Electronic Journal of Applied Statistical Analysis
EJASA, Electron. J. App. Stat. Anal.
http://siba-ese.unisalento.it/index.php/ejasa/index
e-ISSN: 2070-5948
DOI: 10.1285/i20705948v15n2p300

Benrabia distribution: properties and applications

By Benrabia, AlZoubi

Published: 20 June 2022

Electronic Journal of Applied Statistical Analysis
Vol. 15, Issue 02, June 2022, 300-317
DOI: 10.1285/i20705948v15n2p300

Benrabia distribution: properties and applications

Mohammed Benrabia[*a] and Loai M. A. AlZoubi[b]

[a] *Department of Mathematics, The University of Jordan , Amman, Jordan*
[b] *Department of Mathematics, Al al-Bayt University , Mafraq (25113), Jordan*

Published: 20 June 2022

In this paper, we propose a new two parameter continuous distribution. It is called a Benrabia distribution. Some statistical properties are derived such as: the moment generating function, the moments and related measures, the reliability analysis and related functions. Also, the distribution of order statistics and the quantile function are presented and the R*é*nyi entropy is derived. The method of maximum likelihood estimation is used to estimate the distribution parameters. A simulation is performed to investigate the performance of MLE, real data applications show that the proposed distribution can provide a better fit than several well-known distributions.

keywords: Mixing distribution, Benrabia distribution, reliability analysis, R*é*nyi entropy, maximum likelihood estimation, moment generating function.

1 Introduction

In statistics, modeling lifetime data is an important issue in many fields including biomedical sciences, economics, finance, engineering. A lot of continuous distributions have introduced for modeling such data, because they can contribute better fit than the based distribution. Some studies have shown the inferiority of some of these distributions in modelling lifetime data sets when compared with some newer models. This is the motivation that lead to the search for other distributions with better fitting to real life data and more flexibility.

A random variable X is said to have a mixture of two or more distributions ($f_1(x)$, $\cdots$, $f_k(x)$), if its probability density function (pdf) $g(x) = \sum_{i=1}^{k} b_i f_i(x)$ with $0 \leq b_i \leq 1$

[*]Corresponding author: mohamedbenrabia41@yahoo.com

ISSN: 2070-5948
http://siba-ese.unisalento.it/index.php/ejasa/index

is the mixing weight, such that $\sum_{i=1}^{k} b_i = 1$. Recently, several distributions have been proposed from mixing distributions, for example, Shraa and Al-Omari (2019) suggested Darna distribution as a mixture of $Exp\left(\frac{\theta}{\alpha}\right)$ and $\Gamma\left(3, \frac{\theta}{\alpha}\right)$ with mixing proportion $\frac{2\alpha^2}{2\alpha^2+\theta^2}$. Shanker (2017) suggested Rama distribution as a mixture of two components of $Exp(\theta)$ and $\Gamma(4, \theta)$ using mixing proportion $\frac{\theta^3}{\theta^3+6}$. Another two components mixture of $Exp(\theta)$ and $\Gamma(3, \theta)$ is proposed using mixing proportion $\frac{\theta^3}{\theta^3+2}$ by Shanker and Shukla (2017) named Ishita distribution. Shanker (2016) suggested Aradhana distribution by mixing $Exp(\theta)$, $\Gamma(2, \theta)$ and $\Gamma(3, \theta)$ with mixing proportions $\frac{\theta^2}{\theta^2+2\theta+2}$, $\frac{2\theta}{\theta^2+2\theta+2}$ and $\frac{2}{\theta^2+2\theta+2}$. Shanker (2015) used the mixture weight $\frac{\theta^2}{\theta^2+1}$ with $Exp(\theta)$ and $\Gamma(2, \theta)$ to propose Shanker distribution. Gharaibeh (2021) proposed Gharaibeh distribution as a four components mixture of $exp(\beta)$, $\Gamma(2, \beta)$, $\Gamma(4, \beta)$ and $\Gamma(6, \beta)$ with mixing proportions $\frac{\beta^6}{\beta^6+\beta^4+\beta^2+1}$, $\frac{\beta^4}{\beta^6+\beta^4+\beta^2+1}$, $\frac{\beta^2}{\beta^6+\beta^4+\beta^2+1}$ and $\frac{1}{\beta^6+\beta^4+\beta^2+1}$; respectively. Benrabia and Alzoubi (2021) employed the concept of mixture distributions using the exponential and gamma distributions, with mixture proportions $\frac{\alpha\beta}{\alpha\beta+1}$ and $\frac{1}{\alpha\beta+1}$, to suggest a new two parameters distribution called Alzoubi distribution.

Other ways of proposing new distributions are used, like the transmutation maps. For example, transmuted Mukherjee-Islam distribution (Al-zou′bi, 2017), transmuted Janardan distribution (Al-Omari et al., 2017b), a generalization of the new Weibull Pareto distribution (Al-Omari et al., 2017a) and transmuted Shanker distribution (Al-Zoubi et al., 2021). Some other distributions using this map were generated by AzZwideen and Al-Zou'bi (2020); Alsikeek (2018); Rabaiah (2018); Saadeh (2019); Almawajdeh (2019); Almousa (2019).

In this article, we employed the concept of mixture distributions to suggest a new two parameters distribution called Benrabia distribution. This new distribution is a mixture of two components of $Exp(\beta)$ and $\Gamma(\alpha-1, \beta)$ with mixing proportions $\frac{\alpha}{\alpha+\beta}$ and $\frac{\beta}{\alpha+\beta}$, respectively. Also, we want to prove that the suggested distribution is more flexible than the base distribution based on some real lifetime data.

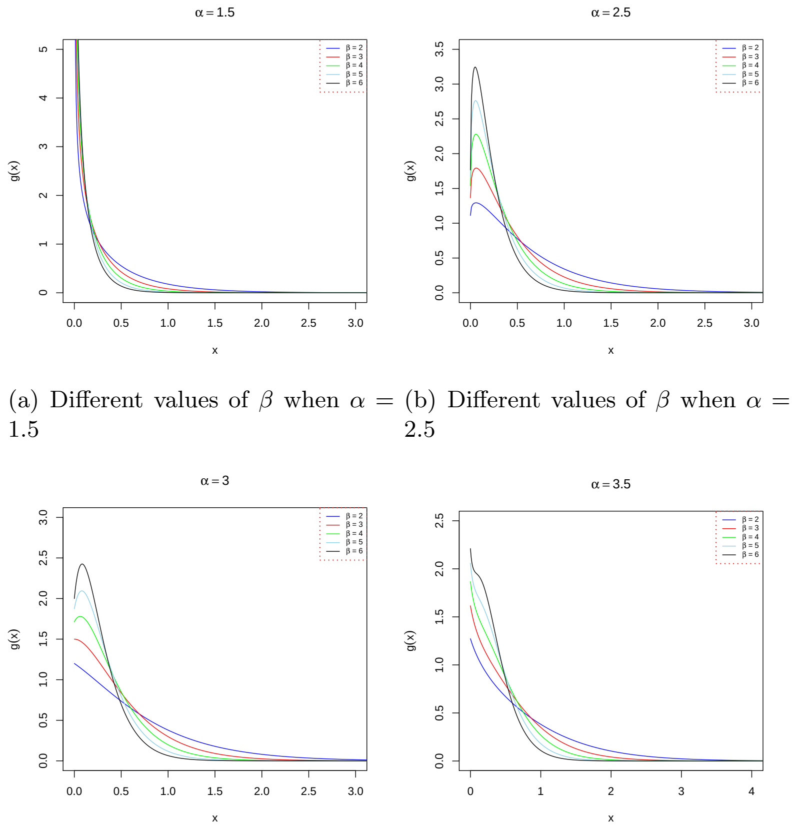
This paper is organized as follows, in Section 2 we define the probability density and the cumulative distribution function of Benrabia distribution. In Section 3, we consider some statistical properties including the moments, the moment generating function, skewness, kurtosis, and coefficient of variation. In Section 4, we conduct the reliability analysis including the reliability, hazard rate, cumulative hazard rate, reversed hazard rate and odds ratio functions of Benrabia distribution. In Section 5, we describe the density of order statistics and the quantile function. Sections 6 and 7 derive the Bonferoni and Lorenz curves and the Rényi entropy. In Section 8, we determine the mean deviation about mean and median. In Section 9, we study the estimation of the model parameters using maximum likelihood method. In Section 10, we provide a simulation study. Section 11 present some real lifetime data sets. Finally, in Section 12, we end this research with a conclusion and suggested a future work.

