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Weighted Gharaibeh distribution with real data applications

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In this paper, a new continuous one parameter distribution called weighted Gharaibeh distribution (WGD) is introduced. Its moment generating function, moments and related measures, order statistics, hazard rate function, mean residual life function, Renyi entropy, mean deviations about mean and median, Stochastic ordering, Bonferroni and Lorenz curves are investigated and addressed. The parameter of the WGD is estimated using the maximum likelihood method. A simulation study is conducted to investigate the performance of maximum likelihood estimate. Applications to two real data sets are presented and showed that the WGD has the adequacy of fitting such data better than Gharaibeh distribution and some other existing one parameter distributions.

keywords: Gharaibeh distribution, Reliability, Moments, Mean deviation, Entropy, Bonferroni and Lorenz curves.

1 Introduction

The idea of weighted distributions is suggested by Fisher (1934). It is adapted by many authors to introduce new flexible distributions for modeling real data in various fields. If the random variable X has probability density function (*pdf*) $g(x;\alpha)$, with unknown parameter α, then the weighted distribution of X is defined as

$$f(x) = \frac{\psi(x)g(x;\alpha)}{E(\psi(x))} \tag{1}$$

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where $\psi(x)$ is a non-negative weight function such that $E[\psi(x)]$ exists. If the weight function is of the form $\psi(x) = x^{\tau}$, the resulted distribution is called size-biased distribution of order τ. If $\tau = 1$, then the obtained weighted distribution is known as the length biased distribution.

Many weighted distributions are proposed in the literature such as: Weighted exponential distribution (Dey et al., 2015), length and area-biased Maxwell distributions (Sharma et al., 2018), size-biased Ishita distribution (Al-Omari et al., 2019), weighted Suja distribution (Alsmairan and Al-Omari, 2020), length-baised Suja distribution (Al-Omari and Alsmairan, 2019), A two-parameter weighted Lindley distribution (Ghitany et al., 2011), weighted power Shanker distribution (Ganaie et al., 2021), sized-biased Mukherjee-Islam distribution (Siddiqui et al., 2016), among others.

Others extended and generalized the existing distributions by adding extra parameters to suggest more flexible distributions such as: transmuted Aradhana distribution (Gharaibeh, 2020), new three-parameter Lindley distribution (El-Monsef, 2016), odd exponentiated half-logistic exponential distribution (Afify et al., 2018), Topp-Leone Mukherjee-Islam distribution (Al-Omari and Gharaibeh, 2018), transmuted Ishita distribution (Gharaibeh and Al-Omari, 2019), Alzoubi distribution (Benrabia and Alzoubi, 2022a), Benrabia distribution (Benrabia and Alzoubi, 2022b), Marshall-Olkin extended inverted Kumaraswamy distribution (Usman and Ahsan ul Haq, 2020), transmuted Shanker distribution (Alzoubi et al., 2022), transmuted Mukherjee-Islam distribution (Al-zou′bi, 2017), transmuted Janardan distribution (Al-Omari et al., 2017), quasi xgamma-geometric distribution (Sen et al., 2019), generalized Ramos-Louzada distribution (Al-Mofleh et al., 2020), among others.

Even though these extended distributions provide better goodness of fit for some real data sets, this approach of increasing the number of parameters to the base models has the drawback of complicated parameters estimation. Therefore, we adapted the idea of weighted distributions which has the privilege of increasing the flexibility of the base models without increasing the number of parameters.

Gharaibeh distribution is a one parameter lifetime distribution introduced by Gharaibeh (2021). Applications to three real data sets are presented and showed that Gharaibeh distribution has the superiority of fitting and modeling such real data than other existing distributions. Its probability density function (*pdf*) is defined as:

$$g(x;\alpha) = \frac{\alpha^6}{120(\alpha^6+\alpha^4+\alpha^2+1)}(x^5+20x^3+120x+120\alpha)e^{-\alpha x}; \qquad x>0; \alpha>0. \quad (2)$$

The k^{th} moment and mean of the Gharaibeh distribution random variable are, respectively, given by

$$E(X^k) = \frac{\alpha^6 k! + \alpha^4(k+1)! + \frac{\alpha^2(k+3)!}{6} + \frac{(k+5)!}{120}}{\alpha^k(\alpha^6+\alpha^4+\alpha^2+1)}; \qquad k = 1,2,3,\cdots \quad (3)$$

and

$$E(X) = \frac{\alpha^6+2\alpha^4+4\alpha^2+6}{\alpha(\alpha^6+\alpha^4+\alpha^2+1)}. \quad (4)$$

In this paper, we employed the idea of weighted distributions to modify Gharaibeh distribution and suggest a new one parameter model called weighted Gharaibeh distribution (WGD). Many properties of the suggested WGD are studied and investigated. Also, applications to two real data sets are presented and showed that WGD outperforms Gharaibeh distribution and some other competitive distributions in modelling such data.

The outline of this paper is as follows. The *pdf* and *cdf* of the proposed distribution (WGD) are introduced in Section 2. Reliability analysis including survival, hazard rate, cumulative hazard, reversed hazard, odds, and mean residual life functions are investigated in Section 3. The distributions of order statistics are given in Section 4. Moments and related measures are addressed in Section 5. The Renyi entropy is derived in Section 6. The Bonferroni and Lorenz curves are given in Section 7. Stochastic ordering and mean deviations are studied in Section 8. The maximum likelihood estimate of the WGD parameter is provided in Section 9. A simulation study to evaluate the behavior of the maximum likelihood estimate is discussed in Section 10. Applications to two real data sets are presented in Section 11. The paper is concluded in Section 12.

2 Weighted Gharaibeh Distribution (WGD)

Based on (1) with $\psi(x) = x$ and using (2) and (4), a random variable X is said to have a weighted Gharaibeh distribution (WGD) if its *pdf* is defined as

$$f(x) \quad = \quad \frac{\alpha^7(x^6+20x^4+120x^2+120\alpha x)e^{-\alpha x}}{120(\alpha^6+2\alpha^4+4\alpha^2+6)}; x>0, \alpha>0. \tag{5}$$

with a corresponding *cdf* given by

$$F(x) = 1 - \left(\frac{\begin{bmatrix} \alpha^6x^6 + 6\alpha^5x^5 + (20\alpha^2+30)\alpha^4x^4 \\ + (80\alpha^2+120)\alpha^3x^3 + 120(\alpha^4+2\alpha^2+3)\alpha^2x^2 \end{bmatrix}}{120(\alpha^6+2\alpha^4+4\alpha^2+6)} + \alpha x + 1 \right) e^{-\alpha x}. \tag{6}$$

Figure 1 shows the *pdf* and *cdf* of the WGD with different values of the distribution parameter. It can be seen that WGD is skewed to the right. Also, the cdf of WGD is an increasing function of the parameter α.

