Electronic Journal of Applied Statistical Analysis
EJASA, Electron. J. App. Stat. Anal.
http://siba-ese.unisalento.it/index.php/ejasa/index
e-ISSN: 2070-5948
DOI: 10.1285/i20705948v17n3p472

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15 December 2024

Electronic Journal of Applied Statistical Analysis
Vol. 17, Issue 03, December 2024, 472-497
DOI: 10.1285/i20705948v17n3p472

Weibull-Exponential Pareto Distribution: Order Statistics and their Properties and Application to Nigeria Covid-19 Active Cases

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15 December 2024

The Weibull Exponential Pareto (WEP) distribution is a convolution of the Weibull and Exponential-Pareto distributions using the Weibull-X technique. The distribution generalizes some existing models in the literature. This article presents a new dimension to the study of convoluted distribution by exploring the statistical tools and properties of order statistics from the WEP distribution. Distribution of sample median, extreme order statistics, joint density of two order statistics $X_{(r:n)}$ and $X_{(s:n)}$ for $0 < r < s < n$ and the sample range statistics $R_n = X_{(n:n)} - X_{(1:n)}$ and the explicit expressions for the distributions of rth order statistics was derived and their respective moments. The study demonstrated the use of the beta-G procedure for deriving distributions of the extreme order statistics. The mean and variance of $X_{(r:n)}$, and $X_{(1:n)}$ and the mean value of the sample range R_n were derived, and the recurrence relation for the moment of order statistics was investigated. Result from the application of the WEP distribution to Nigeria COVID-19 data was used to predict the expected occurrences for the maximum and minimum number of COVID-19 active cases of patients on admission for any sample of size n. Some numerical computations for the mean of order statistics of WEP distribution were tabulated for a random sample of size $n = 4$ and numerical results for the variance, skewness, and kurtosis of order statistics were obtained by the Monte Carlo simulation.

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ISSN: 2070-5948
http://siba-ese.unisalento.it/index.php/ejasa/index

keywords: Weibull Exponential Pareto distribution, order statistics, extreme order statistics, Nigeria Covid-19 active cases, moment of order statistics. Monte Carlo simulation

1 Introduction

Order statistics and its associated functions from the generalized distributions is an emerging area of study that is yet to gain desirable studies among researchers. Many convoluted distributions have been developed and applied to real-life datasets as revealed in Famoye et al. (2005); Cordeiro et al. (2010); Al-Kadim and Boshi (2013); Alzaatreh et al. (2013) among several authors. Nevertheless, interests and motivations of researchers continue to grow unabated as seen in the works of Khaleel et al. (2020); Rashwan and Kamel (2020) and recently in new distributions developed by Al-khazaleh (2021); Adeyemi et al. (2021); Benchiha and Al-Omari (2021) and Sindhu et al. (2021). Despite commendable efforts of researchers in generalizing new distributions, the study of order statistics and its associated functions from most of the new distributions have not been considered making this study one of the important areas open for research.
Although some properties of order statistics have been employed by some notable authors to characterize the classical distributions such as Weibull, Pareto, Logistics, and exponential as revealed Balakrishnan and Malik (1986); Khan and Abu-Salih (1988); David and Nagaraja (2004); Balakrishnan and Cohen (2014). Convoluted distributions by the tools of order statistics are yet to be explored.
Order statistics has immense potential that makes it a useful tool for the characterization of probability distributions (Khan and Abu-Salih, 1988; Kumar and Kumar, 2023) and it is an important area of study in probability and statistics with applications in many fields of studies such as actuarial and insurance, climatology, hydrology, sports, medicine, and reliability analysis. Some existing works on order statistics include the recurrence relations obtained for moments of order statistics of some basic probability distributions including Joshi (1978) for exponential and truncated exponential distributions, Balakrishnan and Malik (1986) established some form of recurrence relations based on order statistics from the linear-exponential distribution, Kamps (1991) derived a general recurrence relation for moments of order statistics for some distributions including exponential, power function, Pareto, Lomax, and logistic distributions. Khan et al. (1983) derived some recurrence relations between moments of order statistics for some basic distributions including Weibull and exponential. See also Joshi and Balakrishnan (1982); Kumar et al. (2018) and recently from Gul and Mohsin (2021). Tippett (1925) estimated the difference between the maximum and minimum order statistics in a given sample. Greenberg and Sarhan (1958) applied the tools of order statistics to the study of health data.
David and Nagaraja (1981) investigated properties of order statistics and its applications on the estimation of parameters of the exponential distribution, Khan and Khan (1987) investigated the moments of order statistics and some characterization from the Burr distribution. Khan and Abu-Salih (1988) employed the properties of order statistics to

characterize the complimentary Weibull and Weibull distributions. Detail studies of order statistics and estimation methods are contained in the book of Arnold et al. (1992), Arnold et al. (2008) and Balakrishnan and Cohen (2014).
The extreme order statistics of two-parameter Lomax distribution was investigated by Dar and Al-Hossain (2015), Abdul-Moniem (2017) studied Power Lomax distribution based on order statistics. Kumar and Dey (2017) and Kumar et al. (2018) studied order statistics from the power Lindley and power Lomax distributions respectively and one of the most recent studies is published by Kumar and Kumar (2023)

This paper focused on the application of useful theories of order statistics to characterize the convolution of Weibull (W) and exponential Pareto (EP) distributions. The remaining parts of the study are outlined as follows; section 2 contains some relevant materials including the WeIbull-Exponential Pareto distribution. In section 3, some distributional properties of order statistics of the WEP distribution are derived. The beta-G procedure was employed to generate the distributions of extreme order statistics in section 4. The recurrence relations, the moments of order statistics, the moment of extreme order statistics, and the moment of the sample range were investigated in section 5. Some real-life application to Nigerian COVID-19 cases of patients on admission was carried out in section 6, simulation studies of the statistical properties of order statistics was conducted in section 7 and the study was concluded in section 8.

2 Materials and Methods

Relevant materials associated with the conception of the research are defined and presented in this section

2.1 Weibull-Exponential Pareto Distribution

The cumulative distribution function (cdf) of exponential Pareto (EP) distribution by Al-Kadim and Boshi (2013) is

$$G(x) = 1 - e^{-\lambda(\frac{x}{k})^{\theta}}; \lambda, \theta, k > 0; x > 0 \tag{1}$$

The Weibull-X approach for generating new flexible distributions by Alzaatreh et al. (2013) has the cdf defined for continuous distributions as follows;

$$\begin{aligned} F(x) &= \alpha\beta \int_0^{-log(1-G(x))} t^{\alpha-1} e^{-\beta t^{\alpha}} dt. \\ &= 1 - e^{-\beta(log(1-G(x))^{\alpha}}; x > 0; \alpha, \beta > 0 \end{aligned} \tag{2}$$

The corresponding pdf is the derivative given by

$$f(x) = \alpha\beta \frac{g(x)}{1-G(x)} e^{-\beta(log(1-G(x))^{\alpha}} [-log(1-G(x))]^{\alpha-1}; x > 0; \alpha, \beta > 0 \tag{3}$$

By taking X to be a random variable from the EP model combined with the Weibull-X families of distribution for T Weibull random variable having the cdf of the form $1 - e^{-(t^{\alpha})}, \beta = 1, \alpha > 0$; a four parameter WEP distribution (Adeyemi et al., 2023), has cumulative distribution function (cdf) obtained by substituting (1) into (2) to get

$$F(x) = 1 - exp\left(-\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha}\right) \tag{4}$$

The derivative yields the pdf of the WEP distribution obtained as,

$$f(x) = \frac{\alpha\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1}\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha-1} exp\left(-\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha}\right) \tag{5}$$

α, θ are shape parameters and λ, k are scale parameters. $\alpha, \lambda, \theta, k > 0; x > 0$

