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A New Inverse Rayleigh Distribution with Applications of COVID-19 Data: Properties, Estimation Methods and Censored Sample

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This paper aims at modeling the COVID-19 spread in the United Kingdom and United States of America, by specifying an optimal statistical univarite model. A new lifetime distribution with three-parameters is introduced by a combination of inverse Rayleigh distribution and odd Weibull family of distributions to formulate the odd Weibull inverse Rayleigh (OWIR) distribution. Some of the mathematical properties of the OWIR distribution are discussed as linear representation, quantile, moments, function of moment production, hazard rate, stress-strength reliability, and order statistics. Maximum likelihood, maximum product spacing, and Bayesian estimation method are applied to estimate the unknown parameters of OWIR distribution. The parameters of the OWIR distribution are estimated under progressive type-II censoring scheme with random removal. A numerical results of a Monte Carlo simulation is obtained to assess the use of estimation methods.

keywords: Odd Weibull family, inverted Rayleigh distribution, maximum product spacing, Bayesian estimation, COVID-19, censored sample.

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1 Introduction

Lifetime distributions have been received a lot of attention over years. Thus researchers interest have been grown over time for modeling different data. Researchers create anew lifetime distribution either by introducing a new parameter to make the distribution of interest more flexible or perhaps producing a new distribution family or modeling data in several areas such as economics, engineering, reliability and medical sciences, among others [Anake et al. Anake et al. (2015)]. The inverted distributions are of great importance due to their applicability in many areas such as; biological sciences, life test problems, medical, etc. The density and hazard ratio forms of the inverted distributions show a different structure from the conformation of non-inverted distributions. The applications of inverted distributions have been discussed with many researchers, and the reader can refer to Abd AL-Fattah et al. Abd AL-Fattah et al. (2017), Barco et al. **?**, Hassan and Abd-Allah Hassan and Abd-Allah (2019), Muhammed Muhammed (2019), Chesneau et al. Chesneau et al. (2020), Usman and ul Haq Usman and ul Haq (2020), Eferhonore et al. Eferhonore et al. (2020) among others.

Let X be a random variable following inverse Rayleigh (IR) distribution with scale parameter $\lambda \geq 0$. Then the cumulative distribution function (CDF) and probability density function (PDF) are as followig:

$$G(x;\lambda) = e^{-\frac{\lambda}{x^2}}; \quad x \geq 0, \lambda \geq 0 \tag{1}$$

and,

$$g(x;\lambda) = \frac{2\lambda}{x^3} e^{-\frac{\lambda}{x^2}}; \quad x, \lambda > 0 \tag{2}$$

Generalization of different distributions of IR were discussed in different statistical writings by different authors, mostly applied in reliability estimation. For example: Rao and Mbwambo Rao et al. (2019) introduced a generalization of the IR distribution known as exponentiated IR (EIR) distribution. Malik and Ahmad Malik and Ahmad (2019) introduced transmuted alpha power IR (TAPIR) distribution. Al-Omari et al. Al-Omari et al. (2017) suggested exponentiated generalized IR (EGIR) distribution. Almarashi et al. Almarashi et al. (2020) proposed a new extension of IR distribution by using half-logistic transformation. Ahsan ul Haq ul Haq (2016) introduced Kumaraswamy exponentiated IR (KEIR) distribution. Mahdy et al. Mahdy et al. (2019) introduced elicitation IR distribution referred to as compound IR distribution.

Alzaatreh et al. Alzaatreh et al. (2013) discussed a general method of generalized families by using the transformed-transformer (T-X) approach. Based on this approach, Bourguignon et al. Bourguignon et al. (2014) proposed a flexible family called the Weibull-G family, which can be called odd Weibull-G (OW) family. The OW family has received increased attention, which accommodates all five major hazard shapes: increasing, decreasing, constant, unimodal, and bathtub failure rates. The cdf of the OW family is given by

$$F(x;\Theta) = 1 - e^{-\beta\left[\frac{G(x;\lambda)}{1-G(x;\lambda)}\right]^{\alpha}}, x > 0, \alpha > 0, \tag{3}$$

where Θ is vector parameter (α, β, λ). The corresponding PDF of (3) is given by

$$f(x;\Theta) = \alpha\beta \frac{g(x;\lambda)G(x;\lambda)^{\alpha-1}}{[1-G(x;\lambda)]^{\alpha+1}} e^{-\beta\left[\frac{G(x;\lambda)}{1-G(x;\lambda)}\right]^{\alpha}}, x > 0, \alpha > 0, \quad (4)$$

The three-parameters odd Weibull inverse Rayleigh (OWIR) distribution, which has many desirable properties, is obtained in this paper. The OWIR distribution has a very flexible PDF, can be positively skewed, symmetrical and negative skewed, and can allow tails to be more flexible. It is able to model declining, rising, bathtub, upside-down bathtub, and reversed-J hazard rates monotonically. In addition, it has a closed-form CDF and it is very simple to manage, make the distribution a candidate to be used in various fields, such as life testing, durability, biomedical studies and study of survival.

The proposed distribution is very competitive with some conventional distributions with scale and shape parameters as X-Gamma inverse Weibull (XGIW) which was introduced by Ibrahim and Almetwally Ibrahim and Almetwally (2021), the Alpha power inverse Weibull (APIW) which was introduced by Basheer Basheer (2019), generalized inverse Weibull (GIW) which was introduced by De Gusmao et al. De Gusmao et al. (2011), Exponentiated generalized inverse Weibull (EGIW) which was introduced by Elbatal and Muhammed Elbatal and Muhammed (2014), and Exponential Lomax (EL) which was introduced by El-Bassiouny et al. El-Bassiouny et al. (2015).

Different types of censoring schemes exist, including right, left, interval censoring, single or multiple censoring, and type-I or type-II censoring, but conventional type-I and type-II censoring systems do not have the flexibility to allow units to be removed at a point other than the experiment terminal point. The hybrid censoring scheme, which was first introduced by Epstein Epstein (1960), is known as a mixture of type-I and type-II schemes. For this reason, a more general censoring system called the progressive censoring schemes type II is considered here. In Balakrishnan and Aggarwala Balakrishnan and Aggarwala (2000), Balakrishnan Balakrishnan (2007), and Almetwally et al. Almetwally et al. (2018), and Alshenawy et al. Alshenawy et al. (2020) more details on the progressive type-II censoring system can be explored.

The remainder of this paper is structured as follows: We get the OWIR distribution in Section 2. In Section 3, we discuss some of the OWIR distribution's mathematical properties. In Section 4, OWIR distribution methods of estimation are obtained. Censored sample for OWIR distribution is obtained in section 5. A simulation of OWIR distribution results is obtained in section 6. Two implementations of real data analytics were obtained in Section 7. In Section 8, the paper is summarized and concluded.

2 OWIR Distribution

Consider the IR distribution with the positive scale parameter λ and the CDF given (for $x > 0$) by Equations (1,2). We define the CDF of the OWIR distribution, by inserting the CDF of the IR distribution in Equation (3), such as

$$F(x;\Theta) = 1 - e^{-\beta\left[\frac{e^{-\frac{\lambda}{x^2}}}{1-e^{-\frac{\lambda}{x^2}}}\right]^{\alpha}}, x > 0, \alpha, \beta, \lambda > 0, \tag{5}$$

$$f(x;\Theta) = 2\alpha\beta\frac{\lambda}{x^3}\frac{e^{-(\alpha-1)\frac{\lambda}{x^2}}}{\left[1-e^{-\frac{\lambda}{x^2}}\right]^{\alpha+1}}e^{-\beta\left[\frac{e^{-\frac{\lambda}{x^2}}}{1-e^{-\frac{\lambda}{x^2}}}\right]^{\alpha}}, x > 0, \alpha, \beta, \lambda > 0. \tag{6}$$

The hazard rate (HR) function of the OWIR distribution is shown as

$$h(x;\Theta) = 2\alpha\beta\frac{\lambda}{x^3}\frac{e^{-(\alpha-1)\frac{\lambda}{x^2}}}{\left[1-e^{-\frac{\lambda}{x^2}}\right]^{\alpha+1}}. \tag{7}$$

Figures 1 and 2 show some plots of the PDF and HR function for OWIR distribution, respectively, of the OWIR distribution for the values defined for α, β and λ. The diagrams shown in Figures 2 indicate that the HR function in the OWIR distribution can be augmented, decreased, and formed a bathtub. One advantage of the OWIR distribution over an IR distribution is that the last of them cannot model a phenomenon showing increasing, decreasing shapes, bathtub failure rates, and thus becomes more flexible to analyze lifetime data.