3 Reliability Analysis

For the WGD with *pdf* $f(x)$ in (5) and *cdf* $F(x)$ in (6), the survival function, $S(x)$, hazard function, $h(x)$, cumulative hazard function, $H(x)$, reversed hazard rate function, $rh(x)$, odds function, $O(x)$, and mean residual life function, $MRL(x)$, are , respectively, defined

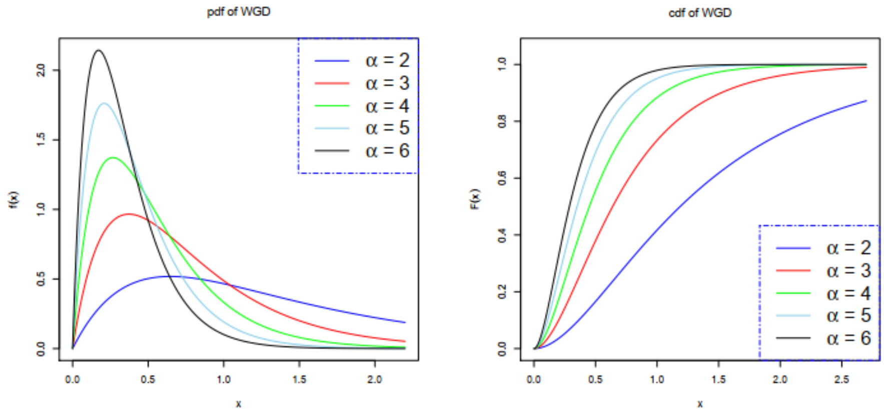

Figure 1: The *pdf* and *cdf* of WGD with different values of the paremeter α

as

$$
\begin{aligned}
S(x) &= 1 - F(x) = \frac{e^{-\alpha x}\begin{bmatrix}\alpha^6x^6+6\alpha^5x^5+(20\alpha^2+30)\alpha^4x^4\\ +(80\alpha^2+120)\alpha^3x^3+120(\alpha^4+2\alpha^2+3)\alpha^2x^2\end{bmatrix}}{120(\alpha^6+2\alpha^4+4\alpha^2+6)} + \alpha x e^{-\alpha x} + e^{-\alpha x} \\
h(x) &= \frac{f(x)}{1-F(x)} = \frac{\alpha^7(x^6+20x^4+120x^2+120\alpha x)}{\begin{bmatrix}\alpha^6x^6+6\alpha^5x^5+(20\alpha^2+30)\alpha^4x^4+(80\alpha^2+120)\alpha^3x^3\\ +120(\alpha^4+2\alpha^2+3)\alpha^2x^2+120(\alpha^6+2\alpha^4+4\alpha^2+6)(\alpha x+1)\end{bmatrix}} \\
H(x) &= -ln\,(1-F(x)) \\
&= \alpha x - ln\left(\frac{\begin{bmatrix}\alpha^6x^6+6\alpha^5x^5+(20\alpha^2+30)\alpha^4x^4\\ +(80\alpha^2+120)\alpha^3x^3+120(\alpha^4+2\alpha^2+3)\alpha^2x^2\end{bmatrix}}{120(\alpha^6+2\alpha^4+4\alpha^2+6)} + \alpha x + 1\right) \\
rh(x) &= \frac{f(x)}{F(x)} \\
&= \frac{\alpha^7(x^6+20x^4+120x^2+120\alpha x)}{\begin{bmatrix}120(\alpha^6+2\alpha^4+4\alpha^2+6)(e^{\alpha x}-\alpha x-1)-\alpha^6x^6-6\alpha^5x^5\\ -(20\alpha^2+30)\alpha^4x^4-(80\alpha^2+120)\alpha^3x^3-120(\alpha^4-2\alpha^2+3)\alpha^2x^2\end{bmatrix}} \\
O(x) &= \frac{F(x)}{1-F(x)} = \frac{120(\alpha^6+2\alpha^4+4\alpha^2+6)e^{\alpha x}}{\begin{bmatrix}\alpha^6x^6+6\alpha^5x^5+(20\alpha^2+30)\alpha^4x^4+(80\alpha^2+120)\alpha^3x^3\\ +120(\alpha^4+2\alpha^2+3)\alpha^2x^2+120(\alpha^6+2\alpha^4+4\alpha^2+6)(\alpha x+1)\end{bmatrix}} - 1
\end{aligned}
$$

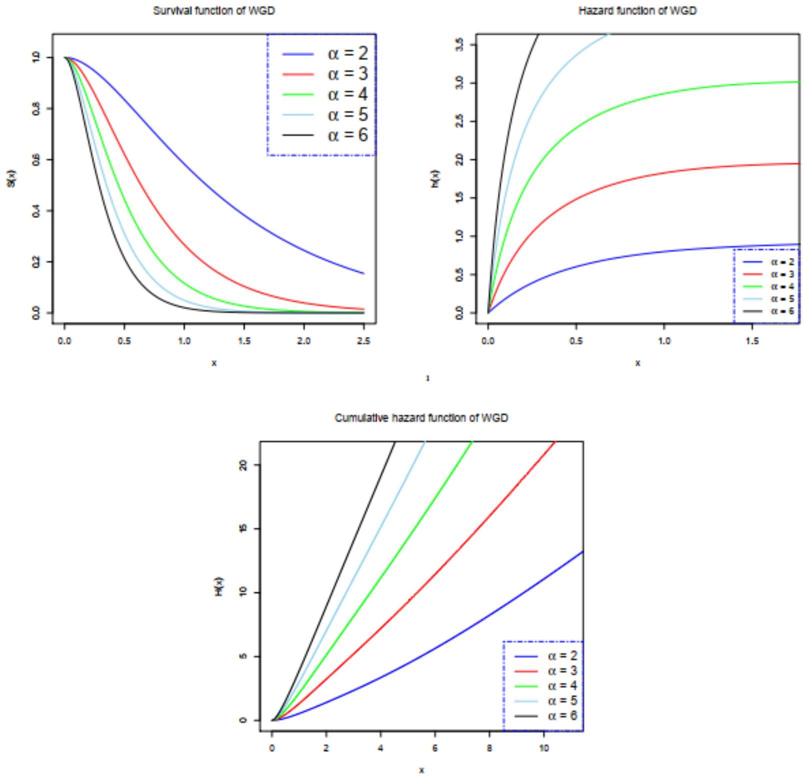

Figure 2: The survival, hazard, and cumulative hazard functions of WGD with different values of the parameter α