2.2 Order Statistics and their Functions

Let $X_1, X_2, ..., X_n$ be a random sample of size n from the WEP distribution and let the corresponding order statistics realized from the random samples be represented by $X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)}$. Then the density function of $X_{(r:n)}$ which has been defined by many authors including (David and Nagaraja, 1981; Arnold et al., 1992) is presented as follows;

$$f_{(r:n)}(x) = C_{r:n}\left(\left[F(x)\right]^{r-1}\left[1 - F(x)\right]^{n-r} f(x)\right); 0 < x < \infty \tag{6}$$

$$C_{r:n} = \frac{n!}{(r-1)!(n-r)!}$$

The distribution of minimum and maximum order statistics at $r = 1$ and $r = n$ is given respectively as follows;

$$f_{(1:n)}(x) = n\left(\left[1 - F(x)\right]^{n-1} f(x)\right); 0 < x < \infty \tag{7}$$

$$f_{(n:n)}(x) = n\left(\left[F(x)\right]^{n-1} f(x)\right); 0 < x < \infty \tag{8}$$

The cumulative distribution function of $X_{(r:n)}$ has been defined by several authors . See David and Nagaraja (1981);

$$F_{(r:n)}(x) = P(X_{(r)} < x) = \sum_{j=i}^{n} \binom{n}{i} [F(x)]^{i}[1 - F(x)]^{n-i} \tag{9}$$

The joint density of order statistics $X(r)$ and $X(s)$ for $r < s; 0 < x_r < x_s; r = 1, 2, ..., n$, is defined by;

$$f_{x_{(r)},x_{(s)}}(x_r, x_s) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}[F(x_r)]^{r-1} f(x_r) f(x_s)$$
$$[1 - F(x_s)]^{n-s}[F(x_s) - F(x_r)]^{s-r-1} \tag{10}$$

3 Distributional Properties of Order Statistics from the WEP Distribution

This sub-section is used to derive expressions for some functions of order statistics notably; for the cdf, pdf, and joint distribution of order statistics.

3.1 The Density Function of $X_{(r:n)}$ from WEP Distribution

Theorem 3.1: *Let $X_1, X_2, ..., X_n$ be a random sample of size n from the WEP distribution with cdf $F(x)$, and pdf $f(x)$, and let $X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)}$ be the corresponding order statistics from the sample. Then the density function of the r^{th} order statistics $f_{r:n}(x)$ is given by*

$$C_{r:n}\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\frac{\alpha\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1}\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha-1}\left[exp\left(-\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha}\right)\right]^{m} \quad (11)$$

Proof: The pdf of order statistics of WEP distribution is obtained by inserting the cdf and pdf in equations (4) and (5) into equation (6) and is given by;

$$\begin{aligned} f_{(r:n)}(x) &= C_{r:n}\left[1-exp\left(-\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha}\right)\right]^{r-1}\left[exp\left(-\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha}\right)\right]^{n-r} \\ &\times \quad \frac{\alpha\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1}\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha-1}exp\left(-\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha}\right) \\ &= C_{r:n}\left[1-exp\left(-\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha}\right)\right]^{r-1}\left[exp\left(-\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha}\right)\right]^{n-r+1} \\ &\times \quad \frac{\alpha\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1}\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha-1} \end{aligned} \quad (12)$$

By applying binomial expansion, we obtain

$$C_{r:n}\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\frac{\alpha\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1}\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha-1}\left[exp\left(-\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha}\right)\right]^{m} \quad \square \quad (13)$$

where $m = n - r + i + 1$ and $C_{r:n} = \frac{n!}{(r-1)!(n-r)!}$; $\alpha, \lambda, \theta, k > 0; x > 0$

The graphical structures are investigated for the Weibull Exponential Pareto order statistics (left), maximum order statistics (middle), and the minimum order statistics (right) for sample size $n = 20$ and $r = 1, 2, 5, 10, 20$. The plots for PDFs are shown in Figure 1 The plots of pdf preserved the uni-modal property of the WEP distribution in Adeyemi et al. (2023). It shows the distribution of order statistics is skewed to the right and the kurtosis decreases as the sample size increases from the minimum $X_{(1:20)}$ to the maximum $X_{(20:20)}$.

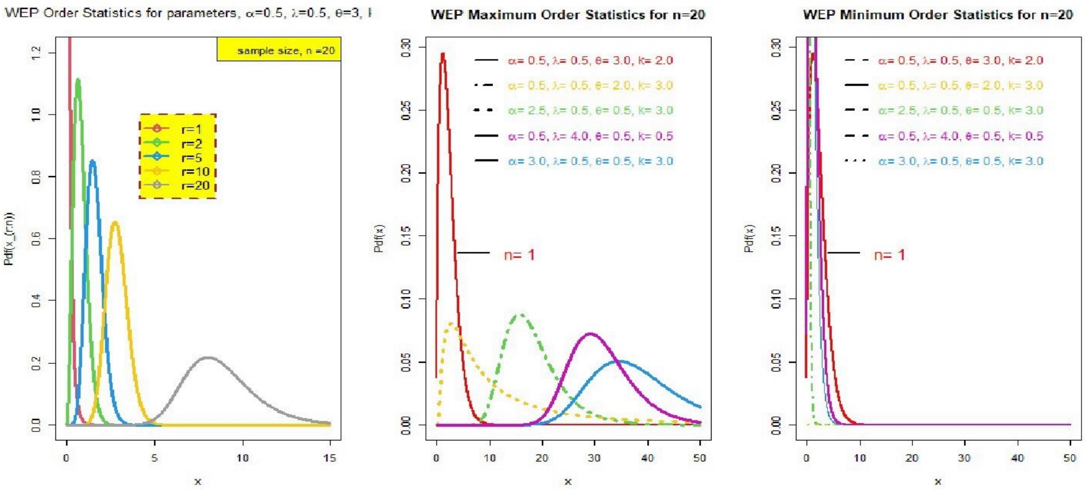

Figure 1: pdf of WEP order statistics for some value of the parameters

3.2 The CDF of Order Statistics from WEP Distribution

Lemma 3.1: *Let F(x) and f(x) be the cdf and pdf of a random variable from the WEP distribution respectively, the cdf of the order statistics r^{th} is given by;*

$$F_{(r:n)}(x) = \sum_{j=i}^{n}\sum_{l=0}^{i}(-1)^i \binom{n}{i}\binom{i}{l}[1-F(x)]^{n-i+l} \tag{14}$$

Proof: The cdf $F_r(x)$ of the r^{th} order statistics can be derived as a sum of binomial random variables Y with n independent trials and probability p

$$F(y) = P(Y = x) = \binom{n}{x} p^x (1-p)^{n-x} \tag{15}$$

$$F_r(x) = \sum_{r=x}^{n} F(y) = \sum_{r=x}^{n}\binom{n}{x} p^x (1-p)^{n-x} \tag{16}$$

Then using $F(x)$ as the probability of success in the cumulative function which also satisfies the condition that $0 \le F(x) \le 1$, equation (16) becomes;

$$F_r(x) = P(X_{(r)} \le x) = \sum_{r=x}^{n} F(y) = \sum_{r=x}^{n}\binom{n}{x}[F(x)]^x[1-F(x)]^{n-x}, -\infty \le x \le \infty \tag{17}$$

by applying binomial expansion on equation (17), the result is obtained.