2 Benrabia Distribution

In this section, we define the probability density function (pdf) and the cumulative distribution function (cdf) of the proposed distribution with graphic illustration for both of them.

Definition A random variable X is said to have a Benrabia distribution with parameters α and β (it is denoted by $X \sim Br(\alpha, \beta)$), if its pdf is defined as:

$$g(x|\alpha, \beta) = \frac{\beta}{\alpha + \beta} \left(\alpha + \frac{x^{\alpha-2}\beta^{\alpha-1}}{\Gamma(\alpha - 1)} \right) e^{-\beta x} \quad x > 0, \quad \alpha > 1, \beta > 0 \tag{1}$$



(a) Different values of β when $\alpha = 1.5$

(b) Different values of β when $\alpha = 2.5$

(c) Different values of β when $\alpha = 3$

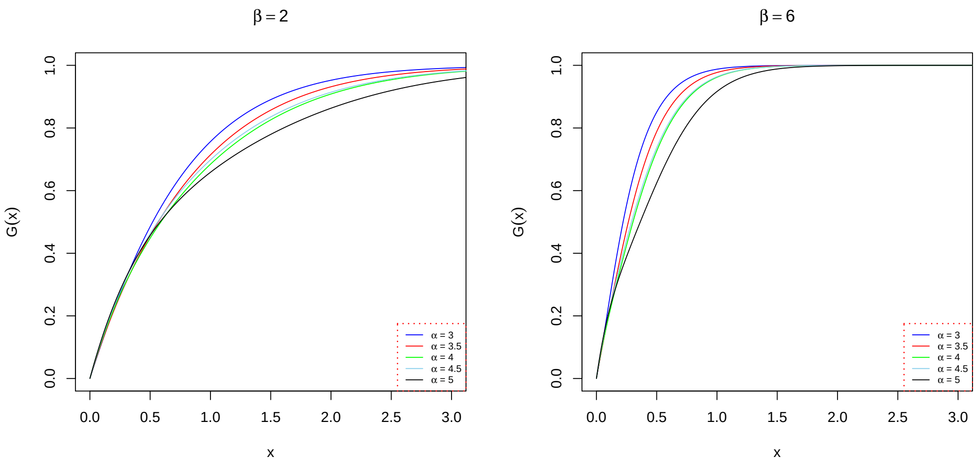
(d) Different values of β when $\alpha = 3.5$

Figure 1: Plots of Benrabia probability density function with different parameters values (a), (b), (c) and (d).

The cumulative distribution function of Benrabia distribution is given by

$$G(x|\alpha, \beta) = \frac{1}{\alpha + \beta} \left[\alpha(1 - e^{-\beta x}) + \beta \ P(\alpha - 1, \beta x) \right], \tag{2}$$

where $P(\alpha, x) = \frac{\gamma(\alpha,x)}{\Gamma(\alpha)}$ is the lower regularized gamma function with $\gamma(\alpha, x) = \int_0^x t^{\alpha-1}e^{-t}dt$, is the lower incomplete gamma function.



(a) Different values of α when $\beta = 2$ (b) Different values of α when $\beta = 6$

Figure 2: Plots of Benrabia distribution function with different parameters values (a) and (b).

Figure 1 shows the graph of the pdf of Benrabia distribution for different values of α and β. We show that the distribution is skewed right.

3 Moments and Moment Generating Function

In this section, the moment generating function and the r^{th} moment are presented. Also the mean, variance, kurtosis, skewness and coefficient of variation are calculated.

Theorem 1 *The moment generating function of the proposed distribution is defined by*

$$M_X(t) = \frac{\beta}{\alpha+\beta}\left[\frac{\alpha}{\beta-t} + \left(1-\frac{t}{\beta}\right)^{-(\alpha-1)}\right] \quad t < \beta \tag{3}$$

Proof:

$$\begin{aligned} M_X(t) = E\left(e^{tX}\right) &= \int_0^\infty e^{tx}g(x)dx \\ &= \frac{\alpha\beta}{\alpha+\beta}\int_0^\infty e^{-(\beta-t)x}dx + \frac{\beta}{\alpha+\beta}\int_0^\infty \frac{x^{\alpha-2}\beta^{\alpha-1}e^{-(\beta-t)x}}{\Gamma(\alpha-1)}dx \\ &= \frac{\alpha\beta}{\alpha+\beta}\left(\frac{-1}{\beta-t}\right)e^{-(\beta-t)x}\Big|_0^\infty + \frac{\beta}{\alpha+\beta}\left(\frac{\beta}{\beta-t}\right)^{(\alpha-1)} \\ &= \frac{\alpha\beta}{(\alpha+\beta)(\beta-t)} + \frac{\beta}{\alpha+\beta}\left(\frac{\beta}{\beta-t}\right)^{(\alpha-1)} \end{aligned}$$

Theorem 2 *The r^{th} moment of Benrabia distribution can be expressed as follows*

$$E(X^r) = \frac{1}{(\alpha+\beta)\beta^r}\left[\alpha\Gamma(r+1) + \beta\frac{\Gamma(\alpha+r-1)}{\Gamma(\alpha-1)}\right] \tag{4}$$

Proof: Let X have a $Br(\alpha, \beta)$, then the r^{th} moment is

$$\begin{aligned}
E(X^r) &= \int_0^\infty x^r g(x)dx \\
&= \frac{\alpha\beta}{\alpha+\beta}\int_0^\infty x^r e^{-\beta x}dx + \frac{\beta}{\alpha+\beta}\int_0^\infty \frac{x^{r+\alpha-2}\beta^{\alpha-1}e^{-\beta x}}{\Gamma(\alpha-1)}dx \\
&= \frac{\alpha\beta}{\alpha+\beta}\frac{\Gamma(r+1)}{\beta^{r+1}} + \frac{\beta}{\alpha+\beta}\frac{\Gamma(\alpha+r-1)}{\Gamma(\alpha-1)\beta^r} \\
&= \frac{1}{\beta^r(\alpha+\beta)}\left[\alpha\ \Gamma(r+1) + \beta\ \frac{\Gamma(\alpha+r-1)}{\Gamma(\alpha-1)}\right] \qquad \square
\end{aligned}$$