and

$$
\begin{aligned}
MRL(x) &= E(X-x|X>x) = \frac{1}{1-F(x)}\int_x^\infty (1-F(t))\,dt \\
&= \frac{\left[\begin{array}{l} \alpha^6x^6 + 12\alpha^5x^5 + \left(20\alpha^6+90\alpha^4\right)x^4 + \left(160\alpha^5+480\alpha^3\right)x^3 \\ + \left(120\alpha^6+720\alpha^4+1800\alpha^2\right)x^2 + \left(120\alpha^7+480\alpha^5+1920\alpha^3+4320\alpha\right)x \\ + 240\alpha^6+720\alpha^4+2400\alpha^2+5040 \end{array}\right]}{\alpha\left[\begin{array}{l} \alpha^6x^6+6\alpha^5x^5+(20\alpha^2+30)\alpha^4x^4+(80\alpha^2+120)\alpha^3x^3 \\ +120(\alpha^4+2\alpha^2+3)\alpha^2x^2+120(\alpha^6+2\alpha^4+4\alpha^2+6)(\alpha x+1) \end{array}\right]}
\end{aligned}
$$

The graphs of these functions for some values of the WGD parameter are shown in Figures 2. It is obvious that as α increases, the values of hazard, cumulative hazard, and odds functions increase while the survival, reversed hazard, and mean residual life functions values decrease. Note that $MRL(0) = \frac{2\alpha^6+6\alpha^4+20\alpha^2+42}{\alpha(\alpha^6+2\alpha^4+4\alpha^2+6)} = E(X) = \mu_1$ (given in (12)).

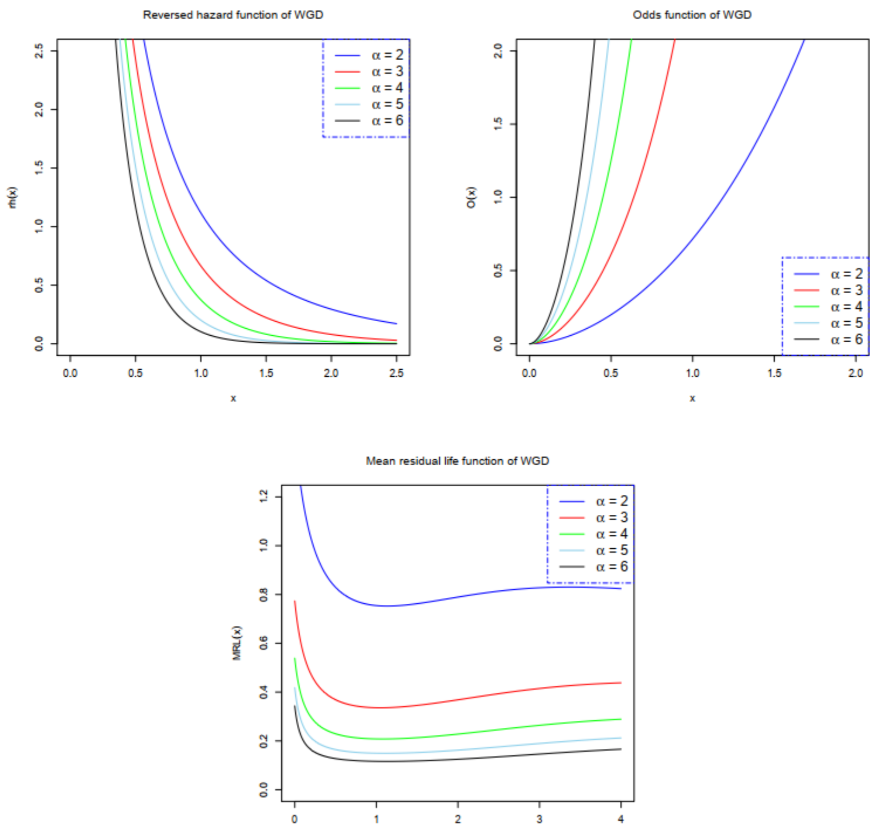

Figure 3: The reversed hazard, odds, and mean residual life functions of WGD with different values of the parameter α

4 Order Statistics

Let $X_{(1)}$, $X_{(2)}$, ... $X_{(n)}$ be the order statistics of the random sample X_1, X_2, ... X_n selected from WGD. Then the *pdf* of the i^{th} order statistics, $X_{(i)}$, (see David and Nagaraja (2005)) is defined as

$$f_{(i)}(x) = i\binom{n}{i} f(x)[F(x)]^{i-1}[1-F(x)]^{n-i} \tag{7}$$

By using binomial series, we have

$$[F(x)]^{i-1} = [1-(1-F(x))]^{i-1} = \sum_{j=0}^{i-1}\binom{i-1}{j}(-1)^j(1-F(x))^j \tag{8}$$

By plugging (5),(6) and (8)in (7), we have

$$\begin{aligned} f_{(i)}(x) &= \frac{i\binom{n}{i}\alpha^7(x^6+20x^4+120x^2+120\alpha x)}{120(\alpha^6+2\alpha^4+4\alpha^2+6)}\sum_{j=0}^{i-1}\binom{i-1}{j}(-1)^j e^{-\alpha x(n+j-i+1)} \\ &\times\left(\frac{\begin{bmatrix}\alpha^6x^6+6\alpha^5x^5+(20\alpha^2+30)\alpha^4x^4 \\ +(80\alpha^2+120)\alpha^3x^3+120(\alpha^4+2\alpha^2+3)\alpha^2x^2\end{bmatrix}}{120(\alpha^6+2\alpha^4+4\alpha^2+6)}+\alpha x+1\right)^{n+j-i} \end{aligned} \tag{9}$$

Hence, the *pdf*s of the smallest order statistic, $X_{(1)}$, and the largest order statistic, $X_{(n)}$, are, respectively, given by

$$\begin{aligned} f_{(1)}(x) &= \frac{n\alpha^7(x^6+20x^4+120x^2+120\alpha x)}{120(\alpha^6+2\alpha^4+4\alpha^2+6)}e^{-n\alpha x} \\ &\times\left(\frac{\begin{bmatrix}\alpha^6x^6+6\alpha^5x^5+(20\alpha^2+30)\alpha^4x^4 \\ +(80\alpha^2+120)\alpha^3x^3+120(\alpha^4+2\alpha^2+3)\alpha^2x^2\end{bmatrix}}{120(\alpha^6+2\alpha^4+4\alpha^2+6)}+\alpha x+1\right)^{n-1} \end{aligned}$$

and

$$\begin{aligned} f_{(n)}(x) &= \frac{n\alpha^7(x^6+20x^4+120x^2+120\alpha x)}{120(\alpha^6+2\alpha^4+4\alpha^2+6)}\sum_{j=0}^{n-1}\binom{n-1}{j}(-1)^j e^{-\alpha x(j+1)} \\ &\times\left(\frac{\begin{bmatrix}\alpha^6x^6+6\alpha^5x^5+(20\alpha^2+30)\alpha^4x^4 \\ +(80\alpha^2+120)\alpha^3x^3+120(\alpha^4+2\alpha^2+3)\alpha^2x^2\end{bmatrix}}{120(\alpha^6+2\alpha^4+4\alpha^2+6)}+\alpha x+1\right)^{j} \end{aligned}$$