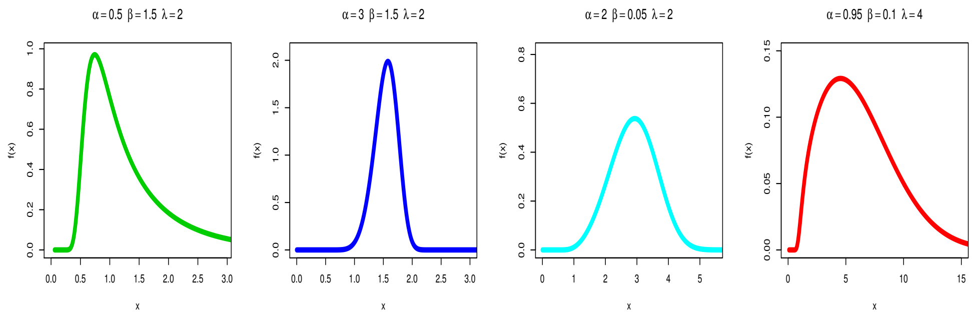

Figure 1: PDF of the OWIR distribution for certain parameter values

3 Some Mathematical Properties of The OWIR Distribution

3.1 Linear Representation for The OWIR Distribution

We provide a useful linear representation for the OW family and use it to provide a useful linear representation for the OWIR distribution. A mixture representation of the OW family can be provided as follows,

$$f(x;\Theta) = \alpha\beta\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}g(x;\lambda)\frac{[G(x;\lambda)]^{\alpha(j+1)-1}}{[1-G(x;\lambda)]^{\alpha(j+1)+1}}. \tag{8}$$

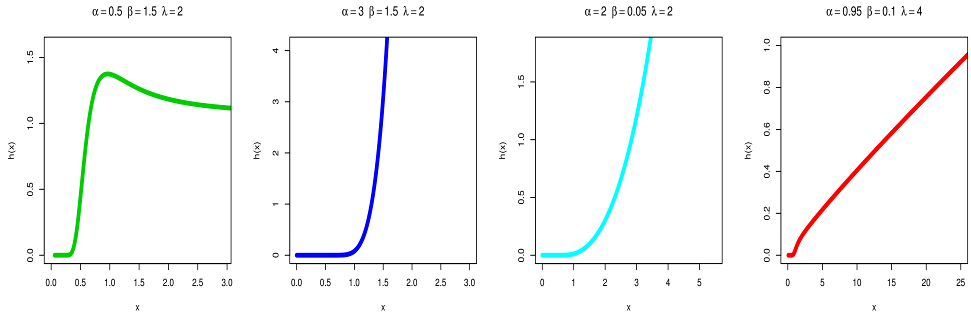

Figure 2: HF of the OWIR distribution for certain parameter values

Using the PDF and CDF of the IR distribution, the last equation of OWIR distribution can be rewritten as

$$f(x;\Theta)=\sum_{j,k=0}^{\infty}\epsilon_{j,k}\frac{2\lambda[\alpha(k+1)+j]}{x^3}e^{-\frac{\lambda[\alpha(k+1)+j]}{x^2}}, \tag{9}$$

where $\epsilon_{j,k}=\frac{\alpha\beta}{\alpha(k+1)+j}\frac{(-1)^{k+j}}{k!}\binom{\alpha(k+1)+j}{j}$. Equation (9) denotes the IR density with parameter $\lambda[\alpha(k+1)+j]$.

3.2 Quantile for The OWIR Distribution

The quantile function of the OWIR distribution, say $x=Q(x)=F(x,\Theta)^{-1}(Q)$ is derived by inverting (5) as follows:

$$x_Q=\sqrt{\frac{\lambda}{\ln\left\{1+\left[\frac{-1}{\beta}\ln\left(1-Q\right)\right]^{\frac{-1}{\alpha}}\right\}}};\ 0<Q<1 \tag{10}$$

In particular, the first quartile, say Q1, the second quartile, say Q2, and the third quartile, say Q3 are obtained by setting $Q=0.25,0.5,0.75$, respectively, in Equation (10).

3.3 Moments for The OWIR Distribution

The r^{th} moment of X follows simply from Equation (9) as

$$\begin{aligned}\acute{\mu}_r&=E(X^r)\\&=\sum_{j,k=0}^{\infty}\epsilon_{j,k}\Gamma\Big(1-\frac{r}{2}\Big)\lambda^{\frac{r}{2}-1}\left(\alpha(k+1)+j\right)^{\frac{r}{2}-1}\end{aligned} \tag{11}$$

The moment generating function is given by

$$\begin{aligned} M_X^{\prime}(t) &= E(e^{xt}) \\ &= \sum_{j,k,r=0}^{\infty} \frac{t^r}{r!} \epsilon_{j,k} \Gamma\Big(1-\frac{r}{2}\Big) \lambda^{\frac{r}{2}-1} \left(\alpha(k+1)+j\right)^{\frac{r}{2}-1} \end{aligned} \tag{12}$$

3.4 Stress-Strength Reliability

Let X and Y are the independent strength and stress random variables observed from OWIR distribution. Then, the stress-strength reliability R is calculated as;

$$\begin{aligned} R = P(Y < X) &= \int_0^{\infty} f(x;\Theta_1) \int_0^{x} f(y;\Theta_2) dy \; dx \\ &= \int_0^{\infty} f(x;\Theta_1) F(x;\Theta_2) dx \end{aligned} \tag{13}$$

Then, we have

$$\begin{aligned} R =& 1 - 2\alpha_1\lambda_1 \sum_{k_1,k_2=0}^{\infty} \sum_{j_1,j_2=0}^{\infty} \frac{(-1)^{k_1+k_2+j_1+j_2}}{k_1!k_2!} \binom{\alpha_1(k_1+1)+j_1}{j_1} \binom{\alpha_2 k_2 + j_2 - 1}{j_2} \\ & \beta_1^{k_1+1} \beta_2^{k_2} \int_0^{\infty} \frac{1}{x^3} e^{-\frac{\lambda_1(\alpha_1(k_1+1)+j_1)+\lambda_2(\alpha_2 k_2+j_2)}{x^2}} dx \end{aligned} \tag{14}$$

The traditional reliability of the stress-strength model for OWIR distribution can be concluded as

$$\begin{aligned} R = 1 - \sum_{k_1,k_2=0}^{\infty} \sum_{j_1,j_2=0}^{\infty} & \frac{(-1)^{k_1+k_2+j_1+j_2}}{k_1!k_2!} \binom{\alpha_1(k_1+1)+j_1}{j_1} \binom{\alpha_2 k_2 + j_2 - 1}{j_2} \\ & \frac{\alpha_1\lambda_1\beta_1^{k_1+1}\beta_2^{k_2}}{\lambda_1\left(\alpha_1(k_1+1)+j_1\right)+\lambda_2\left(\alpha_2 k_2+j_2\right)} \end{aligned} \tag{15}$$

Figure 3 display plots of the stress-strength reliability measure for the OWIR distribution for different values of the parameters as following

3.5 Order Statistics

In life testing, censoring and reliability, moments of order statistics play a great role in predicting time to failure of a certain object by taking into account a few early failures. Suppose $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$ are ordered statistics of a random sample

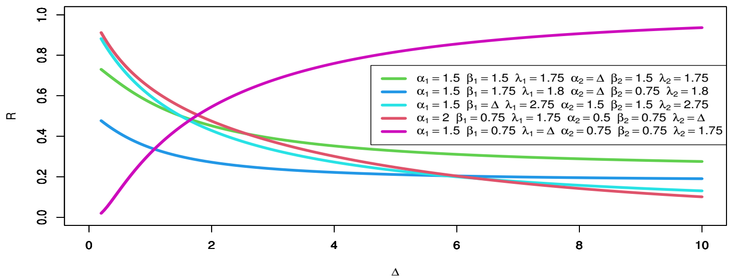

Figure 3: stress-strength reliability of the OWIR distribution for different parameter values

$X_1 < X_2 < ... < X_n$ drawn from OWIR distribution, then the PDF of $X_{k:n}$ is given by

$$f_{X_{k:n}}(x) = 2\alpha\beta \frac{n!}{(k-1)!(n-k)!} \frac{\lambda}{x^3} \frac{e^{-\frac{\lambda(\alpha-1)}{x^2}}}{\left(1-e^{-\frac{\lambda}{x^2}}\right)^{\alpha+1}} e^{-\beta(n-k+1)\left[\frac{e^{-\frac{\lambda}{x^2}}}{1-e^{-\frac{\lambda}{x^2}}}\right]^\alpha} \times \left[1 - e^{-\beta\left[\frac{e^{-\frac{\lambda}{x^2}}}{1-e^{-\frac{\lambda}{x^2}}}\right]^\alpha}\right]^{k-1} \tag{16}$$

4 Estimation Methods

In this section, we study the estimation problem of the OWIR distribution parameters using three different estimation methods called: the maximum likelihood estimators (MLE), maximum product of spacing estimators (MPSE), and Bayesian estimation based on square error loss function.