Corollary 3.1: Let F(x) and f(x) be the cdf and pdf of a random variable from the

WEP distribution respectively, the explicit expression for cdf of the order statistics r^{th} is given by;

$$F_{(r:n)}(x) = \sum_{j=i}^{n}\sum_{l=0}^{i}(-1)^i\binom{n}{i}\binom{i}{l}[exp\left(-\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha}\right)]^{n-i+l} \tag{18}$$

$\alpha, \lambda, \theta, k > 0; x > 0$

3.3 The Joint PDF of two Order Statistics from WEP Distribution

The joint density of order statistics $X_{(r:n)}$ and $X_{(s:n)}$ for $r < s < n$ is derived by substituting (4) and (5) into (10) as follows;

$$\begin{aligned} f_{x_{(r)},x_{(s)}}(x,y) = & \frac{n!}{(r-1)!(s-r-1)!(n-s)!}\left[1 - exp\left(-\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha}\right)\right]^{r-1} \\ & \left[exp\left(-\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha}\right) - exp\left(-\left(\lambda\left(\frac{y}{k}\right)^{\theta}\right)^{\alpha}\right)\right]^{s-r-1} \\ & \left[exp\left(-\left(\lambda\left(\frac{y}{k}\right)^{\theta}\right)^{\alpha}\right)\right]^{n-s}\frac{\alpha\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1}\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha-1} \\ & exp\left(-\left(\lambda\left(\frac{y}{k}\right)^{\theta}\right)^{\alpha}\right)exp\left(-\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha}\right)\frac{\alpha\lambda\theta}{k}\left(\frac{y}{k}\right)^{\theta-1}\left(\lambda\left(\frac{y}{k}\right)^{\theta}\right)^{\alpha-1} \\ = & \frac{n!}{(r-1)!(m_1)!(n-s)!}\left[1 - exp\left(-\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha}\right)\right]^{r-1} \\ & \left[exp\left(-\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha}\right) - exp\left(-\left(\lambda\left(\frac{y}{k}\right)^{\theta}\right)^{\alpha}\right)\right]^{m_1} exp\left(-\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha}\right) \\ & \left[exp\left(-\left(\lambda\left(\frac{y}{k}\right)^{\theta}\right)^{\alpha}\right)\right]^{m_2}\left[\frac{\alpha\lambda\theta}{k}\right]^2\left(\frac{xy}{k}\right)^{\theta-1}\left(\lambda\left(\frac{xy}{k}\right)^{\theta}\right)^{\alpha-1} \end{aligned} \tag{19}$$

$m_1 = s - r - 1$, $m_2 = n - s + 1$ and $\alpha, \lambda, \theta, k > 0; x > 0$

Corollary 3.2: The joint density of the (smallest) $X_{(1:n)}$ and (largest) $X_{(n:n)}$ order statistics is given by

$$\begin{aligned} f_{x_{(1)},x_{(n)}}(x,y) = & \frac{n!}{(n-2)!}[exp\left(-\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha}\right) - exp\left(-\left(\lambda\left(\frac{y}{k}\right)^{\theta}\right)^{\alpha}\right)]^{n-2} \\ & \left[\frac{\alpha\lambda\theta}{k}\right]^2\left(\frac{xy}{k}\right)^{\theta-1}\left(\lambda\left(\frac{xy}{k}\right)^{\theta}\right)^{\alpha-1} exp\left(-\left(\lambda\left(\frac{x+y}{k}\right)^{\theta}\right)^{\alpha}\right) \end{aligned} \tag{20}$$

$\alpha, \lambda, \theta, k > 0; x > 0$

3.4 Order Statistics of Extreme Random Variables from WEP Distribution

The order statistics of extreme random observations from WEP distribution can be obtained as special cases of the $X_{(r:n)}$ in equation (12) as follows;

The minimum order statistics $X_{(1:n)}$ of WEP distribution has the pdf derived from equation (12) when $r = 1$ as a special case given by;

$$f_{(1:n)}(x) = \frac{n\alpha\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1}\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha-1}\left[exp\left(-\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha}\right)\right]^{n} \tag{21}$$

The maximum order statistics $X_{(n:n)}$ of WEP distribution has the pdf obtained as a sub model of equation (12) when $r = n$ and is given by;

$f_{(n:n)}(x) =$

$$\sum_{i=0}^{n-1}(-1)^{i}\binom{n-1}{i}\frac{n\alpha\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1}\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha-1}\left[exp\left(-\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha}\right)\right]^{i+1} \tag{22}$$

3.5 The Distribution of the Sample Median Statistic X_{med}

The pdf of the median statistics (Med) in Arnold *et al* (1992) is derivable using, Med= $X_{(m+1)}$ if $n = 2m + 1$ for odd integers given by

$$f_{Med:n}(x) = \frac{n!}{\left[((n-1)/2)!\right]^{2}}\left[F(x)\right]^{(n-1)/2}\left[1-F(x)\right]^{(n-1)/2}f(x); -\infty < x < \infty \tag{23}$$

The pdf follows for even integers using Med= $X_{(m)}$ if $n = 2m$.

Theorem 3.2 Let $X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)}$ be the order statistics from a random sample of size n from the WEP distribution, the pdf of the median for **o**dd sample size n is given by;

$$f_{X_{(\frac{n+1}{2}:n)}}(x) = \begin{cases} \frac{n!}{\left[((n-1)/2)!\right]^{2}}\sum_{i=0}^{\frac{n-1}{2}}(-1)^{i}\binom{\frac{n-1}{2}}{i}\left[exp\left(-\lambda\left(\frac{x}{k}\right)^{\theta}\right)\right]^{(n+1)/2+i} \\ X\frac{\alpha\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1}\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha-1}; \alpha,\lambda,\theta,k>0; x>0 \end{cases} \tag{24}$$

Proof

$$f_{X_{(\frac{n+1}{2}:n)}}(x) = \frac{n!}{\left[((n-1)/2)!\right]^{2}}\left[1-exp\left(-\lambda\left(\frac{x}{k}\right)^{\theta}\right)\right]^{(n-1)/2}\left[exp\left(-\lambda\left(\frac{x}{k}\right)^{\theta}\right)\right]^{(n-1)/2}$$

$$X\ \frac{\alpha\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1}\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha-1}exp\left(-\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha}\right) \tag{25}$$

The proof is obtained after some mathematical operations.

3.6 Distribution of the Range of a Random Sample

Riffi (2015) had obtained the distributional property of the sample range for the exponential distribution with parameter λ as follows;

$$f_{i,j}(r) = \frac{(n-i)!}{(n-j)!(j-i-1)!}\lambda exp(-\lambda(n-j+1)r)\Big[1-exp(-\lambda r)\Big]^{j-i-1}, r > 0 \quad (26)$$

By carrying out some modification on Theorem 3.2 Riffi (2015); the density function of the statistics $X_{(s:n)} - X_{r:n}$ of WEP after some algebraic operation can be obtained using;

$$f_{X_{s:n}-X_{r:n}}(x_{s:n}, x_{r:n}) = \frac{(n-r)!}{(n-s)!(s-r-1)!}\Big[F(x)\Big]^{s-r-1}\Big[1-F(x)\Big]^{n-s} f(x) \quad (27)$$

Theorem 3.3: Let $X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)}$ be the order statistics from a random sample of size n from the class of distribution with cdf of the form $F(x) = 1 - e^{-(bx)}$. If the relation $f(x) = h(x)[1 - F(x)]$ exist where $h(x)$ is the hazard rate function; then the distribution of the sample range $R = X_{(n:n)} - X_{(1:n)}$ is given by

$$\bar{f}_{R,n}(x) = (n-1)\sum_{i=0}^{n-2}(-1)^i\binom{n-2}{i}h(x)[1-F(x)]^{i+1} \quad (28)$$

Proof:
Substitute $f(x) = h(x)[1 - F(x)]$ into equation (28) to get

$$f_{X_{s:n}-X_{r:n}}(x_{s:n}, x_{r:n}) = \frac{(n-r)!}{(n-s)!(s-r-1)!}\Big[F(x)\Big]^{s-r-1}\Big[1-F(x)\Big]^{n-s+1} h(x) \quad (29)$$

setting $s = n, r = 1$ in (29) followed by binomial expansion,

$$\begin{aligned}\bar{f}_{R,n}(x) = f_{X_{n:n}-X_{1:n}}(x_{n:n}, x_{1:n}) &= \frac{(n-1)!}{(n-n)!(n-2)!}\Big[F(x)\Big]^{n-2}\Big[1-F(x)\Big]h(x) \\ &= (n-1)\sum_{i=0}^{n-2}(-1)^i\tbinom{n-2}{i}h(x)[1-F(x)][1-F(x)]^i \quad (30) \\ &= (n-1)\sum_{i=0}^{n-2}(-1)^i\binom{n-2}{i}h(x)[1-F(x)]^{i+1} \quad \square\end{aligned}$$

Remark 3.1: The novelty in Theorem 3.3 is that the result represents a distributional function of the range order statistics from the general class of probability distributions with cdf of the form $F(x) = 1 - e^{-(bx)}$ and the existence of a relation between PDF and CDF given by $f(x) = h(x)[1 - F(x)]$; where $h(x)$ is the hazard rate function.