5 Moments and Related Measures of the WGD

5.1 Moment Generating Function

Theorem 1 *For the WGD random variable, the moment generating function is given by*

$$M_X(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!\alpha^r(\alpha^6+2\alpha^4+4\alpha^2+6)}\left[\alpha^6(r+1)!+\alpha^4(r+2)!+\frac{\alpha^2(r+4)!}{6}+\frac{(r+6)!}{120}\right] \tag{10}$$

Using the *pdf* $f(x)$ in (5) and the binomial series, the moment generating function of the WGD can be proved as

$$\begin{aligned}
M_X(t) &= E(e^{tX}) = \int_0^{\infty} e^{tx} f(x)dx \\
&= \frac{\alpha^7(x^6+20x^4+120x^2+120\alpha x)}{120(\alpha^6+2\alpha^4+4\alpha^2+6)} e^{-x(\alpha-t)}dx \\
&= \frac{\alpha^7}{120(\alpha^6+2\alpha^4+4\alpha^2+6)}\left[\frac{720}{(\alpha-t)^7}+\frac{480}{(\alpha-t)^5}+\frac{240}{(\alpha-t)^3}+\frac{120\alpha}{(\alpha-t)^2}\right] \\
&= \frac{1}{(\alpha^6+2\alpha^4+4\alpha^2+6)}\left[\frac{6}{(1-\frac{t}{\alpha})^7}+\frac{4\alpha^2}{(1-\frac{t}{\alpha})^5}+\frac{2\alpha^4}{(1-\frac{t}{\alpha})^3}+\frac{\alpha^6}{(1-\frac{t}{\alpha})^2}\right] \\
&= \frac{1}{(\alpha^6+2\alpha^4+4\alpha^2+6)}\left[\alpha^6\sum_{r=0}^{\infty}\binom{r+1}{r}(t/\alpha)^r+2\alpha^4\sum_{r=0}^{\infty}\binom{r+2}{r}(t/\alpha)^r\right. \\
&\qquad \left.+4\alpha^2\sum_{r=0}^{\infty}\binom{r+4}{r}(t/\alpha)^r+6\sum_{r=0}^{\infty}\binom{r+6}{r}(t/\alpha)^r\right] \\
&= \sum_{r=0}^{\infty}\frac{t^r}{r!\alpha^r(\alpha^6+2\alpha^4+4\alpha^2+6)}\left[\alpha^6(r+1)!+\alpha^4(r+2)!+\frac{\alpha^2(r+4)!}{6}+\frac{(r+6)!}{120}\right].
\end{aligned}$$

5.2 Moments and Related Measures

For the WGD random variable, the r^{th} moment about the origin (μ_r) is given as the coefficient of $\frac{t^r}{r!}$ in the moment generating function $M_X(t)$ in (10), which is given by

$$\mu_r = E(X^r) = \frac{\alpha^6(r+1)!+\alpha^4(r+2)!+\alpha^2\frac{(r+4)!}{6}+\frac{(r+6)!}{120}}{\alpha^r\left(\alpha^6+2\alpha^4+4\alpha^2+6\right)}; \qquad r=1,2,3,... \tag{11}$$

Thus, the first four moments about the origin of the WGD are

$$\mu_1 = \frac{2\left(\alpha^6+3\alpha^4+10\alpha^2+21\right)}{\alpha\left(\alpha^6+2\alpha^4+4\alpha^2+6\right)}, \tag{12}$$

$$\mu_2 = \frac{6\left(\alpha^6+4\alpha^4+20\alpha^2+56\right)}{\alpha^2\left(\alpha^6+2\alpha^4+4\alpha^2+6\right)}, \tag{13}$$

$$\mu_3 = \frac{24\left(\alpha^6+5\alpha^4+35\alpha^2+126\right)}{\alpha^3\left(\alpha^6+2\alpha^4+4\alpha^2+6\right)}, \tag{14}$$

$$\mu_4 = \frac{120\left(\alpha^6+6\alpha^4+56\alpha^2+252\right)}{\alpha^4\left(\alpha^6+2\alpha^4+4\alpha^2+6\right)}. \tag{15}$$

Using these moments, the variance (σ^2), coefficient of variation (CV), skewness (Sk), kurtosis (Ku) and index of dispersion (Dis) of the WGD random variable are defined, respectively, as:

$$\sigma^2 = \mu_2-\mu_1^2 = \frac{2\left(\alpha^{12}+6\alpha^{10}+38\alpha^8+150\alpha^6+196\alpha^4+192\alpha^2+126\right)}{\alpha^2\left(\alpha^6+2\alpha^4+4\alpha^2+6\right)^2},$$

Table 1: The mean, variance, coefficient of variation, skewness, kurtosis and index of dispersion for WGD with different values of the parameter α.

α	μ_1	σ^2	***CV***	***Sk***	***Ku***	***Dis***
0.25	27.660	113.415	0.385	0.741	3.836	4.100
0.50	13.278	29.614	0.410	0.692	3.762	2.230
0.75	8.168	14.157	0.461	0.638	3.622	1.733
1.00	5.385	8.391	0.538	0.685	3.535	1.558
1.25	3.648	5.174	0.623	0.867	3.760	1.418
1.50	2.559	3.139	0.692	1.110	4.378	1.227
1.75	1.886	1.911	0.733	1.329	5.207	1.013
1.766	1.853	1.853	0.734	1.341	5.261	1.000
2.00	1.466	1.206	0.749	1.484	5.987	0.823
2.25	1.193	0.803	0.751	1.568	6.546	0.673
2.50	1.007	0.565	0.746	1.599	6.848	0.561
2.75	0.874	0.418	0.740	1.598	6.947	0.478
3.00	0.774	0.322	0.734	1.581	6.921	0.416
3.25	0.696	0.257	0.728	1.558	6.832	0.369
3.50	0.634	0.210	0.724	1.535	6.723	0.332
3.75	0.582	0.176	0.720	1.515	6.614	0.302
4.00	0.539	0.150	0.718	1.497	6.516	0.278
4.25	0.503	0.129	0.716	1.483	6.431	0.257
4.50	0.471	0.113	0.714	1.471	6.359	0.240
4.75	0.443	0.100	0.713	1.461	6.300	0.225
5.00	0.419	0.089	0.712	1.453	6.252	0.212