4.1 Maximum Likelihood Estimators

Let $x_1, ..., x_n$ be a random sample from the OWIR distribution with parameters α, β and λ. the log-likelihood function for OWIR distribution is given by

$$\begin{aligned} l(\Theta) =& n\left[\log(2)+\log(\alpha)+\log(\beta)+log(\lambda)\right] - 3\sum_{j=1}^{n} log(x_j) - \lambda(\alpha-1)\sum_{j=1}^{n} x_j^{-2} \\ &- (\alpha-1)\sum_{j=1}^{n}\log\left[1-e^{-\frac{\lambda}{x_j^2}}\right] - \beta\sum_{j=1}^{n}\left(\frac{e^{-\frac{\lambda}{x_j^2}}}{1-e^{-\frac{\lambda}{x_j^2}}}\right)^\alpha \end{aligned} \tag{17}$$

The partial derivatives of $l(\Theta)$ with respect to the model parameters α, β and λ are

$$\frac{\partial l(\Theta)}{\partial \alpha} = \frac{n}{\alpha} - \lambda \sum_{j=1}^{n} x_j^{-2} - \sum_{j=1}^{n} \log\left[1 - e^{-\frac{\lambda}{x_j^2}}\right] - \beta \sum_{j=1}^{n} \left(\frac{e^{-\frac{\lambda}{x_j^2}}}{1 - e^{-\frac{\lambda}{x_j^2}}}\right)^{\alpha} \log\left(\frac{e^{-\frac{\lambda}{x_j^2}}}{1 - e^{-\frac{\lambda}{x_j^2}}}\right) \quad (18)$$

$$\frac{\partial l(\Theta)}{\partial \beta} = \frac{n}{\beta} - \sum_{j=1}^{n} \left(\frac{e^{-\frac{\lambda}{x_j^2}}}{1 - e^{-\frac{\lambda}{x_j^2}}}\right)^{\alpha} \quad (19)$$

and

$$\begin{aligned} \frac{\partial l(\Theta)}{\partial \lambda} = & \frac{n}{\lambda} - (\alpha - 1) \sum_{j=1}^{n} x_j^{-2} + 2\lambda (\alpha - 1) \sum_{j=1}^{n} x_j^{-3} \frac{e^{-\frac{\lambda}{x_j^2}}}{1 - e^{-\frac{\lambda}{x_j^2}}} \\ & - \beta\alpha \sum_{j=1}^{n} \frac{2\lambda}{x_j^3} \frac{e^{-\frac{\lambda}{x_j^2}} - 2}{1 - e^{-\frac{\lambda}{x_j^2}}} \left(\frac{e^{-\frac{\lambda}{x_j^2}}}{1 - e^{-\frac{\lambda}{x_j^2}}}\right)^{\alpha} \end{aligned} \quad (20)$$

The MLE of α, β and λ can be obtained by maximizing the Equations 18, 19 and 20 with respect to α, β and λ and equaling zero. The R packages can be used to maximize the log-likelihood function to obtain the estimators by using Newton-Rapshon method.

4.2 Maximum Product of Spacings Method

The maximum product of spacings (MPS) method is used to estimate the parameters of continuous univariate models as an alternative to the MLE method which was introduced by Cheng and Amin Cheng and Amin (1983) and used of censored application by Singh et al. Kumar Singh et al. (2016), Almetwally and Almongy Almetwally and Almongy (2019), Basu et al. Basu et al. (2019), Almetwally et al. Almetwally et al. (2019), El-Sherpieny et al. El-Sherpieny et al. (2020) and Alshenawy et al. Alshenawy et al. (2020). Let $x_{(1)} < ... < x_{(n)}$ be a random sample of size n, the uniform spacing of the OWIR distribution can be defined by

$$D_j(\Theta) = F(x_{(i)}, \Theta) - F(x_{(i-1)}, \Theta);\ j = 1, ..., n+1 \quad (21)$$

where $D_j(\Theta)$ denotes to the uniform spacings, $F(x_{(0)}, \Theta) = 0$, $F(x_{(n+1)}, \Theta) = 1$ and $\sum_{j=1}^{n+1} D_j(\Theta) = 1$. The MPS estimators (MPSE) of the OWIR parameters can be obtained by maximizing

$$G(\Theta) = \frac{1}{n+1} \sum_{j=1}^{n+1} \log\left[e^{-\beta\left[\frac{e^{-\frac{\lambda}{x_{(i-1)}^2}}}{1-e^{-\frac{\lambda}{x_{(i-1)}^2}}}\right]^{\alpha}} - e^{-\beta\left[\frac{e^{-\frac{\lambda}{x_{(i)}^2}}}{1-e^{-\frac{\lambda}{x_{(i)}^2}}}\right]^{\alpha}} \right] \quad (22)$$

with respect to α, β and λ. Further, the MPSE of the OWIR parameters can also be obtained by first derivatives with parameters and equal zero.

4.3 Bayesian estimation

The Bayesian approach deals with the parameters as random and parameter uncertainties are represented by a prior joint distribution that is formed prior to the data collected on the failure. The ability to integrate prior knowledge into the research makes the Bayesian approach very useful in the analysis of reliability since the limited availability of data is one of the key problems associated with reliability analysis. In parameters (α, β, λ), we have used informative prior as independent gamma distributions. The independent joint prior density function of α, β, λ can be written as follows:

$$\Pi(\alpha, \beta, \lambda) \propto \alpha^{a_1-1}\beta^{a_2-1}\lambda^{a_3-1}e^{-(b_1\alpha+b_2\beta+b_3\lambda)} \tag{23}$$

To determine elicit hyper-parameters of the independent joint prior, we can use estimate and variance-covariance matrix of MLE method. By equating mean and variance of gamma priors, the estimated hyper-parameters can be written as

$$a_j = \frac{\left[\frac{1}{L}\sum_{j=1}^{L}\hat{\Theta}_j^i\right]^2}{\frac{1}{L-1}\sum_{j=1}^{L}\left[\hat{\Theta}_j^i - \frac{1}{L}\sum_{j=1}^{L}\hat{\Theta}_j^i\right]^2};\ j = 1, ..., p-1,$$

$$b_j = \frac{\frac{1}{L}\sum_{j=1}^{L}\hat{\Theta}_j^i}{\frac{1}{L-1}\sum_{j=1}^{L}\left[\hat{\Theta}_j^i - \frac{1}{L}\sum_{j=1}^{L}\hat{\Theta}_j^i\right]^2};\ j = 1, ..., p-1,$$

where Θ is vector of parameter for OWIR distribution and L is the number of Iteration. The joint posterior density function of Θ is obtained from likelihood function and joint prior function. Then the joint posterior of OWIR distribution can be written as

$$\Pi(\Theta|x) \propto \frac{e^{-\lambda\left(b_3+(\alpha-1)\sum_{j=1}^{n}x_j^{-2}\right)}}{\prod_{j=1}^{n}\left[1-e^{-\frac{\lambda}{x_j^2}}\right]^{\alpha+1}} e^{-\beta\left[\sum_{j=1}^{n}\left(\frac{e^{-\frac{\lambda}{x_j^2}}}{1-e^{-\frac{\lambda}{x_j^2}}}\right)^{\alpha}+b_2\right]}\alpha^{n+a_1-1}\beta^{n+a_2-1}\lambda^{n+a_3-1}e^{-b_1\alpha}, \tag{24}$$