Using (4), the first four moments of the suggested distribution are

$$\begin{aligned}
\mu = E(X) &= \frac{\alpha-\beta+\alpha\beta}{\beta(\alpha+\beta)} \\
E(X^2) &= \frac{\alpha^2\beta-\alpha\beta+2\alpha}{\beta^2(\alpha+\beta)} \\
E(X^3) &= \frac{\alpha^3\beta-\alpha\beta+6\alpha}{\beta^3(\alpha+\beta)} \\
E(X^4) &= \frac{\alpha^4\beta+2\alpha^3\beta-\alpha^2\beta-2\alpha\beta+24\alpha}{\beta^4(\alpha+\beta)}
\end{aligned}$$

Based on these moments; the variance, standard deviation, coefficient of variation, and coefficients of skewness and kurtosis of Benrabia distribution are, respectively, defined as

$$\begin{aligned}
\sigma^2 &= E(X^2) - \mu^2 = \frac{\alpha^2\beta-\alpha\beta+2\alpha}{\beta^2(\alpha+\beta)} - \left[\frac{\alpha-\beta+\alpha\beta}{\beta(\alpha+\beta)}\right]^2 \\
&= \frac{\alpha^3\beta+\alpha^2(1-3\beta)+\beta^2(\alpha-1)+4\alpha\beta}{\beta^2(\alpha+\beta)^2} \\
\sigma &= \sqrt{\frac{\alpha^3\beta+\alpha^2(1-3\beta)+\beta^2(\alpha-1)+4\alpha\beta}{\beta^2(\alpha+\beta)^2}} \\
C.V &= \frac{\sigma}{\mu} = \frac{\sqrt{\alpha^3\beta+\alpha^2(1-3\beta)+\beta^2(\alpha-1)+4\alpha\beta}}{\alpha-\beta+\alpha\beta}
\end{aligned}$$

$$
\begin{aligned}
sk(X) &= \frac{E(X^3) - 3\mu E(X^2) + 2\mu^3}{\sigma^3} \\
&= \frac{\left[\begin{array}{c} (\alpha+\beta)^2(\alpha^3\beta - \alpha\beta + 6\alpha) + (\alpha + \alpha\beta - \beta)^3 \\ + (-3\alpha + 3\beta - 3\alpha\beta)(\alpha^2\beta - \alpha\beta + 2\alpha)(\alpha+\beta) \end{array}\right]}{(\alpha^3\beta + \alpha^2(1-3\beta) + \beta^2(\alpha-1) + 4\alpha\beta)^{\frac{3}{2}}} \\
&= \frac{\left[\begin{array}{c} \alpha^5\beta - \alpha^4\beta^2 - 3\alpha^4\beta - \alpha^3\beta^3 + 6\alpha^3\beta^2 - \alpha^3\beta + \alpha^3 \\ + 3\alpha^2\beta^3 - 14\alpha^2\beta^2 + 9\alpha^2\beta - \alpha\beta^3 + 15\alpha\beta^2 - \beta^3 \end{array}\right]}{(\alpha^3\beta + \alpha^2(1-3\beta) + \beta^2(\alpha-1) + 4\alpha\beta)^{\frac{3}{2}}} \\
ku(X) &= \frac{E(X^4) - 4\mu E(X^3) + 6\mu^2 E(X^2) - 3\mu^4}{\sigma^4} \\
&= \frac{\left[\begin{array}{c} (\alpha+\beta)^3(\alpha^4\beta + 2\alpha^3\beta - \alpha^2\beta - 2\alpha\beta + 24\alpha) - 4(\alpha - \beta + \alpha\beta)(\alpha^3\beta - \alpha\beta + 6\alpha)(\alpha+\beta)^2 \\ + 6(\alpha + \alpha\beta - \beta)^2(\alpha^2\beta - \alpha\beta + 2\alpha)(\alpha+\beta) - 3(\alpha + \alpha\beta - \beta)^4 \end{array}\right]}{(\alpha^3\beta + \alpha^2(1-3\beta) + \beta^2(\alpha-1) + 4\alpha\beta)^2}
\end{aligned}
$$

4 Reliability Analysis

If T is a random variable that follows Benrabia distribution, then the survival or reliability function(RF), hazard, cumulative hazard function, the reversed hazard rate and odd functions corresponding to (1) are respectively, defined by

$$
\begin{aligned}
R(t) &= 1 - G(t) = \frac{\alpha e^{-\beta t} + \beta\left[1 - P(\alpha-1, \beta t)\right]}{\alpha+\beta} \\
h(t) &= \frac{g(t)}{1-G(t)} = \frac{\beta e^{-\beta t}\left(\alpha + \frac{t^{\alpha-2}\beta^{\alpha-1}}{\Gamma(\alpha-1)}\right)}{\alpha e^{-\beta t} + \beta\left[1 - P(\alpha-1, \beta t)\right]} \\
H(t) &= -ln(1 - G(t)) \\
&= ln(\alpha+\beta) - ln(\alpha e^{-\beta t} + \beta\left[1 - P(\alpha-1, \beta t)\right]), \\
rh(t) &= \frac{g(t)}{G(t)} = \frac{\beta e^{-\beta t}\left(\alpha + \frac{t^{\alpha-2}\beta^{\alpha-1}}{\Gamma(\alpha-1)}\right)}{\alpha(1 - e^{-\beta t}) + \beta\ P(\alpha-1, \beta t)} \\
O(t) &= \frac{G(t)}{1-G(t)} = \frac{\alpha(1 - e^{-\beta t}) + \beta\ P(\alpha-1, \beta t)}{\alpha e^{-\beta t} + \beta\left[1 - P(\alpha-1, \beta t)\right]}
\end{aligned}
$$

Figure 4 shows that the cumulative hazard rate functions is an increasing function. While the reversed hazard function is a decreasing function.

Table 5.1 shows the values of the mean, standard deviation, skewness, excess kurtosis and the coefficient of variation of Benrabia distribution for values of β of 0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4, and 4.5 and values of α of 1.5, 1.8, 2.1, 2.4, 2.7, 3.0, 3.3, 3.6, 3.9, 4.2 and 4.5. The table shows that the distribution is skewed right regardless the value of α and β. The excess kurtosis = kurtosis − 3, (Joanes and Gill, 1998). The values of excess kurtosis are all positive, which means that the tails of the distribution are heavier than the normal distribution tails. It, also shows that the values of the mean and standard deviation decrease as the values of β increases. They increase as the value of α increases.

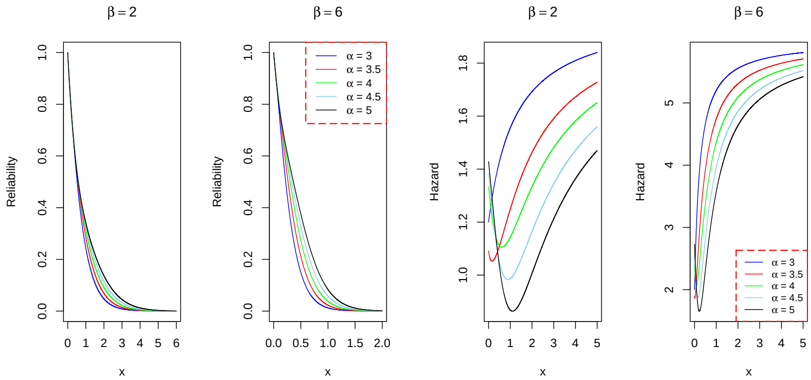


Figure 3: The reliability and hazard rate functions of Br distribution when $\beta = 2$ and $\beta = 6$

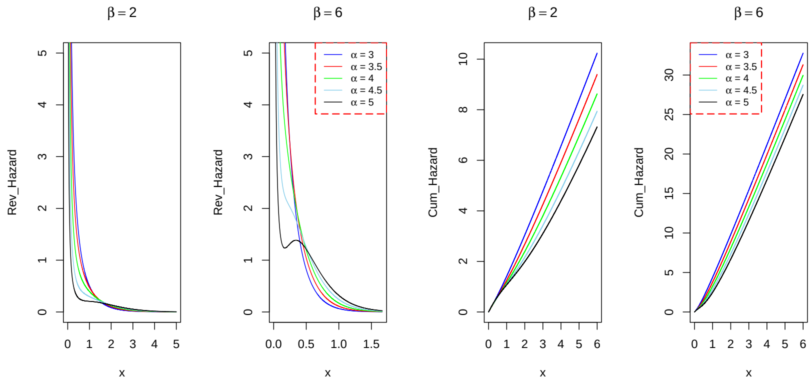


Figure 4: The reversed and cumulative hazard rate functions of Br distribution when $\beta = 2$ and $\beta = 6$

5 Order Statistics and Quantile Function

In this section, we will derive the distribution of order statistics and the quantile function of Benrabia distribution.