$$
\begin{aligned}
CV &= \frac{\sigma}{\mu_1} = \frac{\left(\alpha^{12}+6\alpha^{10}+38\alpha^{8}+150\alpha^{6}+196\alpha^{4}+192\alpha^{2}+126\right)^{1/2}}{\sqrt{2}\left(\alpha^{6}+3\alpha^{4}+10\alpha^{2}+21\right)},\\
Sk(X) &= \frac{\mu_3-3\mu_1\mu_2+2\mu_1^3}{\sigma^3}\\
&= \frac{\sqrt{2}\begin{pmatrix}\alpha^{18}+9\alpha^{16}+90\alpha^{14}+549\alpha^{12}+1272\alpha^{10}\\ +1752\alpha^{8}+1582\alpha^{6}+1620\alpha^{4}+1512\alpha^{2}+756\end{pmatrix}}{\left(\alpha^{12}+6\alpha^{10}+38\alpha^{8}+150\alpha^{6}+196\alpha^{4}+192\alpha^{2}+126\right)^{\frac{3}{2}}},\\
Ku(X) &= \frac{\mu_4-4\mu_1\mu_3+6\mu_1^2\mu_2-3\mu_1^4}{\sigma^4}\\
&= \frac{6\begin{bmatrix}\alpha^{24}+12\alpha^{22}+142\alpha^{20}+1114\alpha^{18}+4246\alpha^{16}+11244\alpha^{14}+23246\alpha^{12}\\ +39748\alpha^{10}+57748\alpha^{8}+65844\alpha^{6}+52344\alpha^{4}+30240\alpha^{2}+10206\end{bmatrix}}{\left(\alpha^{12}+6\alpha^{10}+38\alpha^{8}+150\alpha^{6}+196\alpha^{4}+192\alpha^{2}+126\right)^{2}},\\
Dis &= \frac{\sigma^2}{\mu_1} = \frac{\left(\alpha^{12}+6\alpha^{10}+38\alpha^{8}+150\alpha^{6}+196\alpha^{4}+192\alpha^{2}+126\right)}{\alpha\left(\alpha^{6}+2\alpha^{4}+4\alpha^{2}+6\right)\left(\alpha^{6}+3\alpha^{4}+10\alpha^{2}+21\right)}.
\end{aligned}
$$

Table 1 provides numerical values of the mean, variance, coefficient of variation, skewness, kurtosis and index of dispersion for the WGD with different values of the parameter α. It can be seen that the values of the mean, variance and index of dispersion decrease as the value of α increases. The positive values of skewness emphasize that the WGD is right skewed as shown in Figure 5. Moreover, the values of *Dis* in Table 1 indicate that the WGD is over-dispersed when $\alpha < 1.766$ and under-dispersed when $\alpha > 1.766$.

6 Renyi Entropy

The Renyi entropy is defined as

$$RE(\tau) = \frac{1}{1-\tau} log \int_0^{\infty} (f(x))^{\tau} dx \, ; \tau > 0, \tau \neq 1. \tag{16}$$

Theorem 2 *The Renyi entropy of the WGD random variable X is given by*

$$RE(\tau) = \frac{1}{1-\tau} log \left[(\alpha^6 + 2\alpha^4 + 4\alpha^2 + 6)^{-\tau} \sum_{i=1}^{\tau} \sum_{j=1}^{i} \sum_{k=1}^{\tau-i} \binom{\tau}{i} \binom{i}{j} \binom{\tau-i}{k} 20^{-j} 6^{-i} \alpha^{8\tau-i-k} \frac{(\tau+3i+2j+k)!}{(\alpha\tau)^{\tau+3i+2j+k+1}} \right]$$

Using the *pdf* of the WGD in (5) and plug it in (16), we have

$$\begin{aligned} RE(\tau) &= \frac{1}{1-\tau} log \int_0^{\infty} \left[\frac{\alpha^7}{120(\alpha^6 + 2\alpha^4 + 4\alpha^2 + 6)} (x^6 + 20x^4 + 120x^2 + 120\alpha x) e^{-\alpha x} \right]^{\tau} dx \\ &= \frac{1}{1-\tau} log \left[\left(\frac{\alpha^7}{120(\alpha^6 + 2\alpha^4 + 4\alpha^2 + 6)} \right)^{\tau} \int_0^{\infty} (x^5 + 20x^3 + 120x + 120\alpha)^{\tau} x^{\tau} e^{-\alpha\tau x} \right] dx \end{aligned} \tag{17}$$

Using Binomial series, we have

$$\begin{aligned} (x^5 + 20x^3 + 120x + 120\alpha)^{\tau} &= (20x^3(\frac{x^2}{20} + 1) + 120\alpha(\frac{x}{\alpha} + 1))^{\tau} \\ &= \sum_{i=1}^{\tau} \binom{\tau}{i} (20x^3(\frac{x^2}{20} + 1))^i (120\alpha(\frac{x}{\alpha} + 1))^{\tau-i} \\ &= \sum_{i=1}^{\tau} \binom{\tau}{i} (20x^3)^i (120\alpha)^{\tau-i} \sum_{j=1}^{i} \binom{i}{j} (\frac{x^2}{20})^j \sum_{k=1}^{\tau-i} \binom{\tau-i}{k} (\frac{x}{\alpha})^k \\ &= \sum_{i=1}^{\tau} \sum_{j=1}^{i} \sum_{k=1}^{\tau-i} \binom{\tau}{i} \binom{i}{j} \binom{\tau-i}{k} 20^{i-j} 120^{\tau-i} \alpha^{\tau-i-k} x^{3i+2j+k} \end{aligned} \tag{18}$$

Plugging (18) in (17), we have

$$\begin{aligned} RE(\tau) &= \frac{1}{1-\tau} log \left[\left(\frac{\alpha^7}{120(\alpha^6 + 2\alpha^4 + 4\alpha^2 + 6)} \right)^{\tau} \sum_{i=1}^{\tau} \sum_{j=1}^{i} \sum_{k=1}^{\tau-i} \binom{\tau}{i} \binom{i}{j} \binom{\tau-i}{k} \right. \\ &\qquad \left. \times 20^{i-j} 120^{\tau-i} \alpha^{\tau-i-k} \int_0^{\infty} x^{\tau+3i+2j+k} e^{-\alpha\tau x} dx \right] \\ &= \frac{1}{1-\tau} log \left[(\alpha^6 + 2\alpha^4 + 4\alpha^2 + 6)^{-\tau} \sum_{i=1}^{\tau} \sum_{j=1}^{i} \sum_{k=1}^{\tau-i} \binom{\tau}{i} \binom{i}{j} \binom{\tau-i}{k} \right. \\ &\qquad \left. \times 20^{-j} 6^{-i} \alpha^{8\tau-i-k} \frac{(\tau+3i+2j+k)!}{(\alpha\tau)^{\tau+3i+2j+k+1}} \right]. \end{aligned}$$

