left(\overline{\theta}_n-\theta\right)+\hat{\Gamma}(\overline{\theta})^{-1}\hat{\Delta}^{(n)}(\overline{\theta})-\Gamma_f(\theta)^{-1}\Delta_f^{(n)}(\theta)\\ &= \sqrt{n}\left(\overline{\theta}_n-\theta\right)+\Gamma_f(\theta)^{-1}\left(\hat{\Delta}^{(n)}(\overline{\theta})-\Delta_f^{(n)}(\theta)\right)+\\ &\left(\hat{\Gamma}(\overline{\theta})^{-1}-\Gamma_f(\theta)^{-1}\right)\hat{\Delta}^{(n)}(\overline{\theta})\\ &= \Gamma_f(\theta)^{-1}(\hat{\Delta}^{(n)}(\overline{\theta})-\Delta_f^{(n)}(\overline{\theta}_n)+\Delta_f^{(n)}(\overline{\theta}_n)-\Delta_f^{(n)}(\theta)\\ &+\Gamma_f(\theta)\sqrt{n}\left(\overline{\theta}_n-\theta\right))+\left(\hat{\Gamma}(\overline{\theta})^{-1}-\Gamma_f(\theta)^{-1}\right)\hat{\Delta}^{(n)}(\overline{\theta})\\ &= o_{\mathbb{P}_\theta}(1).\end{aligned}$$

Because $\hat{\Delta}^{(n)}(\overline{\theta})-\Delta_f^{(n)}(\overline{\theta})=o_{\mathbb{P}_\theta}(1)$ and $\hat{\Gamma}(\overline{\theta})^{-1}-\Gamma_f(\theta)^{-1}=o_{\mathbb{P}_\theta}(1)$.
So we get,

$$\sqrt{n}\left(\widetilde{\theta}_n-\theta\right)=\Gamma_f(\theta)^{-1}\Delta_f^{(n)}(\theta)+o_{\mathbb{P}_\theta}(1), \tag{10}$$

which leads to

$$L\left(\sqrt{n}\left(\widetilde{\theta}_n-\theta\right)\right)\Rightarrow\mathcal{N}\left(\mathbf{0},\Gamma_{f_1}(\theta)^{-1}\right).$$

□

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