By using the most common of symmetric loss function, which is a squared error loss function. The Bayes estimators of $\hat{\Theta}$ based on squared error loss function is given by

$$\begin{aligned} S(\tilde{\Theta_j}) =& E\left(\tilde{\Theta}_j - \Theta_j\right)^2 \\ & \int_0^{\infty}\left(\tilde{\Theta}_j - \Theta_j\right)^2\Pi(\Theta_j|x)d\Theta_j; \quad j = 1, 2, 3 \end{aligned} \tag{25}$$

It is noted that it is not possible to obtain the integrals given by Equation (25) directly. As a consequence, to find an estimated value of integrals, we use the MCMC. Gibbs sampling and more general Metropolis within Gibbs samplers are an important subclass of the MCMC techniques. Together with the Gibbs sampling, the MH algorithm is the two most popular instances of the MCMC method. Similar to acceptance-rejection sampling, the MH algorithm believes that a candidate value from a proposal distribution can be generated for each iteration of the algorithm. Within the Gibbs sampling steps, we use the MH to generate random samples from the OWIR distribution family of conditional posterior densities:

$$\Pi(\alpha|\beta,\lambda,x) \propto \frac{e^{-\lambda\left((\alpha-1)\sum_{j=1}^{n} x_j^{-2}\right)}}{\prod_{j=1}^{n}\left[1-e^{-\frac{\lambda}{x_j^2}}\right]^{\alpha+1}} e^{-\beta\left[\sum_{j=1}^{n}\left(\frac{e^{-\frac{\lambda}{x_j^2}}}{1-e^{-\frac{\lambda}{x_j^2}}}\right)^{\alpha}+b_2\right]} \alpha^{n+a_1-1} e^{-b_1\alpha}, \tag{26}$$

$$\begin{aligned}\Pi(\beta|\alpha,\lambda,x) &\propto \beta^{n+a_2-1}\, e^{-\beta\left[\sum_{j=1}^{n}\left(\frac{e^{-\frac{\lambda}{x_j^2}}}{1-e^{-\frac{\lambda}{x_j^2}}}\right)^{\alpha}+b_2\right]}, \\ &\sim \Gamma\left(n+a_2, \left[\sum_{j=1}^{n}\left(\frac{e^{-\frac{\lambda}{x_j^2}}}{1-e^{-\frac{\lambda}{x_j^2}}}\right)^{\alpha}+b_2\right]\right)\end{aligned} \tag{27}$$

and

$$\Pi(\lambda|\alpha,\beta,x) \propto \frac{e^{-\lambda\left(b_3+(\alpha-1)\sum_{j=1}^{n} x_j^{-2}\right)}}{\prod_{j=1}^{n}\left[1-e^{-\frac{\lambda}{x_j^2}}\right]^{\alpha+1}} e^{-\beta\left[\sum_{j=1}^{n}\left(\frac{e^{-\frac{\lambda}{x_j^2}}}{1-e^{-\frac{\lambda}{x_j^2}}}\right)^{\alpha}+b_2\right]} \lambda^{n+a_3-1}. \tag{28}$$

In Highest Posterior Density (HPD) intervals, the method of Chen and Shao Chen and Shao (1999) was commonly used to produce the HPD intervals of unknown benefit distribution parameters. Samples drawn using the proposed MH algorithm should be used in this study to produce time-lapse estimates. A 90% per cent HPD interval can be obtained from the percentile tail points, for example, with two endpoints being the 7^th and 95% per cent percentiles from the MCMC sampling outputs respectively. In order to informally search for potential asymmetry in the posterior density of a parameter, it is often helpful to present the posterior median.

5 Censored Sample for OWIR Distribion

There are also cases where units are withdrawn or lost from the test prior to failure in life-testing and reliability studies. In an industrial laboratory, for example, units may split unintentionally, people may drop out of the study in a clinical trial, or they need to be terminated early due to lack of funds. Due to time and expense constraints involved with the experiment, elimination of units before failure is most much a task in certain cases. The data of such tests or experiments are called censored data.

There are numerous forms of censoring schemes, including correct, interval censoring, single or multiple censoring, and type-I or type-II censoring, but traditional type-I and type-II censoring schemes are not versatile in enabling units to be withdrawn at a point other than the experiment terminal point. The hybrid censoring system is known as a combination of type-I and type-II schemes. For this reason, a more general censoring system called a progressive type-II censoring system is considered here. Under the progressive type-II censored sample with random removal, the following assumptions will be used and can be defined as follows.

1. Suppose that n iid observations are placed on a life testing and OWIR distribution has the lifetimes of these units.

2. Generate progressive sample as $x_{1:m:n} < x_{2:m:n} < ... < x_{m:m:n}$, where m is prefixed where $m \leq n$.

3. $x_1, R_1 \sim binomial(n-m, p)$ units are randomly removed from the remaining $(n-1)$ surviving objects at the time of the first failure, x_2, R_2 units are randomly removed from the remaining $n-2-R_1$ units at the time of the second failure, and so on the test continues until the m^{th} failure at which were time all the remaining $n-m-R_1-R_2...R_{m-1}$ units are removed, where $R_j \sim binomial(n-m-\sum_{j=1}^{m-1} R_j, p); j = 2, ..., m$ and $R_m = n-m-\sum_{(j=1)}^{(m-1)} R_j$.

For a vector of parameters, Θ for OWIR distribution, the joint likelihood function based on the progressive type-II censored sample with binomial removal is defined as

$$\begin{aligned} L\left(x_{j:m:n};\Theta\right) \propto & \prod_{j=1}^{m} f\left(x_{j:m:n};\Theta\right)\left[1-F\left(x_{j:m:n};\Theta\right)\right]^{R_j} Pr(R_j=r_j) \\ & \prod_{j=1}^{m} Ł(\Theta|x)\ Pr(R_j=r_j). \end{aligned} \tag{29}$$

The likelihood function for OWIR distribution based on progressive type-II censored sample is defined as

$$Ł(\Theta|x) \propto \alpha^m \beta^m \lambda^m \frac{e^{-\lambda(\alpha-1)\sum_{j=1}^{m} x_{j:m:n}^{-2}}}{\prod_{j=1}^{m}\left[1-e^{-\frac{\lambda}{x_{j:m:n}^2}}\right]^{\alpha+1}} e^{-\beta\sum_{j=1}^{m}(R_j+1)\left(e^{\frac{\lambda}{x_{j:m:n}^2}}-1\right)^{-\alpha}}. \tag{30}$$

Therefore, if we have a binomial distribution with the following probability mass function (PMF) and a number of units removed at each assumed failure time, there is a binomial distribution with the PMF as follows:

$$Pr(R_1=r_1) = \binom{n-m}{r_1} p^{(r_1)}(1-p)^{(n-m-r_1)}$$

while, for $j = 2, 3, ..., m-1$:

$$Pr\left(R_j = r_j | R_1\right) = \binom{n-m-\sum_{i=1}^{j-1} r_i}{r_j} p^{(r_j)}(1-p)^{\left(n-m-r_j-\sum_{i=1}^{j-1} r_i\right)}$$

In addition, we assume that for each j, R_j is independent of $X_{(j:m:n)}$ and Θ. Since the likelihood function given parameters of OWIR distribution does not include the binomial parameter P, the MLE of P can be derived by directly maximizing $Pr(Rj = rj)$.

The joint product spacing function based on the progressive type-II censored sample with binomial removal is defined as

$$S\left(x_{j:m:n};\Theta\right) \propto \prod_{j=1}^{m} D\left(x_{j:m:n};\Theta\right)\left[1-F\left(x_{j:m:n};\Theta\right)\right]^{R_j} Pr(R_j = r_j). \tag{31}$$

The product spacing function for OWIR distribution based on progressive type-II censored sample is defined as

$$D\left(x_{j:m:n};\Theta\right)\left[1-F\left(x_{j:m:n};\Theta\right)\right]^{R_j} \propto e^{-\beta\sum_{j=1}^{m} R_j\left(e^{\frac{\lambda}{x^2_{j:m:n}}}-1\right)^{-\alpha}} \prod_{j=1}^{m+1}\left[e^{-\beta\left[e^{\frac{\lambda}{x^2_{(i-1:m:n)}}}-1\right]^{-\alpha}} - e^{-\beta\left[e^{\frac{\lambda}{x^2_{(i:m:n)}}}-1\right]^{-\alpha}}\right]. \tag{32}$$

6 Simulation Results

In this section, the performance of three different estimators of the OWIR parameters is assessed by a simulation study. We consider different sample sizes $n = 25, 70, 150$ for different parameters values $\alpha = (0.5,\ 2)$, $\beta = (0.5,\ 2)$, and $\lambda = (0.5,\ 3)$. We generate N = 10,000 random samples from OWIR distribution. For each estimate, we obtain the Bias, their corresponding mean squares error (MSE) and length of confidence interval (L.CI) and length of HDI (L.HDI) for Bayesian estimation. In censored sample, we discuss two different cases when $\Theta = (0.5, 0.5, 0.5)$ and $\Theta = (0.25, 1.9, 1.7)$ see Tables 3 and 4.