7 Bonferroni and Lorenz Curves

Bonferroni and Lorenz curves are useful tools that have applications in economics, reliability, insurance, demography, etc. They are, respectively, defined as

$$B(p) = \frac{1}{p\mu_1}\int_0^q xf(x)dx, \qquad L(p) = \frac{1}{\mu_1}\int_0^q xf(x)dx, \tag{19}$$

where $q = F^{-1}(p); p \in (0,1]$ and $\mu_1 = E(X)$. Using the *pdf* of WGD in (5), we have

$$\int_0^q xf(x)dx = \mu_1 - \frac{\begin{bmatrix}\alpha^7q^7 + 7\alpha^6q^6 + \left(20\alpha^7 + 42\alpha^5\right)q^5 + \left(100\alpha^6 + 210\alpha^4\right)q^4 \\ + \left(120\alpha^7 + 400\alpha^5 + 840\alpha^3\right)q^3 + \left(120\alpha^8 + 360\alpha^6 + 1200\alpha^4 + 2520\alpha^2\right)q^2 \\ + \left(240\alpha^7 + 720\alpha^5 + 2400\alpha^3 + 5040\alpha\right)q + 240\alpha^6 + 720\alpha^4 + 2400\alpha^2 + 5040\end{bmatrix} \mathrm{e}^{-\alpha q}}{120\alpha\left(\alpha^6 + 2\alpha^4 + 4\alpha^2 + 6\right)}$$

Thus, Bonferroni and Lorenz curves for WGD are, respectively, given as

$$B(p) = \frac{1}{p} - \frac{\begin{bmatrix}\alpha^7q^7 + 7\alpha^6q^6 + \left(20\alpha^7 + 42\alpha^5\right)q^5 + \left(100\alpha^6 + 210\alpha^4\right)q^4 \\ + \left(120\alpha^7 + 400\alpha^5 + 840\alpha^3\right)q^3 + \left(120\alpha^8 + 360\alpha^6 + 1200\alpha^4 + 2520\alpha^2\right)q^2 \\ + \left(240\alpha^7 + 720\alpha^5 + 2400\alpha^3 + 5040\alpha\right)q + 240\alpha^6 + 720\alpha^4 + 2400\alpha^2 + 5040\end{bmatrix} \mathrm{e}^{-\alpha q}}{240p\left(\alpha^6 + 3\alpha^4 + 10\alpha^2 + 21\right)},$$

$$L(p) = 1 - \frac{\begin{bmatrix}\alpha^7q^7 + 7\alpha^6q^6 + \left(20\alpha^7 + 42\alpha^5\right)q^5 + \left(100\alpha^6 + 210\alpha^4\right)q^4 \\ + \left(120\alpha^7 + 400\alpha^5 + 840\alpha^3\right)q^3 + \left(120\alpha^8 + 360\alpha^6 + 1200\alpha^4 + 2520\alpha^2\right)q^2 \\ + \left(240\alpha^7 + 720\alpha^5 + 2400\alpha^3 + 5040\alpha\right)q + 240\alpha^6 + 720\alpha^4 + 2400\alpha^2 + 5040\end{bmatrix} \mathrm{e}^{-\alpha q}}{240\left(\alpha^6 + 3\alpha^4 + 10\alpha^2 + 21\right)}$$

8 Stochastic Ordering and Mean Deviations

8.1 Stochastic Ordering

Stochastic ordering can be used to compare two positive continuous distributions. A random variable X is smaller than a random variable Y in
1- Hazard rate order $(X \leq_{HR} Y)$ if $h_X(x) \geq h_Y(x)$ for all x.
2- Likelihood ratio order $(X \leq_{LR} Y)$ if $\frac{f_X(x)}{f_Y(x)}$ decreases in x.
3- Stochastic order $(X \leq_{ST} Y)$ if $F_X(x) \geq F_Y(x)$ for all x.
4- Mean residual life order $(X \leq_{MRL} Y)$ if $MRL_X(x) \leq MRL_Y(x)$ for all x.

Shaked and Shanthikumar (1994) showed that

$$\begin{array}{ccc} X \leq_{LR} Y \Rightarrow & X \leq_{HR} Y & \Rightarrow X \leq_{MRL} Y \\ & \Downarrow & \\ & X \leq_{ST} Y & \end{array}$$

Theorem 3 *Let $X \sim WGD(\alpha_1)$ and $Y \sim WGD(\alpha_2)$. If $\alpha_1 > \alpha_2$, then $X \leq_{LR} Y$ and thus $X \leq_{HR} Y, X \leq_{MRL} Y$ and $X \leq_{ST} Y$.*

Using the pdf of WGD in (5), we have

$$\frac{f_X(x;\alpha_1)}{f_Y(x;\alpha_2)} = \left[\frac{\alpha_1^7(\alpha_2^6+2\alpha_2^4+4\alpha_2^2+6)}{\alpha_2^7(\alpha_1^6+2\alpha_1^4+4\alpha_1^2+6)}\right]\left[\frac{x^5+20x^3+120x+120\alpha_1}{x^5+20x^3+120x+120\alpha_2}\right]e^{-x(\alpha_1-\alpha_2)}$$

Hence,

$$log\frac{f_X(x;\alpha_1)}{f_Y(x;\alpha_2)} = log\left[\frac{\alpha_1^7(\alpha_2^6+2\alpha_2^4+4\alpha_2^2+6)}{\alpha_2^7(\alpha_1^6+2\alpha_1^4+4\alpha_1^2+6)}\right]+log\left[\frac{x^5+20x^3+120x+120\alpha_1}{x^5+20x^3+120x+120\alpha_2}\right]-x(\alpha_1-\alpha_2)$$

and

$$\begin{aligned}\frac{\partial}{\partial x}\left[log\frac{f_X(x;\alpha_1)}{f_Y(x;\alpha_2)}\right] &= \frac{120(\alpha_2-\alpha 1)(5x^4+60x^2+120)}{(x^5+20x^3+120x+120\alpha_1)(x^5+20x^3+120x+120\alpha_2)}-(\alpha_1-\alpha_2)\\ &= (\alpha_2-\alpha_1)\left[\frac{120(5x^4+60x^2+120)}{(x^5+20x^3+120x+120\alpha_1)(x^5+20x^3+120x+120\alpha_2)}+1\right]\end{aligned}$$

Note that $\frac{\partial}{\partial x}\left[log\frac{f_X(x;\alpha_1)}{f_Y(x;\alpha_2)}\right] < 0$ when $\alpha_1 > \alpha_2$. Therefore, $X \leq_{LR} Y$ and thus $X \leq_{HR} Y$, $X \leq_{MRL} Y$ and $X \leq_{ST} Y$.