5.1 Order statistics

Let $X_{(1)}, X_{(2)}, ...X_{(n)}$ be the order statistics of the random sample $X_1, X_2, \cdots, X_n$ selected from Br distribution. The pdf of the j^{th} order statistics $X_{(j)}$ is defined as

$$g_{(j)}(x) = j\binom{n}{j}[G(x)]^{j-1}[1 - G(x)]^{n-j}g(x) \tag{5}$$

By replacing (1) and (2) in (5) and using binomial theorem, we get

Table 1: The mean, standard deviation, skewness, excess kurtosis and the coefficient of variation of Benrabia distribution with different values of α and β

β	α	$E(X)$	$\sigma(X)$	$sk(X)$	$ekur(X)$	$C.V$	β	α	$E(X)$	$\sigma(X)$	$sk(X)$	$ekur(X)$	$C.V$	β	α	$E(X)$	$\sigma(X)$	$sk(X)$	$ekur(X)$	$C.V$
0.5	1.5	1.750	1.920	2.123	6.700	109.731	2.0	1.5	0.357	0.440	2.367	8.278	123.288	3.5	1.5	0.186	0.239	2.493	9.194	128.947
0.5	1.8	1.913	1.963	2.044	6.260	102.611	2.0	1.8	0.447	0.476	2.114	6.693	106.306	3.5	1.8	0.248	0.268	2.147	6.905	107.893
0.5	2.1	2.038	2.021	1.981	5.886	99.128	2.0	2.1	0.524	0.513	1.953	5.719	97.763	3.5	2.1	0.304	0.295	1.940	5.645	97.121
0.5	2.4	2.138	2.090	1.939	5.622	97.748	2.0	2.4	0.591	0.553	1.840	5.049	93.518	3.5	2.4	0.354	0.323	1.795	4.809	91.293
0.5	2.7	2.219	2.167	1.921	5.485	97.668	2.0	2.7	0.649	0.595	1.758	4.554	91.739	3.5	2.7	0.399	0.352	1.683	4.180	88.241
0.5	3.0	2.286	2.250	1.925	5.472	98.425	2.0	3.0	0.700	0.640	1.699	4.173	91.473	3.5	3.0	0.440	0.382	1.592	3.673	86.891
0.5	3.3	2.342	2.336	1.950	5.569	99.737	2.0	3.3	0.745	0.687	1.658	3.872	92.174	3.5	3.3	0.477	0.413	1.517	3.248	86.640
0.5	3.6	2.390	2.424	1.990	5.757	101.423	2.0	3.6	0.786	0.735	1.631	3.633	93.508	3.5	3.6	0.511	0.445	1.456	2.889	87.123
0.5	3.9	2.432	2.514	2.043	6.019	103.362	2.0	3.9	0.822	0.783	1.616	3.445	95.264	3.5	3.9	0.542	0.478	1.408	2.584	88.109
0.5	4.2	2.468	2.603	2.107	6.342	105.474	2.0	4.2	0.855	0.832	1.611	3.298	97.301	3.5	4.2	0.571	0.511	1.370	2.325	89.443
0.5	4.5	2.500	2.693	2.177	6.713	107.703	2.0	4.5	0.885	0.880	1.614	3.186	99.526	3.5	4.5	0.598	0.544	1.341	2.106	91.020
1.0	1.5	0.800	0.927	2.224	7.325	115.920	2.5	1.5	0.275	0.346	2.418	8.636	125.639	4.0	1.5	0.159	0.207	2.522	9.413	130.149
1.0	1.8	0.929	0.968	2.075	6.447	104.287	2.5	1.8	0.353	0.378	2.128	6.779	106.960	4.0	1.8	0.216	0.233	2.155	6.953	108.237
1.0	2.1	1.032	1.017	1.969	5.811	98.529	2.5	2.1	0.422	0.411	1.948	5.688	97.503	4.0	2.1	0.266	0.258	1.937	5.628	96.977
1.0	2.4	1.118	1.073	1.896	5.363	95.986	2.5	2.4	0.482	0.446	1.822	4.949	92.633	4.0	2.4	0.313	0.284	1.786	4.758	90.774
1.0	2.7	1.189	1.134	1.851	5.066	95.354	2.5	2.7	0.535	0.483	1.727	4.395	90.370	4.0	2.7	0.354	0.310	1.667	4.105	87.401
1.0	3.0	1.250	1.199	1.831	4.888	95.917	2.5	3.0	0.582	0.522	1.655	3.957	89.704	4.0	3.0	0.393	0.337	1.570	3.576	85.763
1.0	3.3	1.302	1.266	1.831	4.802	97.247	2.5	3.3	0.624	0.562	1.599	3.599	90.062	4.0	3.3	0.428	0.365	1.488	3.132	85.251
1.0	3.6	1.348	1.335	1.846	4.785	99.080	2.5	3.6	0.662	0.603	1.559	3.305	91.094	4.0	3.6	0.461	0.394	1.420	2.754	85.495
1.0	3.9	1.388	1.405	1.873	4.821	101.245	2.5	3.9	0.697	0.645	1.530	3.062	92.580	4.0	3.9	0.491	0.423	1.363	2.430	86.260
1.0	4.2	1.423	1.475	1.908	4.897	103.630	2.5	4.2	0.728	0.687	1.511	2.862	94.373	4.0	4.2	0.518	0.453	1.318	2.154	87.390
1.0	4.5	1.455	1.544	1.950	5.005	106.158	2.5	4.5	0.757	0.730	1.502	2.700	96.374	4.0	4.5	0.544	0.483	1.283	1.919	88.779
1.5	1.5	0.500	0.601	2.304	7.846	120.185	3.0	1.5	0.222	0.283	2.459	8.938	127.476	4.5	1.5	0.139	0.182	2.546	9.602	131.149
1.5	1.8	0.606	0.639	2.097	6.586	105.451	3.0	1.8	0.292	0.313	2.139	6.848	107.476	4.5	1.8	0.190	0.207	2.161	6.994	108.525
1.5	2.1	0.694	0.681	1.960	5.758	98.094	3.0	2.1	0.353	0.343	1.944	5.664	97.294	4.5	2.1	0.237	0.230	1.935	5.615	96.854
1.5	2.4	0.769	0.728	1.864	5.181	94.610	3.0	2.4	0.407	0.374	1.807	4.871	91.904	4.5	2.4	0.280	0.253	1.778	4.716	90.329
1.5	2.7	0.833	0.778	1.798	4.768	93.381	3.0	2.7	0.456	0.407	1.703	4.275	89.219	4.5	2.7	0.319	0.277	1.654	4.044	86.673
1.5	3.0	0.889	0.831	1.756	4.469	93.541	3.0	3.0	0.500	0.441	1.620	3.796	88.192	4.5	3.0	0.356	0.301	1.551	3.499	84.779
1.5	3.3	0.937	0.887	1.733	4.254	94.589	3.0	3.3	0.540	0.476	1.554	3.399	88.230	4.5	3.3	0.389	0.327	1.464	3.041	84.031
1.5	3.6	0.980	0.943	1.725	4.101	96.216	3.0	3.6	0.576	0.512	1.502	3.066	88.977	4.5	3.6	0.420	0.353	1.389	2.649	84.058
1.5	3.9	1.019	1.000	1.728	3.997	98.226	3.0	3.9	0.609	0.549	1.462	2.787	90.203	4.5	3.9	0.448	0.379	1.327	2.312	84.622
1.5	4.2	1.053	1.058	1.740	3.933	100.488	3.0	4.2	0.639	0.586	1.433	2.552	91.759	4.5	4.2	0.475	0.407	1.276	2.023	85.566
1.5	4.5	1.083	1.115	1.760	3.901	102.916	3.0	4.5	0.667	0.624	1.413	2.357	93.541	4.5	4.5	0.500	0.434	1.234	1.775	86.781