In terms of Bias, MSE, and L.CI, the output of various estimators is evaluated, which is the most effective way of estimating those whose values of Bias, MSE, and L.CI are closer to zero. Results of the simulation are obtained through the R program. For MLE, MPS, and Bayesian estimation, Tables 1, 2, 3 and 4 demonstrate the Bias, MSE, and L.CI.

Table 1: Bias, MSE and L.CI of the MLE, MPS, and Bayesian estimate for OWIR distribution when $\alpha = 0.5$.

				MLE			MPS			Bayesian		
β	λ	n		Bias	MSE	L.CI	Bias	MSE	L.CI	Bias	MSE	L.HDI
0.5	0.5	25	α	0.0004	0.0131	0.2145	-0.0211	0.0120	0.4207	0.0003	0.0030	0.2145
			β	0.0746	0.0988	0.5876	0.0294	0.0844	1.1335	0.0744	0.0280	0.5875
			λ	0.1074	0.1615	0.7749	0.0380	0.1368	1.4428	0.1072	0.0505	0.7750
		70	α	0.0000	0.0044	0.1121	-0.0054	0.0043	0.2553	0.0000	0.0008	0.1121
			β	0.0345	0.0287	0.3109	-0.0046	0.0292	0.6696	0.0344	0.0075	0.3109
			λ	0.0513	0.0329	0.4028	-0.0083	0.0335	0.7174	0.0513	0.0132	0.4027
		150	α	-0.0002	0.0020	0.0795	-0.0051	0.0020	0.1738	-0.0003	0.0004	0.0795
			β	0.0178	0.0137	0.1903	0.0003	0.0138	0.4610	0.0178	0.0027	0.1903
			λ	0.0244	0.0144	0.2252	-0.0051	0.0145	0.4722	0.0244	0.0039	0.2251
	3	25	α	0.0004	0.0135	0.2278	-0.0196	0.0076	0.3329	0.0003	0.0034	0.2278
			β	0.0851	0.1114	0.6731	0.0049	0.0258	0.6295	0.0854	0.0368	0.6735
			λ	0.3822	3.4526	3.4408	-0.0551	0.5139	2.8032	0.3622	2.5986	2.4382
		70	α	0.0001	0.0044	0.1124	-0.0102	0.0025	0.1928	0.0001	0.0008	0.1123
			β	0.0409	0.0287	0.3076	0.0056	0.0090	0.3722	0.0409	0.0078	0.3074
			λ	0.3262	0.9216	2.4276	-0.0477	0.1609	1.5620	0.0833	0.1489	1.4264
		150	α	-0.0012	0.0021	0.0774	-0.0090	0.0012	0.1331	-0.0013	0.0004	0.0774
			β	0.0178	0.0140	0.1894	0.0029	0.0044	0.2606	0.0178	0.0026	0.1893
			λ	0.1449	0.5371	1.3686	-0.0356	0.0682	1.0150	0.0291	0.0591	1.0037
3	0.5	25	α	0.0122	0.0461	0.5130	-0.0390	0.0340	0.7082	0.0127	0.0172	0.5126
			β	0.2958	0.2160	3.3020	-0.1848	0.2022	1.6108	0.2941	0.7898	3.2953
			λ	0.1032	0.0271	0.6917	0.0832	0.0602	0.9073	0.1022	0.0415	0.6921
		70	α	-0.0026	0.0164	0.2932	-0.0333	0.0116	0.4008	-0.0026	0.0056	0.2931
			β	0.1519	0.3393	1.7751	-0.0466	0.0598	0.9414	0.1517	0.2277	1.7745
			λ	0.0620	0.0286	0.4830	0.0516	0.0225	0.5529	0.0620	0.0190	0.4828
		150	α	-0.0005	0.0054	0.1744	-0.0224	0.0052	0.2676	-0.0005	0.0020	0.1744
			β	0.0655	0.0339	1.0605	-0.0167	0.0200	0.5502	0.0654	0.0773	1.0602
			λ	0.0292	0.0059	0.2314	0.0306	0.0073	0.3128	0.0292	0.0043	0.2313
	3	25	α	0.0224	0.0526	0.5877	0.0127	0.0273	0.6475	0.0217	0.0229	0.5881
			β	0.3742	0.9504	3.3391	-0.3363	0.7096	3.0346	0.3757	0.8605	3.3324
			λ	0.5945	0.8529	4.1995	-0.1475	0.6550	3.1266	0.6010	1.5074	4.2064
		70	α	0.0053	0.0084	0.2779	-0.0142	0.0067	0.3172	0.0052	0.0050	0.2776
			β	0.1235	0.2381	1.7818	-0.1174	0.1875	1.6364	0.1234	0.2205	1.7789
			λ	0.2937	0.3006	2.2663	0.0658	0.2751	2.0432	0.2949	0.4194	2.2640
		150	α	0.0071	0.0044	0.1718	-0.0096	0.0030	0.2111	0.0071	0.0019	0.1711
			β	0.0946	0.1462	1.0918	-0.0190	0.0898	1.1768	0.0931	0.0853	1.0894
			λ	0.1138	0.1574	1.2054	0.0391	0.1102	1.2974	0.1137	0.1060	1.2006

Table 2: Bias, MSE and L.CI of the MLE, MPS, and Bayesian estimate for OWIR distribution when $\alpha = 2$.

				MLE			MPS			Bayesian		
β	λ	n		Bias	MSE	L.CI	Bias	MSE	L.CI	Bias	MSE	L.HDI
0.5	0.5	25	α	-0.1354	0.5635	0.7178	-0.2014	0.0555	0.5115	-0.1135	0.0414	0.7075
			β	0.3356	0.2581	1.8882	-0.0128	0.0632	1.0522	0.2685	0.1421	1.1092
			λ	0.0756	0.5728	0.2062	-0.0659	0.0241	0.5887	0.0505	0.0090	0.3368
		70	α	-0.0159	0.2723	0.7123	-0.0553	0.0252	0.5889	-0.0211	0.0334	0.7190
			β	0.1877	0.2377	1.5218	-0.0003	0.0775	1.1025	0.1820	0.1200	1.5767
			λ	0.0340	0.8829	0.4421	-0.0175	0.0083	0.3544	0.0375	0.0142	0.4476
		150	α	-0.0149	0.1168	0.5186	-0.0492	0.0182	0.4950	-0.0131	0.0175	0.5192
			β	0.1131	0.1956	1.1710	0.0069	0.0290	0.6714	0.1086	0.1005	1.1749
			λ	0.0135	0.1034	0.3553	-0.0091	0.0044	0.2585	0.0117	0.0084	0.3589
	2	25	α	0.0026	0.5897	0.1531	-0.0944	0.1117	1.2576	0.0026	0.0015	0.1530
			β	0.1085	0.7122	0.9021	0.0444	0.1403	1.4585	0.1084	0.0646	0.9017
			λ	0.0907	0.5777	1.2194	-0.0475	0.2922	2.1117	0.0907	0.1048	1.2188
		70	α	0.0034	0.2389	0.2099	-0.0424	0.0360	0.7251	0.0034	0.0029	0.2099
			β	0.0862	0.5260	0.8130	0.0117	0.0627	0.9812	0.0862	0.0504	0.8126
			λ	0.0565	0.3066	1.0771	-0.0394	0.1304	1.4078	0.0566	0.0786	1.0766
		150	α	0.0009	0.1195	0.2195	-0.0258	0.0169	0.5003	0.0009	0.0031	0.2194
			β	0.0744	0.3636	0.7936	-0.0076	0.0316	0.6971	0.0741	0.0465	0.7938
			λ	0.0449	0.2101	1.0770	-0.0467	0.0732	1.0449	0.0446	0.0774	1.0770
2	0.5	25	α	-0.0078	0.1270	2.0084	-0.1304	0.1168	1.2393	-0.0087	0.1262	2.0093
			β	0.7248	0.1963	6.6316	-0.0394	0.2171	1.8213	0.7321	0.1843	6.6734
			λ	0.0704	0.0031	0.9196	0.0058	0.0038	0.2413	0.0712	0.0604	0.9228
		70	α	0.0141	0.0396	1.1983	-0.0677	0.0391	0.7293	0.0144	0.0393	1.1975
			β	0.5553	0.0114	5.9155	-0.0436	0.0962	1.2051	0.5531	0.0106	5.9132
			λ	0.0312	0.0005	0.6625	0.0011	0.0015	0.1496	0.0310	0.0006	0.6621
		150	α	0.0157	0.0195	0.8666	-0.0486	0.0209	0.5347	0.0146	0.0149	0.8609
			β	0.3736	0.0274	4.8779	0.0043	0.0596	0.9578	0.3723	0.0268	4.8704
			λ	0.0103	0.0005	0.4451	0.0038	0.0008	0.1087	0.0102	0.0004	0.4448
	2	25	α	0.0238	0.1486	1.7177	-0.0876	0.1012	1.2040	0.0243	0.1891	1.7099
			β	0.6919	1.3508	3.2970	-0.0947	0.4444	2.5985	0.5676	0.9587	3.0967
			λ	0.2369	0.2411	3.3572	0.0017	0.1428	1.4883	0.2314	0.1777	2.3493
		70	α	-0.0048	0.0862	1.6405	-0.0529	0.0425	0.7818	-0.0051	0.1742	1.6384
			β	0.6395	0.9054	2.5553	-0.0582	0.2095	1.7821	0.6379	0.8189	1.9548
			λ	0.2236	0.3560	2.3469	-0.0118	0.0415	0.7980	0.2231	0.1273	2.0341
		150	α	-0.0770	0.0218	0.9683	-0.0364	0.0175	0.5037	-0.0734	0.0645	0.9632
			β	0.9296	0.1931	1.9973	-0.0653	0.1089	1.2810	0.9089	0.1209	1.6167
			λ	0.2936	0.0319	1.5456	-0.0134	0.0173	0.5179	0.2862	0.0310	1.4385