8.2 Mean Deviations about Mean and Median

Mean deviations about mean (μ_1) and median $(\tilde{d})$ are, respectively, defined as

$$MD(\mu_1) = \int_0^\infty |x-\mu_1|f(x)dx = 2\mu_1F(\mu_1)-2\int_0^{\mu_1} xf(x)dx, \tag{20}$$

$$MD(\tilde{d}) = \int_0^\infty |x-\tilde{d}|f(x)dx = \mu_1-2\int_0^{\tilde{d}} xf(x)dx. \tag{21}$$

Using (5),(6) and (20) in (20) and (21), the mean deviations about mean and median of WGD are, respectively, given as

$$MD(\mu_1) = \frac{e^{-\alpha\mu_1}}{60\alpha\left(\alpha^6+2\alpha^4+4\alpha^2+6\right)}\left[\begin{aligned}&\alpha^6\mu_1^6+12\alpha^5\mu_1^5+\left(20\alpha^6+90\alpha^4\right)\mu_1^4\\&+\left(160\alpha^5+480\alpha^3\right)\mu_1^3+\left(120\alpha^6+720\alpha^4+1800\alpha^2\right)\mu_1^2\\&+\left(120\alpha^7+480\alpha^5+1920\alpha^3+4320\alpha\right)\mu_1\\&+240\alpha^6+720\alpha^4+2400\alpha^2+5040\end{aligned}\right],$$

and

$$MD(\tilde{d}) = \frac{\left[\begin{aligned}&\alpha^7\tilde{d}^7+7\alpha^6\tilde{d}^6+\left(20\alpha^7+42\alpha^5\right)\tilde{d}^5+\left(100\alpha^6+210\alpha^4\right)\tilde{d}^4\\&+\left(120\alpha^7+400\alpha^5+840\alpha^3\right)\tilde{d}^3+\left(120\alpha^8+360\alpha^6+1200\alpha^4+2520\alpha^2\right)\tilde{d}^2\\&+\left(240\alpha^7+720\alpha^5+2400\alpha^3+5040\alpha\right)\tilde{d}+240\alpha^6+720\alpha^4+2400\alpha^2+5040\end{aligned}\right]e^{-\alpha\tilde{d}}}{60\alpha\left(\alpha^6+2\alpha^4+4\alpha^2+6\right)}-\mu_1.$$

9 Maximum Likelihood Estimate

Let $X_1, X_2, \ldots, X_n$ be a random sample from WGD with a *pdf* $f(x)$ in (5) and parameter α . The likelihood function is defined as

$$\begin{aligned} L(\alpha|x_1, x_2, \ldots, x_n) &= \prod_{i=1}^{n} \frac{\alpha^7}{120(\alpha^6 + 2\alpha^4 + 4\alpha^2 + 6)} (x_i^6 + 20x_i^4 + 120x_i^2 + 120\alpha x_i) e^{-\alpha x_i} \\ &= \left(\frac{\alpha^7}{120(\alpha^6 + 2\alpha^4 + 4\alpha^2 + 6)} \right)^n \prod_{i=1}^{n} (x_i^6 + 20x_i^4 + 120x_i^2 + 120\alpha x_i) e^{-\alpha \sum_{i=1}^{n} x_i}. \end{aligned}$$

Hence, the log-likelihood function is given by

$$\begin{aligned} L^* &= lnL(\alpha|x_1, x_2, \ldots, x_n) \\ &= 7n\, ln\alpha - n\, ln\left(120(\alpha^6 + 2\alpha^4 + 4\alpha^2 + 6)\right) + \sum_{i=1}^{n} ln\left(x_i^6 + 20x_i^4 + 120x_i^2 + 120\alpha x_i\right) - \alpha \sum_{i=1}^{n} x_i. \end{aligned}$$

Thus, the maximum likelihood estimate (MLE) of α is the solution of the following equation

$$\frac{\partial L^*}{\partial \alpha} = \frac{7n}{\alpha} - \frac{n(6\alpha^5 + 8\alpha^3 + 8\alpha)}{\alpha^6 + 2\alpha^4 + 4\alpha^2 + 6} + \sum_{i=1}^{n} \frac{120}{x_i^5 + 20x_i^3 + 120x_i + 120\alpha} - \sum_{i=1}^{n} x_i = 0,$$

which can be solved by numerical methods.

10 Simulation Study

In this section, we present a simulation study to investigate the behavior of the MLE in terms of bias and standard error. We generated random samples from the WGD with different values of the parameter ($\alpha = 2, 3, 5, 7, 10$) and sample sizes (n=40, 80, 100, 150, 250, 500) using R software. For each sample, the MLE ($\hat{\alpha}$) of the parameter α, bias and the standard error of the MLE are obtained and the results are summarized in Table 2.

From Table 2, it can be seen that the values of the bias and the standard error decrease as the sample size increases in all cases. This shows accuracy and consistency of the MLE of the parameter.

11 Real Data Applications

In this section, two real data sets are analyzed for illustrative purposes. The first data set represents runoff amounts at Jug Bridge, Maryland. It is obtained from (Folks and Chhikara, 1978) and studied by (Gadde et al., 2019). The data are as follows: 0.17, 1.19, 0.23, 0.33, 0.39, 0.39, 0.40, 0.45, 0.52, 0.56, 0.59, 0.64, 0.66, 0.70, 0.76, 0.77, 0.78, 0.95, 0.97, 1.02, 1.12, 1.24, 1.59, 1.74, 2.92.