$$\begin{aligned}
g_{(j)}(x) &= j\binom{n}{j}\frac{\beta}{(\alpha+\beta)^n}\sum_{k=0}^{j-1}\binom{j-1}{k}[\alpha(1-e^{-\beta x})]^k[\beta\ P(\alpha-1,\beta x)]^{j-k-1}\\
&\times\sum_{l=0}^{n-j}\binom{n-j}{l}[\alpha e^{-\beta x}]^l[\beta(1-P(\alpha-1,\beta x))]^{n-j-l}.\left(\alpha+\frac{x^{\alpha-2}\beta^{\alpha-1}}{\Gamma(\alpha-1)}\right)e^{-\beta x}\\
&= j\binom{n}{j}\frac{\beta}{(\alpha+\beta)^n}\sum_{k=0}^{j-1}\sum_{t=0}^{k}\binom{j-1}{k}\binom{k}{t}(-1)^{k-t}e^{-\beta(k-t)x}\alpha^k[\beta\ P(\alpha-1,\beta x)]^{j-k-1}\\
&\times\sum_{l=0}^{n-j}\sum_{s=0}^{n-j-l}\binom{n-j}{l}\binom{n-j-l}{s}(-1)^{n-j-s-l}[P(\alpha-1,\beta x)]^{n-j-s-l}\\
&\times\beta^{n-j-l}[\alpha e^{-\beta x}]^l.\left(\alpha+\frac{x^{\alpha-2}\beta^{\alpha-1}}{\Gamma(\alpha-1)}\right)e^{-\beta x}
\end{aligned}$$

5.2 Quantile Function

The quantile function of a probability distribution with cdf, $G(x)$, is defined by $q = G^{-1}(x_q)$, where $0 < q < 1$. Then, the quantile function of Benrabia distribution is given by

$$Q_p = \frac{1}{\beta}\left[\gamma^{-1}\left((\alpha-1),\frac{\Gamma(\alpha-1)}{\beta}\left[(\alpha+\beta)p+\frac{\alpha\beta}{log(1-p)}\right]\right)\right], \tag{6}$$

where $\gamma^{-1}(.,.)$ is the inverse of the lower incomplete gamma function.

Proof: By using (2), we have

$$\begin{aligned}
p = G(x) &= \frac{1}{\alpha+\beta}\left[\alpha(1-e^{-\beta x})+\beta\ P(\alpha-1,\beta x)\right]\\
p(\alpha+\beta) &= \left[\alpha(1-e^{-\beta x})+\beta\ \frac{\gamma(\alpha-1,\beta x)}{\Gamma(\alpha-1)}\right]\\
p(\alpha+\beta)-\alpha F(x) &= \beta\ \frac{\gamma(\alpha-1,\beta x)}{\Gamma(\alpha-1)},
\end{aligned}$$

where $F(x)$ is the cdf of the exponential distribution. So

$$\gamma(\alpha-1,\beta x) = \frac{\Gamma(\alpha-1)}{\beta}\left[p(\alpha+\beta)-\frac{\alpha}{F^{-1}(x)}\right]$$

By exerting the idea of Samir et al. (2018) which is also used by Nosakhare et al. (2020), we obtain

$$\beta x = \gamma^{-1}\left((\alpha-1),\frac{\Gamma(\alpha-1)}{\beta}\left[p(\alpha+\beta)-\frac{\alpha}{F^{-1}(x)}\right]\right),$$

with $F^{-1}(x) = -\frac{log(1-p)}{\beta}$, which is the quantile function of the exponential distribution, then Equation (6) becomes

$$Q_p = \frac{1}{\beta}\left[\gamma^{-1}\left((\alpha-1),\frac{\Gamma(\alpha-1)}{\beta}\left[(\alpha+\beta)p+\frac{\alpha\beta}{log(1-p)}\right]\right)\right]$$

6 Bonferroni and Lorenz Curves

The Bonferroni and Lorenz curves have importance in many domains such as economics, demography, (Kakwani and Podder, 1976). The Bonferroni and Lorenz curves for a random variable X are, respectively, defined as

$$B(p) = \frac{1}{p\mu}\int_0^q xg(x)dx \qquad\qquad L(p) = \frac{1}{\mu}\int_0^q xg(x)dx,$$

where $q = F^{-1}(p);\;\; p \in [0,1]$ *and* $\mu = E(X)$. Hence the Bonferroni and Lorenz curves of Benrabia distribution are, respectively, given by:

$$\begin{aligned} B(p) &= \frac{1}{p(\alpha-\beta+\alpha\beta)}\left[\frac{\alpha}{\beta}(1-(1+\beta q)e^{-\beta q}) + (\alpha-1)P(\alpha,\beta q)\right] \\ L(p) &= \frac{1}{(\alpha-\beta+\alpha\beta)}\left[\frac{\alpha}{\beta}(1-(1+\beta q)e^{-\beta q}) + (\alpha-1)P(\alpha,\beta q)\right] \end{aligned}$$

7 Rényi Entropy

The entropy was first introduced by Shannon (1948). It describes the amount of information in a signal or event in information theory. It is defined as a measure of uncertainty of the probability distribution of a random variable X in statistics (Wang, 2008). It is used in many fields such as statistics, engineering. Rényi (1961) defined the Rényi entropy of a random variable X as:

$$R_\delta = \frac{1}{1-\delta}log\int_0^\infty [g(x)]^\delta dx; \qquad \delta > 0 \quad \delta \neq 1 \tag{7}$$

Theorem 3 *The Rényi entropy of the random variable $X \sim Br(\alpha,\beta)$ is defined by*

$$R_\delta = \frac{1}{1-\delta}log\left[\left(\frac{\beta}{\alpha+\beta}\right)^\delta \sum_{i=1}^{\delta}\binom{\delta}{i}\alpha^i\beta^{\delta-i+1}[\Gamma(\alpha-1)]^{i-\delta}\frac{\Gamma(\alpha\delta-2\delta-\alpha i+2i+1)}{\delta^{\alpha\delta-2\delta-\alpha i+2i+1}}\right] \tag{8}$$