Table 3: The MLE, MPS, and Bayesian of the OWIR parameters under progressive type-II censoring scheme with binomial removals for Case I.

$a\alpha = 0.5; \beta = 0.5; \lambda = 0.5$												
				MLE			MPS			Bayesian		
n	p	m		Bias	MSE	L.CI	Bias	MSE	L.CI	Bias	MSE	L.HDI
25	0.3	18	α	0.0197	0.0241	0.2960	-0.0052	0.0196	0.5488	0.0200	0.0061	0.2810
			β	0.0729	0.1897	0.7384	0.0407	0.1271	1.3901	0.0727	0.0406	0.6438
			λ	0.1843	0.4644	1.0145	0.1597	0.3105	2.0956	0.1839	0.1005	0.8934
		22	α	0.0096	0.0177	0.2570	-0.0064	0.0147	0.4741	0.0096	0.0044	0.2059
			β	0.0847	0.1127	0.6893	0.0238	0.0942	1.1999	0.0846	0.0380	0.1768
			λ	0.1458	0.2060	0.9036	0.0636	0.1652	1.5745	0.1459	0.0743	0.1785
25	0.8	18	α	0.0182	0.0223	0.2992	-0.0079	0.0183	0.5299	0.0182	0.0061	0.2682
			β	0.0829	0.1585	0.7044	0.0481	0.1209	1.3506	0.0829	0.0391	0.6910
			λ	0.1902	0.4413	1.0106	0.1554	0.3249	2.1509	0.1900	0.1025	0.9938
		22	α	0.0087	0.0198	0.2465	-0.0107	0.0167	0.5051	0.0087	0.0040	0.2354
			β	0.0844	0.1290	0.6902	0.0389	0.1049	1.2610	0.0845	0.0381	0.6449
			λ	0.1459	0.2583	0.9594	0.0859	0.2048	1.7425	0.1458	0.0810	0.7345
70	0.3	50	α	0.0047	0.0068	0.1518	-0.0078	0.0067	0.3204	0.0047	0.0015	0.1471
			β	0.0505	0.0472	0.4089	0.0273	0.0458	0.8324	0.0505	0.0134	0.3661
			λ	0.1082	0.0790	0.5856	0.0733	0.0744	1.0304	0.1080	0.0340	0.4464
		60	α	0.0025	0.0055	0.1341	-0.0070	0.0054	0.2881	0.0026	0.0012	0.1213
			β	0.0404	0.0363	0.3497	0.0113	0.0362	0.7446	0.0403	0.0096	0.3008
			λ	0.0768	0.0532	0.4724	0.0320	0.0518	0.8836	0.0767	0.0204	0.4127
70	0.8	50	α	0.0030	0.0067	0.1451	-0.0123	0.0067	0.3183	0.0030	0.0014	0.1468
			β	0.0562	0.0538	0.4242	0.0415	0.0515	0.8753	0.0562	0.0148	0.3783
			λ	0.0931	0.0736	0.5308	0.0658	0.0715	1.0164	0.0932	0.0270	0.4962
		60	α	0.0002	0.0053	0.1305	-0.0147	0.0054	0.2831	0.0002	0.0011	0.1306
			β	0.0507	0.0412	0.3656	0.0361	0.0398	0.7690	0.0508	0.0113	0.3138
			λ	0.0801	0.0588	0.4920	0.0508	0.0553	0.9008	0.0801	0.0221	0.4010
150	0.3	120	α	0.0000	0.0025	0.0849	-0.0110	0.0026	0.1952	0.0000	0.0005	0.0834
			β	0.0268	0.0194	0.2302	0.0246	0.0191	0.5333	0.0268	0.0042	0.2139
			λ	0.0474	0.0235	0.2811	0.0359	0.0225	0.5707	0.0473	0.0074	0.2594
		140	α	-0.0014	0.0022	0.0794	-0.0104	0.0022	0.1805	-0.0014	0.0004	0.0781
			β	0.0219	0.0160	0.2004	0.0153	0.0154	0.4831	0.0220	0.0031	0.1948
			λ	0.0337	0.0201	0.2591	0.0171	0.0187	0.5318	0.0336	0.0055	0.2348
	0.8	120	α	-0.0004	0.0026	0.0869	-0.0090	0.0026	0.1965	-0.0004	0.0005	0.0853
			β	0.0280	0.0174	0.2191	0.0174	0.0172	0.5097	0.0280	0.0039	0.2094
			λ	0.0431	0.0202	0.2738	0.0222	0.0196	0.5414	0.0430	0.0067	0.2522
		140	α	-0.0009	0.0020	0.0805	-0.0094	0.0020	0.1735	-0.0009	0.0004	0.0786
			β	0.0226	0.0148	0.2035	0.0131	0.0142	0.4648	0.0226	0.0032	0.1906
			λ	0.0318	0.0173	0.2433	0.0109	0.0162	0.4978	0.0318	0.0049	0.2263

Table 4: The MLE, MPS, and Bayesian of the OWIR parameters under progressive type-II censoring scheme with binomial removals for Case II.