Corollary 3.3: Let $X_1, X_2, ..., X_n$ be a random sample of size n from the WEP distribution with pdf $f(x)$ and cdf $F(x)$ and order statistics from the random sample denoted by $X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)}$. Then the distribution of the sample range $X_{(n:n)} - X_{(1:n)}$ for order statistics from the Weibull exponential Pareto distribution is given by

$$\bar{f}_{R,n}(x) = (n-1)\sum_{i=0}^{n-2}(-1)^i\binom{n-2}{i}\frac{\alpha\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1}\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha-1}\left[exp\left(-\lambda\left(\frac{x}{k}\right)^{\theta}\right)\right]^{i+1} \tag{31}$$

Proof: The proof follows from application of Theorem (3.3).

4 Extreme Order Statistics using the beta-G Framework

The minimum $X_{(1)}$ and the maximum ordered variable $X_{(n)}$ from a set order statistic represented by $X_{(1)}, X_{(2)}, ..., X_{(n)}$ is classified as extreme order statistics which often generates a lot of interest in many fields of studies due to its properties and application in real life situation.
Extreme order statistics can be derived using the old traditional principle of transformation of variables or by directly using the density function of the r^{th} order statistics in equation (12) making $r = 1$ and $r = n$ to get the minimum and maximum order statistics as derived in equations (21) and (22) respectively.
However, Eugene et al. (2002) first introduced the beta distribution as a generator for generalizing flexible probability distributions. Let $F(x)$ be the cumulative distribution function of a random variable X. Then the cdf of the class of beta-G families of distribution is defined by;

$$F(x) = I_{F(x)}(a, b) = \frac{1}{B(a, b)}\int_0^{F(x)} t^{a-1}(1-t)^{b-1}dt \tag{32}$$

Where $a > 0$ and $b > 0$ are shape parameters;
$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a,b)}$ is the beta function and $\Gamma(.)$ is the gamma function.
The pdf of the beta-G distribution has the form;

$$f(a, b; x) = \frac{1}{B(a, b)}\Big[F(x)\Big]^{a-1}\Big[1 - F(x)\Big]^{b-1} f(x) \tag{33}$$

(Eugene et al., 2002; Jones, 2009) has stated that the r^{th} order statistics is a distribution belonging to the special case of the beta-G family of distribution. This study however demonstrates in this section the application of the beta-G framework for constructing the distributions of the minimum $X_{(1:n)}$ and maximum $X_{(n:n)}$ order statistics.

4.1 The Distribution of Minimum Order Statistics

The cdf of $X_{(1:n)}$ is derived by substituting $B(a, b)$ with $B(1, n)$ into the cdf of the class of beta-G families of distribution in equation (32) and is given by;

$$F_{(1:n)}(x) = \frac{1}{B(1, n)}\int_0^{F(x)} (1-t)^{n-1}dt \tag{34}$$

$$F_{(1:n)}(x) = n\int_0^{F(x)} (1-t)^{n-1}dt \tag{35}$$

The solution to the integral in (35) will yield the cdf of Minimum order statistics and is given by;

$$F_{(1:n)}(x) = 1 - \Big(1 - F(x)\Big)^n \tag{36}$$

The pdf can be obtained by finding the first derivative of $F_{(1:n)}(x)$ in (36).
Application of the beta-G for deriving the density function of $X_{(1:n)}$ required replacing $B(a, b)$ with $B(1, n)$ in equation (33) and is given by;

$$f_{(1:n)}(x) = \frac{1}{B(1,n)}\Big[1 - F(x)\Big]^{n-1} f(x) \tag{37}$$

$$f_{(1:n)}(x) = n\Big[1 - F(x)\Big]^{n-1} f(x) \tag{38}$$

Results in equations (36) and (38) strengthen existing results in the literature for the cdf and density function of the minimum order statistics by using other approaches and as previously obtained for $f_{(1:n)}(x)$ in equation (21).

4.2 The distribution of Maximum Order Statistics

We derived the cdf of $X_{(n:n)}$ by substituting $B(a, b)$ for $B(n, 1)$ into the cdf of the class of beta-G families of distribution equation (32) and is given by;

$$F_{(n:n)}(x) = I_{F(x)}(n,1) = \frac{1}{B(n,1)}\int_0^{F(x)} t^{n-1}dt \tag{39}$$

After some mathematical operations, the cdf of $X_{(n:n)}$ from the integral is given as;

$$F_{(n:n)}(x) = \Big(F(x)\Big)^n \tag{40}$$

The pdf of $X_{(n:n)}$ which is simply the derivative of quantity in equation (40) is derived from the beta-G by substituting $B(a, b)$ for $B(n, 1)$ in equation (33) and is given by;

$$f_{(n:n)}(x) = \frac{1}{B(n,1)}\Big[F(x)\Big]^{n-1} f(x) \tag{41}$$

$$f_{(n:n)}(x) = n\Big[F(x)\Big]^{n-1} f(x) \tag{42}$$

Results in equations (40) and (42) strengthen existing results in the literature for deriving the cdf and density function of the maximum order statistics $X_{(n:n)}$ through some other approach and as previously obtained in equation (22).

By inserting the cdf of WEP distributions in equation (5) into equation (40); the cdf of WEP Maximum Order statistics is derived as;

$$F_{(n:n)}(x) = \left[1 - exp\left(-\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha}\right)\right]^{n} \tag{43}$$

Theorem 4.1 *Let $X_1, X_2, ..., X_n$ be a random sample of size n from the WEP distribution with cdf and pdf denoted by $F(x)$ and $f(x)$ respectively. Then the density function of the maximum order statistics $X_{(n:n)}$, of WEP distribution corresponding to cdf in equation (43) is given by;*

$$f_{(n:n)}(x) = \sum_{i=0}^{n-1}(-1)^{i}\binom{n-1}{i}\frac{n\alpha\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1}\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha-1}\left[exp\left(-\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha}\right)\right]^{i+1} \tag{44}$$

Proof: The density function corresponding to the cdf of WEP Maximum Order statistics is derived as;

$$\begin{aligned} f_{(n:n)}(x) = n&\left[1 - exp\left(-\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha}\right)\right]^{n-1} \\ &\frac{\alpha\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1}\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha-1} exp\left(-\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha}\right) \end{aligned} \tag{45}$$

Applying binomial expansion;

$\left[1 - exp\left(-\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha}\right)\right]^{n-1} = \sum_{i=0}^{n-1}(-1)^{i}\binom{n-1}{i}exp\left(-i\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha}\right) = A$

Substitute A into equation (45) to obtain the final proof.

Theorem 4.2 *Let $X_1, X_2, ..., X_n$ be a random sample of size n from the WEP distribution with cdf and pdf denoted by $F(x)$ and $f(x)$ respectively. Then the density function of the minimum order statistics $X_{(1:n)}$, from the WEP distribution is given by;*

$$f_{(1:n)}(x) = \frac{\alpha\lambda\theta}{k}\sum_{i=0}^{\infty}\frac{(-1)^{i}(n)^{i+1}}{i!}\left(\frac{x}{k}\right)^{\theta-1}\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha(i+1)-1} \tag{46}$$

Proof: The density function corresponding to the cdf of WEP Minimum Order statistics is derived as;

$$f_{(1:n)}(x) = n\left[1 - \left\{1 - exp\left(-\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha}\right)\right\}\right]^{n-1}\frac{\alpha\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1}\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha-1} exp\left(-\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha}\right) \tag{47}$$

$$f_{(1:n)}(x) = n\left[exp\left(-\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha}\right)\right]^{n-1}\frac{\alpha\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1}\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha-1} exp\left(-\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha}\right) \tag{48}$$

$$f_{(1:n)}(x) = \left[exp\left(-\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha}\right)\right]^{n}\frac{n\alpha\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1}\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha-1} \tag{49}$$

Remark 4.1: The results posted in (21) and (22) using the density function of the k^{th} order statistics which happens to be the popular and familiar approach is further strengthened by the results obtained for the densities of extreme order statistics from WEP distribution in (44) and (46) using the beta generalized approach