The second data set refers to the time between failures for repairable items which is obtained from (Murthy et al., 2004) and discussed in (Gamedani et al., 2017). The data

Table 2: Bias and standard error of the MLE of the WGD parameter

n	α	$\hat{\alpha}$	**bias**	**S.E.**	**n**	α	$\hat{\alpha}$	**bias**	**S.E.**
40	2	2.0213	0.0213	0.1331	150	2	2.0063	0.0063	0.0676
	3	3.0472	0.0472	0.2625		3	3.0137	0.0137	0.1333
	5	5.0889	0.0889	0.5212		5	5.026	0.026	0.2654
	7	7.1271	0.1271	0.7636		7	7.0373	0.0373	0.3891
	10	10.1836	0.1836	1.1157		10	10.0539	0.0539	0.5686
80	2	2.0102	0.0102	0.093	250	2	2.0038	0.0038	0.0522
	3	3.0227	0.0227	0.1833		3	3.0084	0.0084	0.1029
	5	5.0428	0.0428	0.3647		5	5.0159	0.0159	0.2051
	7	7.0612	0.0612	0.5345		7	7.0228	0.0228	0.3006
	10	10.0885	0.0885	0.7815		10	10.0331	0.0331	0.4395
100	2	2.0065	0.0065	0.0829	500	2	2.0026	0.0026	0.0369
	3	3.015	0.015	0.1633		3	3.0056	0.0056	0.0727
	5	5.0278	0.0278	0.3251		5	5.0107	0.0107	0.1448
	7	7.0394	0.0394	0.4766		7	7.0155	0.0155	0.2124
	10	10.0567	0.0567	0.6966		10	10.0225	0.0225	0.3105

are as follows: 1.43, 0.11, 0.71, 0.77, 2.63, 1.49, 3.46, 2.46, 0.59, 0.74, 1.23, 0.94, 4.36, 0.40, 1.74, 4.73, 2.23, 0.45, 0.70, 1.06, 1.46, 0.30, 1.82, 2.37, 0.63, 1.23, 1.24, 1.97, 1.86, 1.17.

For both data sets, the goodness of fit of the WGD is compared with the following distributions:

- Exponential distribution:
  $f(x) = \theta e^{-x\theta} \quad ; x > 0, \theta > 0.$
- Lindley distribution:
  $f(x) = \frac{\theta^2}{\theta+1}(1+x)e^{-x\theta} \quad ; x > 0, \theta > 0.$
- Aradhana distribution (Shanker (2016)):
  $f(x) = \frac{\theta^3}{\theta^2+2\theta+2}(1+x)^2 e^{-x\theta}; x > 0, \theta > 0.$
- Karam distribution (Gharaibeh and Sahtout (2022)):
  $f(x) = \frac{\eta^6}{\eta^5+2\eta^3+24\eta+120}(x^5+x^4+x^2+1)e^{-\eta x}; x > 0; \eta > 0.$
- Ishita distribution (Shanker and Shukla (2017)):
  $f(x) = \frac{\theta^3}{\theta^3+2}(\theta+x^2)e^{-x\theta}; x > 0, \theta > 0.$
- Size-biased Ishita distribution (SBID) (Al-Omari et al. (2019)):
  $f(x) = \frac{\theta^4}{\theta^3+6}x(\theta+x^2)e^{-x\theta}; x > 0, \theta > 0.$
- Gharaibeh distribution (Gharaibeh (2021)):
  $f(x) = \frac{\beta^6}{120(\beta^6+\beta^4+\beta^2+1)}(x^5+20x^3+120x+120\beta)e^{-\beta x}; x > 0; \beta > 0.$

Table 3: $-2lnL$, AIC, BIC, KS statistic and its p-value for fitted distributions.

	Data 1						Data 2				
Distribution	*-2log L*	*AIC*	*BIC*	*KS*	*p-value*	*Distribution*	*-2log L*	*AIC*	*BIC*	*KS*	*p-value*
Exponential	41.4724	43.4724	44.6913	0.2503	0.0872	*Exponential*	86.0108	88.0108	89.4120	0.1845	0.2590
Lindley	39.6025	41.6025	42.8214	0.2238	0.1635	*Lindley*	83.0946	85.0946	86.4958	0.1407	0.5928
Aradhana	38.1173	40.1173	41.3362	0.2007	0.2665	*Aradhana*	81.3708	83.3708	84.7720	0.1048	0.8964
Karam	49.7321	51.7321	52.9510	0.3013	0.0214	*Karam*	87.8628	89.8628	91.2640	0.1478	0.5287
Ishita	43.0950	45.0950	46.3139	0.2615	0.0655	*Ishita*	84.8328	86.8328	88.2340	0.1526	0.4871
SBID	32.4717	34.4717	35.6906	0.1298	0.7938	*SBID*	79.6360	81.6360	83.0372	0.0743	0.9964
Gharaibeh	42.4046	44.4046	45.6235	0.2618	0.0649	*Gharaibeh*	87.1771	89.1771	90.5783	0.1921	0.2180
WGD	**32.0523**	**34.0523**	**35.2712**	**0.1258**	**0.8238**	*WGD*	**79.2438**	**81.2438**	**82.6450**	**0.0672**	**0.9992**

Table 4: The MLEs of the parameters of the fitted distributions and their confidence intervals.

	Data 1					Data 2			
Distribution	*MLE*	*S.E.*	*CI lower limit*	*CI upper limit*	*Distribution*	*MLE*	*S.E.*	*CI lower limit*	*CI upper limit*
Exponential	1.1860	0.2372	0.7211	1.6509	*Exponential*	0.6482	0.1183	0.4162	0.8802
Lindley	1.6359	0.2575	1.1313	2.1405	*Lindley*	0.9762	0.1345	0.7126	1.2399
Aradhana	2.1019	0.2757	1.5615	2.6423	*Aradhana*	1.3220	0.1490	1.0300	1.6141
Karam	3.2171	0.2401	2.7464	3.6878	*Karam*	2.5190	0.1568	2.2117	2.8263
Ishita	1.7906	0.2041	1.3906	2.1906	*Ishita*	1.2682	0.1153	1.0422	1.4942
SBID	2.8616	0.2958	2.2819	3.4412	*SBID*	1.9042	0.1555	1.5994	2.2089
Gharaibeh	1.7862	0.1954	1.4033	2.1692	*Gharaibeh*	1.3498	0.1037	1.1466	1.5530
WGD	**2.8191**	**0.2943**	**2.2423**	**3.3959**	*WGD*	**1.9443**	**0.1434**	**1.6633**	**2.2254**

For comparison, the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), -2 log likelihood ($-2lnL$), Kolmogorov Smirnov (KS) statistic and its p-value, are evaluated for all fitted distributions and the the results are reported in Table 3. It can be seen that the WGD provides better fit for both data sets since it has the lowest values of the $-2lnL$, AIC, BIC, KS statistic and highest p-value comparing with all fitted distributions.

The maximum likelihood estimates (MLEs), standard errors, and confidence intervals (CI) of the parameters of the fitted distributions are obtained and given in Table 4.

12 Conclusion

This article suggests a new one parameter distribution called weighted Gharaibeh distribution (WGD). Many properties and features of this distribution are studied and investigated such as moments and associated measures, order statistics, reliability analysis, Renyi entropy, maximum likelihood estimate of the distribution parameter, Bonferroni and Lorenz curves, Stochastic ordering and mean deviations about mean and median. The property of consistency for the MLE is illustrated using a simulation study. Applications to real data sets are presented and showed that the proposed distribution outperforms Gharaibeh distribution and some other competitive one parameter distributions in modelling such data.

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