$\alpha = 0.25; \beta = 1.9; \lambda = 1.7$												
				MLE			MPS			Bayesian		
n	p	m		Bias	MSE	L.CI	Bias	MSE	L.CI	Bias	MSE	L.HDI
25	0.3	18	α	0.0277	0.0432	0.2979	0.0337	0.0199	0.5366	0.0276	0.0065	0.2681
			β	0.1713	0.6312	2.1049	-0.2611	0.3183	1.9616	0.1711	0.3171	1.5767
			λ	0.3904	1.5017	2.6249	0.0123	0.4377	2.5942	0.3910	0.6007	2.2776
		22	α	0.0136	0.0263	0.2382	0.0110	0.0114	0.4157	0.0137	0.0039	0.2221
			β	0.1929	0.5195	1.8074	-0.1369	0.2376	1.8347	0.1928	0.2494	1.5311
			λ	0.3446	1.3216	2.4122	0.0447	0.3822	2.4184	0.3443	0.4965	2.0255
25	0.8	18	α	0.0257	0.0434	0.2941	0.0279	0.0195	0.5360	0.0256	0.0063	0.2591
			β	0.2370	0.6431	2.1737	-0.1680	0.2962	2.0304	0.2374	0.3635	1.7952
			λ	0.3914	1.6417	2.5696	0.0593	0.4616	2.6543	0.3916	0.5822	2.1951
		22	α	0.0150	0.0268	0.2385	0.0141	0.0124	0.4336	0.0152	0.0040	0.2179
			β	0.1809	0.5193	1.8038	-0.1493	0.2480	1.8631	0.1817	0.2450	1.4479
			λ	0.3044	1.1792	2.2589	0.0135	0.3965	2.4689	0.3036	0.4242	2.0125
70	0.3	50	α	0.0075	0.0068	0.1466	0.0062	0.0033	0.2251	0.0075	0.0015	0.1466
			β	0.0844	0.1987	0.9508	-0.0964	0.0968	1.1602	0.0844	0.0658	0.9503
			λ	0.2337	0.4986	1.4810	0.0323	0.1290	1.4028	0.2335	0.1970	1.4803
		60	α	0.0038	0.0056	0.1177	0.0023	0.0028	0.2057	0.0039	0.0009	0.1088
			β	0.0690	0.1567	0.7910	-0.0853	0.0722	0.9996	0.0690	0.0454	0.7277
			λ	0.1622	0.4312	1.1641	-0.0019	0.1067	1.2812	0.1618	0.1143	1.0738
70	0.8	50	α	0.0060	0.0070	0.1356	0.0058	0.0034	0.2268	0.0060	0.0012	0.1279
			β	0.0941	0.2131	0.9743	-0.0913	0.0958	1.1597	0.0938	0.0705	0.8650
			λ	0.2124	0.5580	1.4209	0.0073	0.1306	1.4173	0.2123	0.1762	1.1638
		60	α	0.0043	0.0055	0.1160	0.0016	0.0028	0.2076	0.0043	0.0009	0.1153
			β	0.0687	0.1709	0.8511	-0.0818	0.0745	1.0210	0.0689	0.0518	0.7633
			λ	0.1348	0.3814	1.0867	-0.0176	0.0979	1.2249	0.1348	0.0949	0.9184
150	0.3	120	α	0.0004	0.0022	0.0758	-0.0078	0.0010	0.1233	0.0001	0.0004	0.0726
			β	0.0388	0.0792	0.5383	-0.0148	0.0452	0.8314	0.0388	0.0203	0.5244
			λ	0.1048	0.2062	0.7447	0.0832	0.0957	1.1687	0.1047	0.0470	0.7137
		140	α	0.0003	0.0018	0.0665	-0.0035	0.0010	0.1204	0.0003	0.0003	0.0650
			β	0.0274	0.0622	0.4817	-0.0411	0.0374	0.7408	0.0275	0.0158	0.4599
			λ	0.0673	0.1348	0.5887	0.0165	0.0608	0.9651	0.0673	0.0270	0.5591
	0.8	120	α	0.0019	0.0023	0.0764	-0.0062	0.0011	0.1252	0.0019	0.0004	0.0739
			β	0.0334	0.0719	0.5035	-0.0192	0.0419	0.7992	0.0334	0.0176	0.4954
			λ	0.0810	0.1699	0.6478	0.0635	0.0828	1.1007	0.0810	0.0338	0.6228
		140	α	0.0010	0.0022	0.0704	-0.0016	0.0011	0.1293	0.0010	0.0003	0.0715
			β	0.0318	0.0655	0.4666	-0.0355	0.0378	0.7498	0.0320	0.0152	0.4435
			λ	0.0568	0.1477	0.5961	-0.0004	0.0670	1.0154	0.0571	0.0264	0.5539

7 Application of Real Data Analysis

This section is devoted to illustrate the potentiality of the OWIR distribution for two real data sets. OWIR distribution is compared with other competitive models, namely: X-Gamma inverse Weibull (XGIW) [Ibrahim and Almetwally Ibrahim and Almetwally (2021)], the Alpha power inverse Weibull (APIW) [Basheer Basheer (2019)], generalized inverse Weibull (GIW) [De Gusmao et al. De Gusmao et al. (2011)], Exponentiated generalized inverse Weibull (EGIW) [Elbatal and Muhammed Elbatal and Muhammed (2014)], and Exponential lomax (EL) [El-Bassiouny et al. El-Bassiouny et al. (2015)].

Tables 5, 7 provide values of Cramér-von Mises (W*), Anderson-Darling (A*) and Kolmogorov- Smirnov (KS) statistic along with its P-value for the all models fitted based on Two real data sets. In addition, these tables contain the MLE and standard errors (SE) of the parameters for the considered models. In tables 5, 7, the OWIR distribution has the highest p-value and the lowest distance of Kolmogorov-Smirnov(KS), W* and A* value when it compares with all other models that used here to fit the COVID-19 and March precipitation data. Figures 4 and 6 shows the fit empirical, and histogram for the OWIR distribution for March precipitation data and COVID-19 data of the United Kingdom and United States of America. By Tables 6, 8, the Bayesian estimation method of OWIR distribution is best estimation method. History plots, approximate marginal posterior density and MCMC convergence of parameters of OWIR distribution are represented in Figures 5 and 7. Firstly: The data represents a COVID-19 data belong to The United Kingdom of 66 days, from 15 October to 19 December 2020 [https://covid19.who.int/]. these data formed of drought mortality rate. The data are as follows: 0.2240 0.2189 0.2105 0.2266 0.0987 0.1147 0.3353 0.2563 0.2466 0.2847 0.2150 0.1821 0.1200 0.4206 0.3456 0.3045 0.2903 0.3377 0.1639 0.1350 0.3866 0.4678 0.3515 0.3232 0.3678 0.1365 0.1666 0.4491 0.4930 0.4541 0.2969 0.3573 0.1275 0.1591 0.4402 0.3840 0.3579 0.3599 0.2363 0.0858 0.3252 0.4098 0.4630 0.3281 0.3394 0.3095 0.1379 0.1292 0.3805 0.4049 0.2564 0.3091 0.2413 0.1390 0.1127 0.3547 0.3126 0.2991 0.2428 0.2942 0.0807 0.1285 0.2775 0.3311 0.2825 0.2559. .

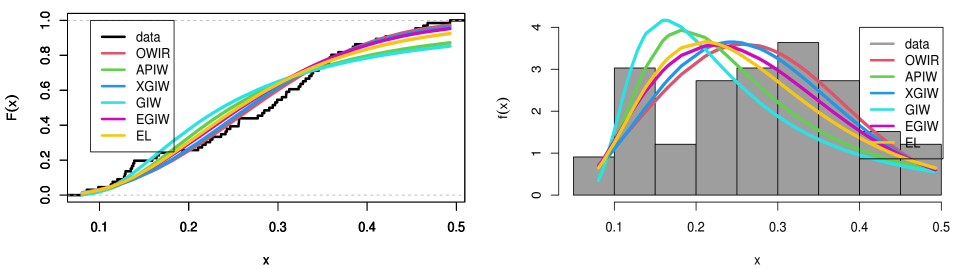

Figure 4: CDF and PDF for COVID-19 data of The United Kingdom with different models.

Table 5: MLE, SE, KS test and P-values for COVID-19 data of The United Kingdom

		α	β	λ	θ	KS	P-Value	W*	A*
OWIR	estimate	1.3408	0.0280	0.0064		0.1040	0.4431	0.1196	0.7318
	SE	0.1337	0.0152	0.0016					
APIW	estimate	22.1227	2.4712	0.0075		0.1457	0.1096	0.5340	2.9409
	SE	25.0265	0.1931	0.0033					
XGIW	estimate	9735.3996	0.2878	6.5239		0.1043	0.4401	0.1654	0.9917
	SE	13039.4641	0.0471	1.2742					
GIW	estimate	0.3611	0.2975	2.0121		0.1640	0.0510	0.6422	3.4744
	SE	4.6196	7.6586	0.1774					
EGIW	estimate	1.9802	53.8731	0.8344	0.4356	0.1252	0.2318	0.2102	1.2108
	SE	0.7855	34.2467	0.2070	0.2608				
EL	estimate	7.2209	57.0067	5.9600		0.1314	0.1873	0.3052	1.7469
	SE	1.7793	68.8318	7.5322					