5 Moments of Order Statistics and Recurrence Relation from WEP

In this section, we extend the study of WEP distribution by deriving an explicit expression for the single moment of order statistics from the WEP distribution.
Lemma 5.1: Let $X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)}$ be order statistics of the random sample from WEP distribution having cdf and pdf $F(x)$ and $f(x)$ respectively, for $a > 0, b > 0$ if

$$I(a,b) = \int_0^{\infty} x^a\left(1 - F(x)\right)^{b+1} h(x)dx \tag{50}$$

then

$$I(a,b) = k^a\left(\frac{1}{\lambda}\right)^{\frac{a}{\theta}}\left(\frac{1}{b+1}\right)^{(\frac{a}{\alpha\theta}+1)}\Gamma\left(\frac{a}{\alpha\theta}+1\right) \tag{51}$$

Proof: substituting the hazard rate, h(x) and survival function of WEP distribution into equation (50) and obtain

$$I(a,b) = \int_0^{\infty} x^a\left(exp\left(-\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha}\right)\right)^{b+1} h(x)dx \tag{52}$$

Let $y = (b+1)\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha}$ by transformation of variable we have the following quantities;
$x = \frac{ky^{\frac{1}{\alpha\theta}}}{\lambda^{\frac{1}{\theta}}(b+1)^{\frac{1}{\alpha\theta}}}$; $\frac{dy}{dx} = \frac{(b+1)\alpha\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1}\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha-1}$=$(b+1)h(x)$
after appropriate substitution the mean becomes;

$$\int_0^{\infty}\frac{1}{(b+1)}\left(\frac{ky^{\frac{1}{\alpha\theta}}}{\lambda^{\frac{1}{\theta}}(b+1)^{\frac{1}{\alpha\theta}}}\right)^a e^{-y}dy = \frac{1}{b+1}\left(\frac{k}{\lambda^{\frac{1}{\theta}}(b+1)^{\frac{1}{\alpha\theta}}}\right)^a\int_0^{\infty} y^{\frac{a}{\alpha\theta}}e^{-y}dy \tag{53}$$

Using the gamma function $\int_0^{\infty} y^r e^{-y}dy = \Gamma\left(r+1\right)$ completes the proof of the lemma.
Remark 5.1 The result from Lemma 5.1 is very important as it will be extremely useful in subsequent investigations.

5.1 Moments of Order Statistics from WEP Distribution

Theorem 5.1: *Let $X_1, X_2, ..., X_n$ be a random sample of size n from the WEP distribution with cdf and pdf denoted by $F(x)$ and $f(x)$ respectively and let $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}$ be corresponding order statistics. Then the expected value of $X_{(r:n)}$; which is the t^{th} moments of the r^{th} order statistics for $t = 1, 2, ...$ denoted by $\mu_{r:n}^{(t)}$ is given by;*

$$\mu_{r:n}^{(t)} = C_{r:n} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} k^t \left(\frac{1}{\lambda}\right)^{\frac{t}{\theta}} \left(\frac{1}{m}\right)^{(\frac{t}{\alpha\theta}+1)} \Gamma\left(\frac{t}{\alpha\theta}+1\right) \tag{54}$$

where Γ is the gamma function and $m = n - r + i + 1$

Proof

$$\mu_{r:n}^{(t)} = \int_0^\infty x^t f_{r:n}(x) dx \tag{55}$$

The WEP distribution (pdf) has a functional relationship with the cdf given by

$$f(x) = \frac{\alpha\lambda\theta}{k} \left(\frac{x}{k}\right)^{\theta-1} \left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha-1} \left(1 - F(x)\right); \alpha, \lambda, \theta > 0; x > 0 \tag{56}$$

Then using equation (56) in equation(55), we have ;

$$\mu_{r:n}^{(t)} = C_{r:n} \int_0^\infty x^t \frac{\alpha\lambda\theta}{k} \left(\frac{x}{k}\right)^{\theta-1} \left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha-1} \left(F(x)\right)^{r-1} \left(1 - F(x)\right)^{n-r+1} dx \tag{57}$$

using binomial expansion of the form $\left(1 - Z\right)^b = \sum_{i=0}^{b} (-1)^i \binom{b}{i} z^i$

$$\mu_{r:n}^{(t)} = \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \frac{\alpha\lambda\theta}{k} \left(\frac{x}{k}\right)^{\theta-1} \left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha-1} C_{r:n} \int_0^\infty x^t \left(1 - F(x)\right)^m dx \tag{58}$$

where $m = n - r + i + 1$

$$\mu_{r:n}^{(t)} = \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} C_{r:n} \int_0^\infty x^t \left(1 - F(x)\right)^m h(x) dx \tag{59}$$

By application of Lemma (5.1); we obtain

$$\mu_{r:n}^{(t)} = C_{r:n} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} I(t, m) \tag{60}$$

$$I(t, m) = k^t \left(\frac{1}{\lambda}\right)^{\frac{t}{\theta}} \left(\frac{1}{m}\right)^{(\frac{t}{\alpha\theta}+1)} \Gamma\left(\frac{t}{\alpha\theta}+1\right); \qquad \square \tag{61}$$

The mean order statistics and the variance of order statistics are derived and given respectively as;

$$\mu_{r:n} = C_{r:n} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} k \left(\frac{1}{\lambda}\right)^{\frac{1}{\theta}} \left(\frac{1}{m}\right)^{(\frac{1}{\alpha\theta}+1)} \Gamma\left(\frac{1}{\alpha\theta}+1\right) \tag{62}$$

and

$\sigma^{(2)}_{r:n} = \mu^{(2)}_{r:n} - \left(\mu_{r:n}\right)^2$

$$\sigma^{(2)}_{r:n} = C_{r:n} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} k^2 \left(\frac{1}{\lambda}\right)^{\frac{2}{\theta}} \left(\frac{1}{m}\right)^{(\frac{2}{\alpha\theta}+1)} \Gamma\left(\frac{2}{\alpha\theta}+1\right) - \left(\mu_{r:n}\right)^2 \tag{63}$$

Remark 5.2: The result in equation (62) generalizes the moments of the WEP distribution and moments of some lifetime distributions existing in the literature.
In addition, some of the statistical properties of classical distributions like Weibull, Rayleigh, and Exponential distributions can be obtained from the result. The results are useful in various fields of application where there is a need to predict the expected maximum duration, expected maximum remission times, expected mean revenue due to additional input of factors of labor, and expected maximum output due to aging/death.
The following results can be obtained by setting r=1=n in Theorem 5.1
Corollary 5.1: For WEP distribution, the t^{th} moment and the mean is deduced and respectively given as ;

$$\frac{k^t}{(\lambda)^{\frac{t}{\theta}}} \Gamma\left(\frac{t}{\alpha\theta}+1\right) \tag{64}$$

$$\frac{k}{(\lambda)^{\frac{1}{\theta}}} \Gamma\left(\frac{1}{\alpha\theta}+1\right) \tag{65}$$

Corollary 5.2: The result in Theorem 5.1 reduces to the explicit expression of the t^{th} moment and the mean of exponential Pareto (EP) distribution studied by Al-Kadim and Boshi (2013), if $\alpha = 1$ as deduced and respectively given by;

$$\frac{k^t}{(\lambda)^{\frac{t}{\theta}}} \Gamma\left(\frac{t}{\theta}+1\right) \tag{66}$$

$$\frac{k}{(\lambda)^{\frac{1}{\theta}}} \Gamma\left(\frac{1}{\theta}+1\right) \tag{67}$$

see Al-Kadim and Boshi (2013) equation (9), page.137.