Proof: Using (1) and (7), we have

$$\begin{aligned} R_\delta &= \frac{1}{1-\delta}log\int_0^\infty \left[\frac{\beta}{\alpha+\beta}\left(\alpha + \frac{x^{\alpha-2}\beta^{\alpha-1}}{\Gamma(\alpha-1)}\right)e^{-\beta x}\right]^\delta dx \\ &= \frac{1}{1-\delta}log\int_0^\infty \left[\left(\frac{\beta}{\alpha+\beta}\right)^\delta\left(\alpha + \frac{x^{\alpha-2}\beta^{\alpha-1}}{\Gamma(\alpha-1)}\right)^\delta e^{-\beta\delta x}\right]dx \end{aligned}$$

Using binomial theorem, we have

$$\begin{aligned} \left(\alpha + \frac{x^{\alpha-2}\beta^{\alpha-1}}{\Gamma(\alpha-1)}\right)^\delta &= \sum_{i=1}^{\delta}\binom{\delta}{i}\alpha^{\delta-i}\left[\frac{x^{\alpha-2}\beta^{\alpha-1}}{\Gamma(\alpha-1)}\right]^i \\ &= \sum_{i=1}^{\delta}\binom{\delta}{i}\alpha^{\delta-i}\left[\frac{\beta^{\alpha-1}}{\Gamma(\alpha-1)}\right]^i x^{i(\alpha-2)} \end{aligned}$$

$$\text{Thus, } R_\delta = \frac{1}{1-\delta}log\left[\int_0^\infty \left(\frac{\beta}{\alpha+\beta}\right)^\delta \sum_{i=1}^{\delta}\binom{\delta}{i}\alpha^{\delta-i}\left[\frac{\beta^{\alpha-1}}{\Gamma(\alpha-1)}\right]^i x^{i(\alpha-2)}e^{-\beta\delta x}\right]dx$$

$$= \frac{1}{1-\delta} log \left[\left(\frac{\beta}{\alpha+\beta} \right)^{\delta} \sum_{i=1}^{\delta} \binom{\delta}{i} \alpha^{\delta-i} \left[\frac{\beta^{\alpha-1}}{\Gamma(\alpha-1)} \right]^{i} \int_0^{\infty} x^{i\alpha-2i} e^{-\beta\delta x} dx \right]$$
$$= \frac{1}{1-\delta} log \left[\left(\frac{\beta}{\alpha+\beta} \right)^{\delta} \sum_{i=1}^{\delta} \binom{\delta}{i} \alpha^{\delta-i} \left[\frac{\beta^{\alpha-1}}{\Gamma(\alpha-1)} \right]^{i} \frac{\Gamma(i\alpha-2i+1)}{(\beta\delta)^{i\alpha-2i+1}} \right]$$
$$= \frac{1}{1-\delta} log \left[\left(\frac{\beta}{\alpha+\beta} \right)^{\delta} \sum_{i=1}^{\delta} \binom{\delta}{i} \alpha^{\delta-i} \frac{\beta^{i-1}}{[\Gamma(\alpha-1)]^{i}} \frac{\Gamma(i\alpha-2i+1)}{\delta^{i\alpha-2i+1}} \right] \qquad \square$$

8 Mean and Median Absolute Deviations

The advantage of using mean deviation about mean or median is giving a better measure of dispersion from the average (Pham-Gia and Hung, 2001). Hence the mean deviation about mean and median for the Br distribution are defined respectively, as

$$\begin{aligned} MD_{mean} = E|X-\mu| &= \int_0^{\infty} |x-\mu| g(x)dx \\ &= \int_0^{\mu} (\mu - x)g(x)dx + \int_{\mu}^{\infty} (x-\mu)g(x)dx \\ &= 2\int_0^{\mu} (\mu - x)g(x)dx \\ &= 2\mu G(\mu) - 2\int_0^{\mu} xg(x)dx \\ &= \frac{1}{\alpha+\beta} \Bigg[\left(2\mu\alpha - \frac{2\alpha}{\beta} \right)(1-e^{\beta\mu}) \\ &+ 2\mu\alpha \ e^{-\beta\mu} - 2(\alpha-1)P(\alpha,\beta\mu) + 2\mu\beta P((\alpha-1),\beta\mu) \Bigg]. \end{aligned}$$

And

$$\begin{aligned} MD_{median} = E|X-M| &= \int_0^{\infty} |x-M| g(x)dx \\ &= \int_0^{M} (M-x)g(x)dx + \int_M^{\infty} (x-M)g(x)dx \\ &= 2\int_0^{M} (M-x)g(x)dx + \int_0^{\infty} (x-M)g(x)dx \\ &= 2MG(M) + \mu - M - 2\int_0^{M} xg(x)dx \\ &= \mu - 2\int_0^{M} xg(x)dx \\ &= \mu + \frac{1}{\alpha+\beta} \left[2M\alpha \ e^{-\beta M} - \frac{2}{\beta}\left(1-e^{-\beta M}\right) - 2(\alpha-1)P(\alpha,\beta M) \right], \end{aligned}$$

where $\mu = \frac{\alpha+\alpha\beta-\beta}{\beta(\alpha+\beta)}$, M is a population median and $P(.,.)$ is the regularized incomplete gamma function.

9 Maximum Likelihood Estimation

Let $X_1, X_2, ...X_n$ be a random sample from Benrabia distribution, then the likelihood function $L(x, \alpha, \beta)$ is defined by

$$
\begin{aligned}
L(x,\alpha,\beta) &= \prod_{j=1}^{n} g(x_j,\alpha,\beta) \\
&= \left(\frac{\beta}{\alpha+\beta}\right)^n e^{-\beta\sum_{j=1}^{n} x_j} \prod_{j=1}^{n}\left(\alpha + \frac{x_j^{\alpha-2}\beta^{\alpha-1}}{\Gamma(\alpha-1)}\right),
\end{aligned}
$$

The log-likelihood is defined as

$$
\ell = ln\ L = n\ ln\left(\frac{\beta}{\alpha+\beta}\right) + \sum_{j=1}^{n} ln\left(\alpha + \frac{x_j^{\alpha-2}\beta^{\alpha-1}}{\Gamma(\alpha-1)}\right) - \beta\sum_{j=1}^{n} x_j \tag{9}
$$

Now, differentiating (9) partially with respect to α and β we have

$$
\begin{aligned}
\frac{\partial\ell}{\partial\alpha} &= -n\left(\frac{1}{\alpha+\beta} + \frac{(n+1)\Gamma'(\alpha-1)}{2\ \Gamma(\alpha-1)}\right) \\
&\quad + \sum_{j=1}^{n}\left[\frac{\Gamma(\alpha-1) + \alpha\Gamma'(\alpha-1) + \beta\ ln(x_j\beta)(x_j\beta)^{\alpha-2}}{\alpha\Gamma(\alpha-1) + \beta(x_j\beta)^{\alpha-2}}\right] \\
\frac{\partial\ell}{\partial\beta} &= \frac{n\alpha}{\beta(\alpha+\beta)} - \sum_{j=1}^{n} x_j + (\alpha-1)\sum_{j=1}^{n}\frac{(x_j\beta)^{\alpha-2}}{\alpha\Gamma(\alpha-1) + \beta(x_j\beta)^{\alpha-2}}
\end{aligned} \tag{10}
$$

The MLE $(\hat{\alpha}, \hat{\beta})$ of (α, β) can be obtained by solving the system of equations $\left\{\frac{\partial\ell}{\partial\alpha} = 0, \frac{\partial\ell}{\partial\beta} = 0\right\}$ The system of equations in (10) has no explicit analytical solution, hence, it can be solved numerically using Newton-Raphson iterative method or any other numerical method.