5.2 Moment of Extreme Order Statistics from WEP Distribution

This section presents some moments of the minimum and maximum order statistics of the WEP distribution
Corollary 5.3: The t^{th} moments of the minimum order statistics $X_{(1:n)}$ of a random variable from the WEP distribution is given by;

$$\mu^{(t)}_{1:n} = \frac{nk^t}{\lambda^{\frac{t}{\theta}} (n)^{\frac{t}{\alpha\theta}+1}} \Gamma\left(\frac{t}{\alpha\theta}+1\right) \tag{68}$$

Corollary 5.4: The t^{th} moments of the maximum order statistics $X_{(n:n)}$ of a random variable from the WEP distribution is given by;

$$\mu_{n:n}^{(t)} = n\sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}\frac{k^t}{\lambda^{\frac{t}{\theta}}(i+1)^{\frac{t}{\alpha\theta}+1}}\Gamma\left(\frac{t}{\alpha\theta}+1\right) \tag{69}$$

Proposition 5.1: The expected value of the minimum order statistics $X_{(1:n)}$ of a random variable from the WEP distribution is given by;

$$\mu_{1:n} = \frac{k}{\lambda^{\frac{1}{\theta}}(n)^{\frac{1}{\alpha\theta}}}\Gamma\left(\frac{1}{\alpha\theta}+1\right) \tag{70}$$

The second moment of minimum order statistics $X_{(1:n)}$ is obtained as;

$$\mu_{1:n}^{(2)} = \frac{k^2}{\lambda^{\frac{2}{\theta}}(n)^{\frac{2}{\alpha\theta}}}\Gamma\left(\frac{2}{\alpha\theta}+1\right) \tag{71}$$

The variance can be obtained using ;

$$Var = E\left(X_{(1:n)}^2\right) - E\left(X_{(1:n)}\right)^2$$

Hence the variance of minimum order statistics $X_{(1:n)}$ for the WEP distribution is obtained as;

$$\sigma_{1:n}^{(2)} = \mu_{1:n}^{(2)} - \left(\mu_{1:n}\right)^2$$

$$\sigma_{1:n}^{(2)} = \frac{k^2}{\lambda^{\frac{2}{\theta}}(n)^{\frac{2}{\alpha\theta}}}\left[\Gamma\left(\frac{2}{\alpha\theta}+1\right) - \Gamma^2\left(\frac{2}{\alpha\theta}+1\right)\right] \tag{72}$$

The mean, second moment and variance of the WEP maximum order statistics can be derived in a similar fashion.

5.3 Moment and Mean Value of the Sample Range Statistics

Theorem 5.2 *Let $X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)}$ be the order statistics from WEP random sample $X_1, X_2, ..., X_n$ of size n, the mean value of the sample range $X_{(n:n)} - X_{(1:n)}$ is obtained and given by;*

$$\mu_R = (n-1)\sum_{i=0}^{n-2}(-1)^i\binom{n-2}{i}k\left(\frac{1}{\lambda}\right)^{\frac{1}{\theta}}\left(\frac{1}{i+1}\right)^{(\frac{t}{\alpha\theta}+1)}\Gamma\left(\frac{1}{\alpha\theta}+1\right) \tag{73}$$

Proof: The t^{th} moments is defined by

$$\mu_R^{(t)} = \int_0^\infty x^t \bar{f}_{R,n}(x)dx \tag{74}$$

Substitute the pdf of range statistics from WEP distribution in (74) to obtain

$$\mu_R^{(t)} = (n-1)\sum_{i=0}^{n-2}(-1)^i\binom{n-2}{i}\frac{\alpha\lambda\theta}{k}\int_0^\infty x^t\left(\frac{x}{k}\right)^{\theta-1}\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha-1}\left[exp\left(-\lambda\left(\frac{x}{k}\right)^{\theta}\right)\right]^{i+1}dx \tag{75}$$

$$\mu_R^{(t)} = (n-1)\sum_{i=0}^{n-2}(-1)^i\binom{n-2}{i}\int_0^\infty x^t\left[exp\left(-\lambda\left(\frac{x}{k}\right)^{\theta}\right)\right]^{i+1}h(x)dx \tag{76}$$

By application of **L**emma 5.1, we have.

$$\mu_R^{(t)} = (n-1)\sum_{i=0}^{n-2}(-1)^i\binom{n-2}{i}I(t,i+1) \tag{77}$$

$$I(t,i+1) = k^t\left(\frac{1}{\lambda}\right)^{\frac{t}{\theta}}\left(\frac{1}{i+1}\right)^{(\frac{t}{\alpha\theta}+1)}\Gamma\left(\frac{t}{\alpha\theta}+1\right) \tag{78}$$

$$\therefore \quad \mu_R^{(t)} = (n-1)\sum_{i=0}^{n-2}(-1)^i\binom{n-2}{i}k^t\left(\frac{1}{\lambda}\right)^{\frac{t}{\theta}}\left(\frac{1}{i+1}\right)^{(\frac{t}{\alpha\theta}+1)}\Gamma\left(\frac{t}{\alpha\theta}+1\right) \tag{79}$$

The mean value of the sample range is obtained and given by;

$$\mu_R = (n-1)\sum_{i=0}^{n-2}(-1)^i\binom{n-2}{i}k\left(\frac{1}{\lambda}\right)^{\frac{1}{\theta}}\left(\frac{1}{i+1}\right)^{(\frac{t}{\alpha\theta}+1)}\Gamma\left(\frac{1}{\alpha\theta}+1\right) \quad \square \tag{80}$$

5.4 Recurrence Relations for Moment of Order Statistics from the WEP Distribution

This section is used to derive an explicit expression for the recurrence relation for the moment of order statistics of WEP distribution through the following theorem;

Theorem 5.3: Let $X_1, X_2, ..., X_n$ be random sample of size n from the WEP distribution whose corresponding order statistics is denoted by $X_{(1:n)}, X_{(2:n)}, ..., X_{(n:n)}$; then for $1 \leq r \leq n$ and $t = 0, 1, 2...$ we have the following result for moment relation

$$\mu_{r:n}^{(t)} = \frac{\alpha\theta\lambda^\alpha(n-r+1)\mu_{r:n}^{(t+\alpha\theta)} - \alpha\theta\lambda^\alpha(r-1)\mu_{r-1:n}^{(t+\alpha\theta)}}{k^{\alpha\theta}(t+\alpha\theta)} \tag{81}$$

Proof: The t^{th} moment of WEP distribution order statistics is defined by;

$$\mu_{r:n}^{(t)} = C_{r:n}\int_0^\infty x^t\left(F(x)\right)^{r-1}\left(1-F(x)\right)^{n-r}f(x)dx \tag{82}$$

$$f(x) = \frac{\alpha\lambda\theta}{k}\frac{\alpha\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1}\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha-1}\left(1-F(x)\right)$$

substituting $f(x)$ into (82) and doing some arithmetic operations we have

$$\mu_{r:n}^{(t)} = \frac{\alpha\lambda\theta}{k}\left(\frac{x}{k}\right)^{\theta-1}\left(\lambda\left(\frac{x}{k}\right)^{\theta}\right)^{\alpha-1} C_{r:n}\int_0^{\infty} x^t\Big(F(x)\Big)^{r-1}\Big(1-F(x)\Big)^{n-r+1}dx \tag{83}$$

$$\mu_{r:n}^{(t)} = \frac{\alpha\theta\lambda^{\alpha}}{k^{\alpha\theta}}C_{r:n}\int_0^{\infty} x^{t+\alpha\theta-1}\Big(F(x)\Big)^{r-1}\Big(1-F(x)\Big)^{n-r+1}dx \tag{84}$$

Using integration by parts we shall obtain;

$$\begin{aligned}
\mu_{r:n}^{(t)} &= \frac{\alpha\theta\lambda^{\alpha}}{k^{\alpha\theta}}C_{r:n}\int_0^{\infty}\Bigg[x^{t+\alpha\theta}\frac{(n-r+1)}{t+\alpha\theta}\Big(F(x)\Big)^{r-1}\Big(1-F(x)\Big)^{n-r}f(x)dx \\
&- x^{t+\alpha\theta}\frac{(r-1)}{t+\alpha\theta}\Big(F(x)\Big)^{r-2}\Big(1-F(x)\Big)^{n-r+1}f(x)dx\Bigg] \\
&= \frac{\alpha\theta\lambda^{\alpha}}{k^{\alpha\theta}}\int_0^{\infty}\Bigg[x^{t+\alpha\theta}\frac{(n-r+1)}{t+\alpha\theta}C_{r:n}\Big(F(x)\Big)^{r-1}\Big(1-F(x)\Big)^{n-r}f(x)dx \\
&- x^{t+\alpha\theta}\frac{(r-1)}{t+\alpha\theta}C_{r:n}\Big(F(x)\Big)^{r-2}\Big(1-F(x)\Big)^{n-r+1}f(x)dx\Bigg] \\
&= \frac{\alpha\theta\lambda^{\alpha}}{k^{\alpha\theta}}\left[\frac{(n-r+1)}{t+\alpha\theta}\mu_{r:n}^{(t+\alpha\theta)} - \frac{(r-1)C_{r:n}}{t+\alpha\theta C_{r-1:n}}\mu_{r-1:n}^{(t+\alpha\theta)}\right] \\
&= \frac{\alpha\theta\lambda^{\alpha}}{k^{\alpha\theta}}\frac{(n-r+1)}{t+\alpha\theta}\left[\mu_{r:n}^{(t+\alpha\theta)} - \mu_{r-1:n}^{(t+\alpha\theta)}\right] \qquad \square
\end{aligned} \tag{85}$$

A special case is when $r = 1$ and the recurrence relation is obtained as;

$$\mu_{1:n}^{(t)} = \frac{n\alpha\theta\lambda^{\alpha}\mu_{1:n}^{(t+\alpha\theta)}}{k^{\alpha\theta}(t+\alpha\theta)} \tag{86}$$

Remark 5.3: Recurrence relation for the mean of order statistics when $t = 1$ from WEP distribution is given by;

$$\mu_{r:n} = \frac{\alpha\theta\lambda^{\alpha}}{k^{\alpha\theta}}\frac{(n-r+1)}{1+\alpha\theta}\left[\mu_{r:n}^{(1+\alpha\theta)} - \mu_{r-1:n}^{(1+\alpha\theta)}\right] \tag{87}$$

$\alpha, \lambda, \theta, k > 0$ and $0 < r < n$

Order statistics from the WEP distribution provide useful characterizations for some lifetime distributions existing in the literature that are yet to be investigated based on order statistics.
The explicit expression for mean order statistics of some lifetime distributions is presented in Table 1.
Where
$\omega_i = C_{r:n}\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}$, and $m = n - r + i + 1$

Table 1: Mean of Order Statistics of WEP sub-models

Distributions	$\alpha,\ \lambda,\ \theta,\ k$	**M**ean $(\mu_{r:n}) =$	**A**uthors
exponential Pareto	$1,-,-,-$	$\omega_i \frac{k^t}{\lambda^{\frac{t}{\theta}} m^{\frac{t}{\theta}+1}} \Gamma\left(\frac{t}{\theta}+1\right)$	Al-Kadim and Boshi (2013)
Weibull-Rayleigh	$-,1/p,2,1$	$\omega_i \frac{p^{t/2}}{m^{\frac{t}{2\alpha}+1}} \Gamma\left(\frac{t}{2\alpha}+1\right)$	Akarawak et al. (2017)
exponential Rayleigh	$1,-,1,\quad -$	$\omega_i \frac{k}{\lambda^{\frac{1}{\theta}} m^{\frac{1}{\alpha\theta}+1}} \Gamma\left(\frac{1}{\alpha\theta}+1\right)$	-
Rayleigh Rayleigh	$2,\frac{1}{8^{1/2}\sigma},2,\beta$	$\omega_i \frac{8^{t/4}\sigma^{t/2}\beta^t}{m^{\frac{t}{4}+1}} \Gamma\left(\frac{t}{4}+1\right)$	Ateeq et al. (2019)
Weibull Rayleigh	$-,\frac{1}{2\gamma},2,-$	$\omega_i \frac{k}{\lambda^{\frac{1}{\theta}} m^{\frac{1}{\alpha\theta}+1}} \Gamma\left(\frac{1}{\alpha\theta}+1\right)$	Ahmad et al. (2017)
Weibull	$1,1,-,\quad -$	$\omega_i \frac{k}{m^{\frac{1}{\theta}+1}} \Gamma\left(\frac{1}{\theta}+1\right)$	Weibull (1951)

6 Application to Nigeria Covid-19 data

6.1 Nigeria Covid-19 Data Analysis

On the 16th of March 2022, 91 new confirmed cases were recorded in Nigeria as reported by https://covid19.ncdc.gov.ng/
The total confirmed active cases as at that date reported on Friday 3:49 pm 18 Mar 2022 is 2493 representing the sum of the datasets:
(197,125,41,8,1,77,11,88,315,3,27, 344,83,37,13,46,335,22,2,340 ,28,51,8,5,62,5,68,130,9,2,10)
The data is applied to WEP distribution and compared with Sindhu et al (2021) Exponentiated transformation of Gumbel type-II (ETGTT) distribution for modeling covid-19 data. The goodness-of-fit estimates of parameters and model selection statistics with the p-values obtained from analysis using the R-software are presented in Table 2.

Table 2: MLEs and Goodness-of-fit statistics for the Nigeria Covid-19 active cases.

Models	$\boldsymbol{\alpha}$	$\boldsymbol{\lambda}$	$\boldsymbol{\theta}$	k	LL	AIC	K-S	PV
WEP	0.7553	0.0794	0.9362	4.0135	163.1919	334.3837	0.0998	0.9170
ETGTT	0.7058	–	0.6849	4.6307	164.8987	335.7975	0.1132	0.8211

Table 2 shows the p-value is close to one, which is an indication of the suitability of the distribution for fitting the data. Figure 2 is the plots for the fitted model which supports the result in Table 2 that the WEP model can be used to fit the data.

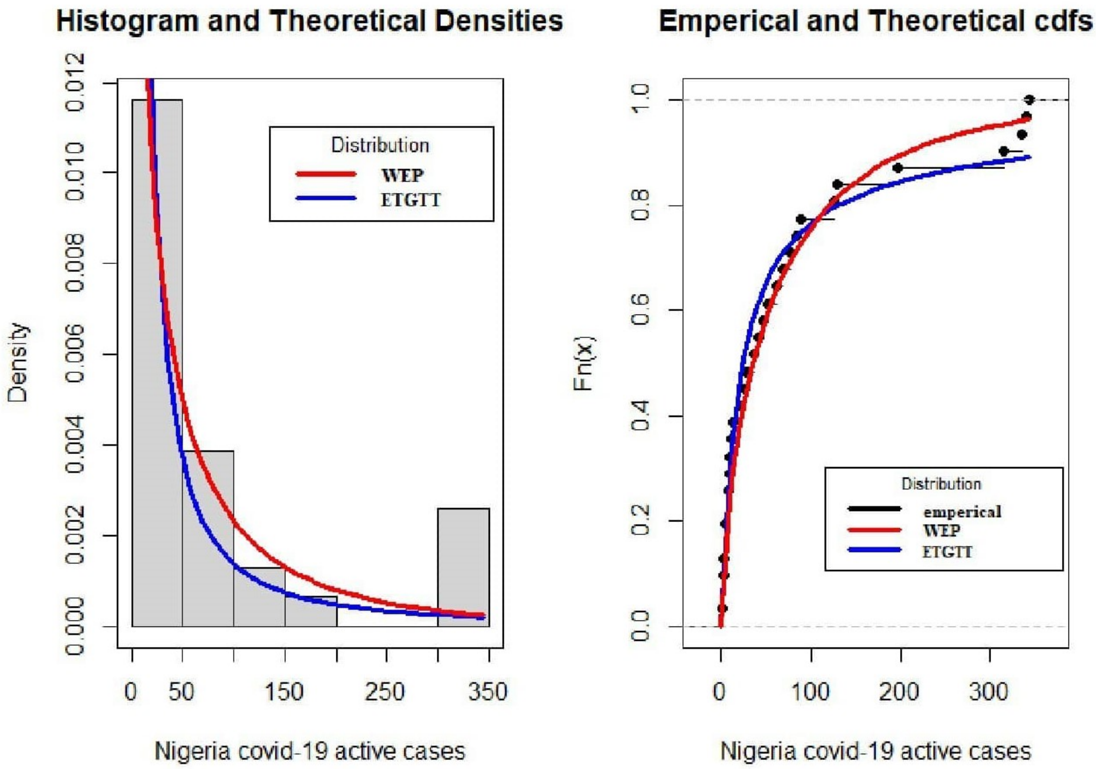

Figure 2: The pdf and cdf of WEP Distribution Fitted to the Nigeria Covid-19 Active Cases on Admission

6.2 Estimation of the Maximum and Minimum Patients on Admission from every n states

The estimated minimum and maximum number of active cases of patients on admission can be determined using the maximum likelihood estimates of the parameters of the WEP distribution in Table 2.
The Monte Carlo estimated values for $N = 10,000$ and sample sizes $n = 5, 10, 20, 30, 40$ for the maximum $X_{(n:n)}$ and the minimum $X_{(1:n)}$ active cases is computed with the corresponding variances and confidence interval (CI) of the estimates presented in Table 3.