10 Simulation study

In this section, we achieve a simulation study to examine the performance and accuracy of the Maximum Likelihood Estimates (MLEs) of the Br distribution with the help of R software R Core Team (2020). For this, we will generate $N = 1000$ samples each of size 50, 100, 300, 500 for different values of α and β using (9). For each sample, the MLE of the parameter space $\phi = (\alpha, \beta)$, mean square error (MSE) and the bias are obtained. Then, we calculate the average bias (AB) and the average of mean squared error (MSEs) for the MLE as follows:

$$
AB(\hat{\phi}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\phi} - \phi)
$$

$$
MSEs = \frac{1}{N}\sum_{i=1}^{N}(\hat{\phi} - \phi)^2
$$

The results of this simulation are summarized in table 2. From table 2, it can be seen that the values of the average bias and the average of mean squared error decrease with increasing sample sizes and thus the estimates behave in a standard manner for different values of α and β. Also, it indicates that the MLEs are asymptotically unbiased and consistent.

n		$\alpha = 1.5$	$\beta = 0.5$	$\alpha = 2$	$\beta = 1$	$\alpha = 3$	$\beta = 2$
50	MLE	1.6311	0.5298	2.2531	1.1126	3.4110	2.3309
	AB	0.1311	0.0298	0.2530	0.1126	0.4110	0.3309
	MSEs	0.1435	0.0098	0.4148	0.1128	1.4376	0.8061
100	MLE	1.5494	0.5094	2.2139	1.0773	3.2481	2.1777
	AB	0.0494	0.0094	0.2139	0.0773	0.2481	0.1777
	MSEs	0.041	0.0037	0.2975	0.0513	0.7357	0.3307
300	MLE	1.5173	0.5060	2.1197	1.0440	3.0987	2.0714
	AB	0.0173	0.0061	0.1197	0.0440	0.0987	0.0714
	MSEs	0.0057	0.0011	0.1438	0.0198	0.3560	0.1429
500	MLE	1.5064	0.5026	2.1110	1.0360	3.0274	2.0250
	AB	0.0064	0.0026	0.1110	0.0360	0.0274	0.0250
	MSEs	0.0030	0.0005	0.1122	0.0130	0.2344	0.0858

Table 2: MLE, average bias and the average of mean squared error for the MLE of the br distribution with different values of parameters.

11 Real Data Applications

In this section, we show the flexibility of the Benrabia distribution by considering real life time data set and compare its goodness of fit with some existing distributions. The data set consists of the repair times (in hours) 46 failures of an airborne communications receiver (Chhikara and Folks, 1977) and used by Meraj et al. (2019), the data is as follows

0.2	0.3	0.5	0.5	0.5	0.5	0.6	0.6	0.7	0.7	0.7	0.8
0.8	1.0	1.0	1.0	1.0	1.1	1.3	1.5	1.5	1.5	1.5	2.0
2.0	2.2	2.5	2.7	3.0	3.0	3.3	3.3	4.0	4.0	4.5	4.7
5.0	5.4	5.4	7.0	7.5	8.8	9.0	10.3	22.0	24.5		

The goodness of fit of the proposed distribution is compared with the following distributions:
- Lindley distribution (Merovci and Elbatal, 2014)

$$f_L(x) = \frac{\alpha^2(1+x)e^{-\alpha x}}{1+\alpha} \quad x > 0,\ \alpha > 0$$

- Transmuted Shanker distribution (Al-Zoubi et al., 2021)

$$f_{TS}(x) = \frac{\alpha^2}{\alpha^2+1}(\alpha+x)e^{-\alpha x}\left(1+\beta-2\beta\left(1-\left(\frac{\alpha^2+\alpha x+1}{\alpha^2+1}\right)e^{-\alpha x}\right)\right) \quad x>0, \alpha>0, \beta>0$$

- Exponential distribution (Kingman, 1982)

$$f_E(x) = \alpha e^{-\alpha x} \quad x>0,\ \alpha>0$$

- Gamma distribution (Johnson et al., 1994)

$$f_G(x) = \frac{\beta^\alpha x^{\alpha-1}e^{-\beta x}}{\Gamma(\alpha)} \quad x>0,\ \alpha>0,\ \beta>0.$$

- Weibull distribution (Weibull, 1951)

$$f_W(x) = \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-(x/\beta)^\alpha} \quad x>0, \alpha>0, \beta>0$$

The criteria of choosing the best model are -2lnL, Akaike Information Criterion (AIC) (Akaike, 1974), Akaike Information Criterion Corrected (AICC) (Akaike, 1974), Bayesian Information Criterion (BIC) (Konishi et al., 2004), Haan Quinn Information Criterion (HQIC) (Hannan and Quinn, 1979), Kolmogorov-Smirnov Statistics (KS Statistics) and its p-value (Chakravarti et al., 1967), where

$$\begin{aligned} AIC &= -2lnL + 2k \\ AICC &= AIC + \frac{2k(k+1)}{n-k-1} \\ BIC &= -2lnL + kln(n) \\ HQIC &= 2ln[ln(n)(k-2lnL)] \\ KS &= \sup_x |F_n(x) - F_0(x)|, \end{aligned}$$

where L is the likelihood function, k is the number of parameter, n is the sample size and $F_n(x)$ is the empirical distribution function. For calculation of the analytical measures, the optimum function optim() R-function with the argument method= “N” (R Core Team, 2020). The best distribution is the one which has lower values of -2lnL, AIC, AICC, BIC, HQIC and K-S statistic and higher p-value, the results are given in the table below From table 3, the values of -2lnL, AIC, AICC, BIC, HQIC and KS statistic demonstrate that Benrabia distribution is more flexible than the other distributions. The p-values show that Benrabia distribution is the best fit of the data.

In order to check that the proposed model is appropriate, we provide two graphic illustration which present the histogram of the data set and the fitted distributions and plots of the empirical and estimated distribution functions of the adapted distributions. Also, table 5 shows that the new distribution provides the best fit for the current data because of lower values of Anderson Darling (A*) and Cramer-Von Mises (W*) statistics. This proves that Benrabia distribution is the best distribution that fits the repair time of an airborne communications.

Distributions	$-2lnL$	*AIC*	*AICC*	*BIC*	*HQIC*	$K-S$	$p-value$
Lindley	219.969	221.969	222.060	223.798	222.654	0.234	0.013
Transmuted Shanker	218.472	222.472	222.751	226.129	223.842	0.222	0.022
Exponential	210.012	212.012	212.103	213.841	212.697	0.160	0.191
Gamma	209.862	213.862	214.141	217.519	215.232	0.145	0.285
Weibull	208.939	212.939	213.219	216.597	214.310	0.121	0.517
Benrabia	204.031	208.031	208.310	211.688	209.401	0.120	0.520

Table 3: -2lnL, AIC, AICC, BIC, HQIC and KS statistic and the p-values of the fitted distributions.