The result shows that the computed values of the maximum with the corresponding variances increase as the sample sizes n increase. On the other hand, the predicted minimum and the corresponding variances decrease as the sample size n increases. The CI of predicted estimates shows a high coverage probability for the population at a confidence level of $Z - value$ of 95 percent.

Table 3: Estimated maximum $X_{(n:n)}$ and the minimum $X_{(1:n)}$ active cases of patients on admission by the number of states

expected maximum $\boldsymbol{\mu}_{(n:n)}$				expected minimum $\boldsymbol{\mu}_{(1:n)}$		
n	MLE	Variance	CI	MLE	Variance	CI
5	205	25609	$(137, 273)$	8	123	$(3, 13)$
10	286	33284	$(231, 340)$	3	17	$(2, 4)$
20	378	38687	$(336, 420)$	1	3	$(0.6, 1.3)$
30	437	41855	$(402, 473)$	1	0.7	$(0.8, 1.1)$
40	481	43510	$(450, 513)$	0	0.3	$(-0.1, 0.1)$

7 Numerical computations of the mean order statistics for arbitrary parameters

Computations for mean of order statistics from WEP distribution for some arbitrary values of parameters are tabulated and presented in Table 4.

Table 4: Mean of order statistics of WEP distribution for some parameters

r	n	$\alpha=\lambda=2$ $\theta=5, k=0.5$	$\alpha=\lambda=2$ $\theta=5, k=1$	$\alpha=\lambda=2$ $\theta=5, k=3.5$	$\alpha=3, \lambda=2$ $\theta=5, k=3.5$	$\alpha=3, \lambda=2$ $\theta=7, k=3.5$
1	1	0.4141	0.8282	2.8987	2.9423	3.0897
1	2	0.3864	0.7727	2.7046	2.8094	2.9894
2	2	0.4418	0.8837	3.0928	3.0752	3.1900
1	3	0.3710	0.7420	2.5971	2.7345	2.9322
2	3	0.4171	0.8341	2.9195	2.9593	3.1037
3	3	0.4542	0.9084	3.1795	3.1331	3.2332
1	4	0.3605	0.7209	2.5235	2.6826	2.8923
2	4	0.4026	0.8052	2.8181	2.8903	3.0519
3	4	0.4316	0.8631	3.0209	3.0282	3.1555
4	4	0.4618	0.9235	3.2323	3.1681	3.2591

Note : David and Nagaraja (1981) established an important property of order statistics defined by $\sum_{i=1}^{n} \mu_{i:n} = n\mu_{1:1}$

This property was also corroborated by Kumar et al. (2018), the computed results displayed in Table 4 are investigated to be consistent with the property.

Analysis of impact of various parameters and the samples size on the mean of order statistics from WEP distribution revealed the following.

- The mean increases with the order statistics for any sample of size n
- The mean increases with increase in parameter k for all order statistics
- The mean increases with increase in parameter α except for $X_{(n:n)}$
- The mean increases with increase in parameter θ for all order statistics
- The mean decreases with increase in the sample size n

7.1 Simulation results for some properties of order statistics of the WEP distribution

The Monte Carlo simulations with N$= 20,000$ replications for small and large sample sizes $(4, 20, 100, 300, 500)$ were used to estimate values for statistical properties of the maximum and minimum order statistics from the WEP distribution. Two sets of arbitrary parameters I=$(\alpha = 2, \lambda = 2, \theta = 1, k = 0.5)$ and II=$(\alpha = 2, \lambda = 2, \theta = 5, k = 0.5)$ were used for the simulation.
Computations for mean, variance, skewness, and kurtosis of order statistics are tabulated and presented in Table 5.

Table 5: Simulation results for statistical properties of order statistics for some parameter values and various sample sizes

	Maximum($X_{(n:n)}$)				Minimum($X_{(1:n)}$)				
n	mean	variance	skewness	kurtosis	mean	variance	skewness	kurtosis	Set
4	0.3456	0.0098	0.4805	0.3816	0.1108	0.0034	0.6294	0.2508	
20	0.4672	0.0063	0.5848	0.4832	0.0496	0.0007	0.6390	0.2497	
100	0.5652	0.0045	0.7139	0.9556	0.0222	0.0001	0.6275	0.2616	I
300	0.6245	0.0038	0.7565	1.0284	0.0128	0.0000	0.6438	0.2195	
500	0.6496	0.0035	0.7715	0.9271	0.0099	0.0000	0.6413	0.1942	
4	0.4613	0.0007	-0.2114	0.1998	0.3605	0.0019	-0.6200	0.5098	
20	0.4921	0.0003	0.1659	-0.0127	0.3069	0.0014	-0.6379	0.5898	
100	0.5120	0.0002	0.4047	0.1981	0.2612	0.0010	-0.6338	0.6169	II
300	0.5223	0.0001	0.5031	0.4139	0.2340	0.0008	-0.6118	0.5793	
500	0.5265	0.0000	0.5277	0.3857	0.2224	0.0007	-0.6382	0.5070	

Remark 5.4:
The parameters set II=$(\alpha = 2, \lambda = 2, \theta = 5, k = 0.5)$ was used in Table 4 and Table 5. It can be seen that the mean of maximum order statistics $\mu_{(4:4)}$ for $X_{(4:4)} = 0.4613$ in

Table 5 and $\mu_{(4:4)}$ for $X_{(4:4)} = 0.4618$ in Table 4.
The mean of minimum order statistics $X_{(1:4)} = 0.3605$ in both Table 5 and Table 4. The Monte Carlo procedure established and strengthened the previous result obtained in Table 4 for the sample size $n = 4$.
The mean increases with the sample size in $X_{(n:n)}$ and decreases with the sample size in $X_{(1:n)}$
The distributions of $X_{(1:n)}$ and $X_{(n:n)}$ exhibits both positive and negative skewness; and their variances consistently decreases as the sample size increases

8 Discussion and Conclusion

Order statistics of many convoluted distributions remain open for some research. This article defined and studied some distributional properties of order statistics from the Weibull-Exponential Pareto distribution. The research provides a new dimension to the study of convoluted distributions based on the useful properties of order statistics from their random samples. The beta-generator was explored for deriving the distributions of extreme order statistics and the results strengthened existing concepts and theorem in the literature. Moment of order statistics from WEP was derived and the results generalizes the moments of Weibull and some recently developed lifetime distributions in literature. The mean and variance of the minimum order statistics as a special case were derived. The study established a recurrence relation for the moment of order statistics from WEP distribution which can be used as a veritable mechanism for investigating several statistical measures for lifetime distributions including Weibull Rayleigh, Exponential Rayleigh, Rayleigh Rayleigh, and Exponential Pareto distributions. The distribution of the range statistics of a random sample of size n for some class of probability distributions was derived. New concepts and theorems introduced represent new knowledge and important frameworks for more research in this area. The study was applied to Nigeria's Covid-19 active cases of patients on admission and the WEP distribution has the capacity to model the datasets. The MLEs of the parameters were used to estimate the mean values for the maximum and minimum order statistics. From the predicted values, we can conclude that the minimum number of expected patients on admission is 0 and the maximum number of expected patients on admission is 481 for $n = 40$. The tabulated results from the Monte Carlo simulation show consistency for the properties of order statistics and agree with the computation for the mean of order statistics in Table 4.

Acknowledgement

The authors would like to express thanks to the Editors, and the suggestions of anonymous reviewers which have improved the final manuscript is highly appreciated.

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