Distributions	Parameter Estimates		Std Error	
Lindley	0.466		0.050	
Transmuted Shanker	0.428	0.590	0.054	0.225
Exponential	0.277		0.041	
Gamma	0.932	0.259	0.170	0.062
Weibull	0.899	3.391	0.096	0.591
Benrabia	9.617	0.373	2.715	0.059

Table 4: Mle Estimates and Standard Errors of the fitted distributions

	Statistic	
Model	A*	W*
Lindley	1.3022	0.1923
Transmuted Shanker	1.0618	0.1573
Exponential	0.9961	0.1436
Gamma	0.9944	0.1433
Weibull	0.9010	0.1298
Br	0.6604	0.1066

Table 5: Goodness of fit using Anderson Darling and Cramer-Von Mises statistics.

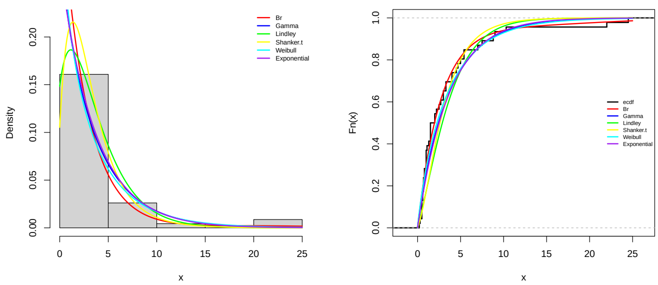


Figure 5: plots of the histogram, pdf of the fitted distributions and the estimated distribution functions of the fitted models.

12 Conclusion

This article proposes a new continuous two parameter distribution called Benrabia distribution. Several statistical properties of this distribution are studied. The moments, moment generating function, reliability analysis. The estimation of model parameters are derived as well as the Rényi entropy and the deviations about the absolute mean and median deviations are presented. An application shows that the suggested distribution is more flexible than some other distributions and provides a better fit for real lifetime data. Corresponding future research related to this work, we may generate new distributions using transmutation map or weighted method. Also, we can calculate the stress-strength reliability for Benrabia distribution

References

Akaike, H. (1974). A new look at the statistical model identification. *IEEE Transactions on Automatic Control*, 19(6):716–723.

Al-Omari, A., Al-khazaleh, A., and Alzoubi, L. (2017a). A generalization of the new weibulll-pareto distribution. *REVISTA INVESTIGACION OPERACIONAL*. accepted for publication.

Al-Omari, A. I., Al-khazaleh, A. M., and Alzoubi, L. M. (2017b). Transmuted janardan distribution: A generalization of the janardan distribution. *Journal of Statistics Applications & Probability*, 5(2):1–11.

Al-zou′bi, L. (2017). Transmuted mukherjee-islam distribution: A generalization of mukherjee-islam distribution. *Journal of Mathematics Research*, 9(4):135–144.

Al-Zoubi, L., Gharaibeh, M., and Alzghool, R. (2021). Transmuted shanker distribution: Properties and applications. *Italian Journal of Pure and Applied Mathematics*, -(-):–. Accepted for publication.

Almawajdeh, N. (2019). Generalization of rani and generalized akash distributions. Master's thesis.

Almousa, B. (2019). Generalization of sujatha and amarendra distributions using quadratic transmutation map. Master's thesis, Al al-Bayt University.

Alsikeek, H. A. (2018). Quadratic transmutation map for reciprocal distribution and two-parameter weighted exponential distribution. Master's thesis, Department of Mathematics, Al al-Bayt University, Mafraq, Jordan.

AzZwideen, R. and Al-Zou'bi (2020). The Transmuted Gamma Gompertz Distribution. *International Journal of Research -GRANTHAALAYAH*, 8(10):236–248.

Benrabia, M. and Alzoubi, L. (2021). Alzoubi distribution: Properties and applications. *Journal of Statistics Applications & Probability: An International Journal*, -(-):1–16. Accepted.

Chakravarti, I., Laha, R., and Roy, J. (1967). *Handbook of Methods of Applied Statistics*, volume I. John Wiley and Sons.

Chhikara, R. and Folks, J. (1977). The inverse gaussian distribution as a lifetime model. *Technometrics*, 19(2):461–468.

Gharaibeh, M. (2021). Gharaibeh Distribution and Its Applications. *Journal of Statistics Applications & Probability*, 10(2):1–12.

Hannan, E. J. and Quinn, B. G. (1979). The determination of the order of an autoregression. *Journal of the Royal Statistical Society. Series B (Methodological)*, 41(2):190–195.

Joanes, D. and Gill, C. (1998). Comparing Measures of Sample Skewness and Kurtosis. *Journal of the Royal Statistical Society. Series D (The Statistician)*, 47(1):183–189.

Johnson, N., Kotz, S., and Balakrishnan, N. (1994). *Continuous Univariate Distributions*, volume 1. Wiley and Sons, Inc., New York, 2nd edition.

Kakwani, N. and Podder, N. (1976). Efficient estimation of the lorenz curve and associated inequality measures from grouped observations. *Econometrica*, 44(1):137–148.

Kingman, J. (1982). The coalescent. *Stochastic Processes and their Applications*, 13(3):235–248.

Konishi, S., Ando, T., and Imoto, S. (2004). Bayesian information criteria and smoothing parameter selection in radial basis function networks. *Biometrika*, 91(1):27–43.

Meraj, A., Asgharzadeh, A., and Hassan, S. (2019). A new compound gamma and lindley distribution with application to failure data. *Austrian Journal of Statistics*, 48:54–75.

Merovci, F. and Elbatal, I. (2014). Transmuted Lindley-Geometric Distribution and its Applications. *Journal of Statistics Applications & Probability*, 3(1):77–91.

Nosakhare, F., Nzei, L., and Opone, F. (2020). A new mixture of exponential-gamma distribution. *Journal of Science*, 33(2):548–564.

Pham-Gia, T. and Hung, T. (2001). The mean and median absolute deviations. *Mathematical and Computer Modelling*, 34(7):921–936.

R Core Team (2020). *R: A Language and Environment for Statistical Computing.* R Foundation for Statistical Computing, Vienna, Austria.

Rabaiah, F. S. (2018). Generalizations of power function and type-i half logistic distributions using quadratic transmutation map. Master's thesis, Department of Mathematics, Al al-Bayt University, Mafraq, Jordan.

Rényi, A. (1961). On measures of entropy and information. *Proceedings of the 4th Berkeley Symposium on Mathematical Statistics and Probability*, 1:547–561.

Saadeh, F. (2019). Generalization to some distributions using transmutation maps. Master's thesis, Al al-Bayt University.

Samir, A., Darwish, D., and Ahmad, S. (2018). Log-gamma-rayleigh distribution: Properties and applications. *GU Journal of Science*, 31(3):967–983.

Shanker, R. (2015). Shanker distribution and its applications. *International Journal of Statistics and Applications*, 5(6):338–348.

Shanker, R. (2016). Aradhana Distribution and its Applications. *International Journal of Statistics and Applications*, 6(1):23–34.

Shanker, R. (2017). Rama distribution and its application. *International Journal of Statistics and Applications*, 7(1):26–35.

Shanker, R. and Shukla, K. K. (2017). Ishita distribution and its applications. *Biometrics & Biostatistics International Journal*, 5(2):1–9.

Shannon, C. E. (1948). A Mathematical Theory of Communication. *Bell System Technical Journal*, 27:379–423, 623–656.

Shraa, D. and Al-Omari, A. (2019). Darna distribution: properties and application. *Electronic Journal of Applied Statistical Analysis*, 12(2):520–541.

Wang, Q. (2008). Probability distribution and entropy as a measure of uncertainty. *Journal of Physics A Mathematical General*, 41(6):1–12.

Weibull, W. (1951). A statistical distribution function of wide applicability. *Journal of Applied Mechanics*, 18:292–297.