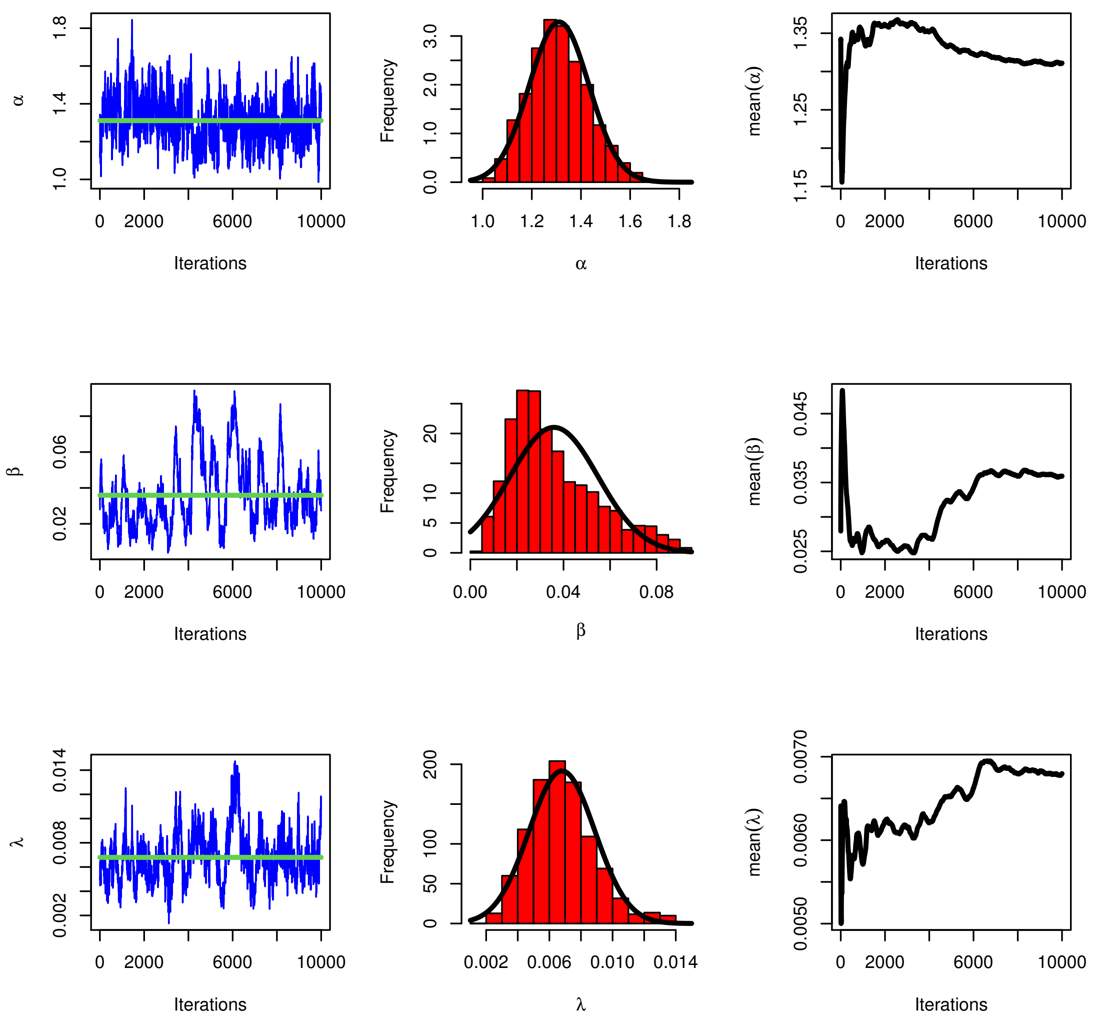

Figure 5: Convergence of MCMC estimation of OWIR for COVID-19 data of The United Kingdom under complete sample.

Table 6: MLE, MPS, and Bayesian under progressive censored sample for COVID-19 data of The United Kingdom

m	P		MLE		MPS		Bayesian	
			coef	SE	coef	SE	coef	SE
50	0.3	α	1.3005	0.1615	1.3408	0.1360	1.2804	0.1329
		β	0.0790	0.0741	0.0280	0.0155	0.0960	0.0506
		λ	0.0119	0.0060	0.0063	0.0017	0.0130	0.0051
	0.8	α	1.3411	0.1378	1.3408	0.1459	1.3294	0.1166
		β	0.0313	0.0179	0.0280	0.0170	0.0392	0.0172
		λ	0.0069	0.0020	0.0064	0.0018	0.0075	0.0019
60	0.3	α	1.3020	0.1354	1.3408	0.1321	1.2773	0.1187
		β	0.0410	0.0255	0.0280	0.0151	0.0514	0.0224
		λ	0.0076	0.0024	0.0064	0.0016	0.0083	0.0024
	0.8	α	1.3408	0.1260	1.3408	0.1363	1.3588	0.1093
		β	0.0280	0.0137	0.0279	0.0156	0.0307	0.0129
		λ	0.0064	0.0016	0.0064	0.0017	0.0069	0.0016
66 complete sample		α	1.3408	0.1337	1.3408	0.1361	1.3111	0.1213
		β	0.0280	0.0152	0.0279	0.0157	0.0359	0.0149
		λ	0.0064	0.0016	0.0065	0.0016	0.0068	0.0015

Secondly: The data represents a COVID-19 data belong to United States of America of 49 days, from 1 November to 19 December 2020 [https://covid19.who.int/]. these data formed of drought mortality rate. The data are as follows: 0.0498 0.0500 0.0564 0.0611 0.0665 0.0695 0.0702 0.0731 0.0775 0.0875 0.0911 0.0923 0.0941 0.0944 0.1006 0.1056 0.1100 0.1109 0.1114 0.1117 0.1152 0.1154 0.1200 0.1214 0.1236 0.1255 0.1285 0.1304 0.1438 0.1463 0.1467 0.1606 0.1621 0.1631 0.1634 0.1737 0.1743 0.1794 0.1818 0.1818 0.1822 0.1834 0.1867 0.1951 0.2039 0.2073 0.2075 0.2090 0.2272.

8 Summary

In this paper, we propose a new three-parameters model, called the OWIR distribution which can be denoted as OWIR distribution. The OWIR distribution is motivated by the wide utilization of the IR model in life testing and provides more flexibility to analyze lifetime data. Survival function, hazard function, linear representation, quantile, moments, stress-strength, and order statistics of the OWIR distribution are provided. We compare MLE, MPS, and Bayesian estimation methods and we conclude that the alternative methods of MLE are better than MLE method. The Bayesian estimation

Table 7: MLE, SE, KS test and P-values for COVID-19 data of United States of America

		α	β	λ	θ	KS	P-Value	W*	A*
OWIR	estimate	1.3019	0.1242	0.0039		0.1065	0.6092	0.0644	0.4365
	SE	0.1531	0.0495	0.0006					
APIW	estimate	1.9660	2.5200	0.0024		0.1342	0.3121	0.2168	1.4351
	SE	1.2125	0.0954	0.0004					
XGIW	estimate	71.5267	0.6419	1.2150		0.1172	0.4756	0.0788	0.5702
	SE	95.1866	0.1887	0.8295					
GIW	estimate	0.2743	0.0876	2.3834		0.1350	0.3056	0.2406	1.5686
	SE	0.9584	0.7285	0.2469					
EGIW	estimate	0.6871	130.8556	1.3118	0.1548	0.1086	0.5721	0.0521	0.3633
	SE	0.0357	1.1072	0.0443	0.0410				
EL	estimate	10.5424	38.9771	1.6659		0.1112	0.5423	0.0973	0.7077
	SE	3.3741	47.7724	2.1815					

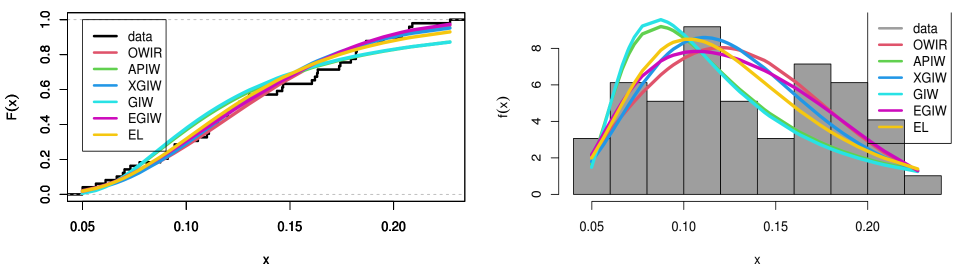

Figure 6: CDF and PDF for COVID-19 data of United States of America with different models.

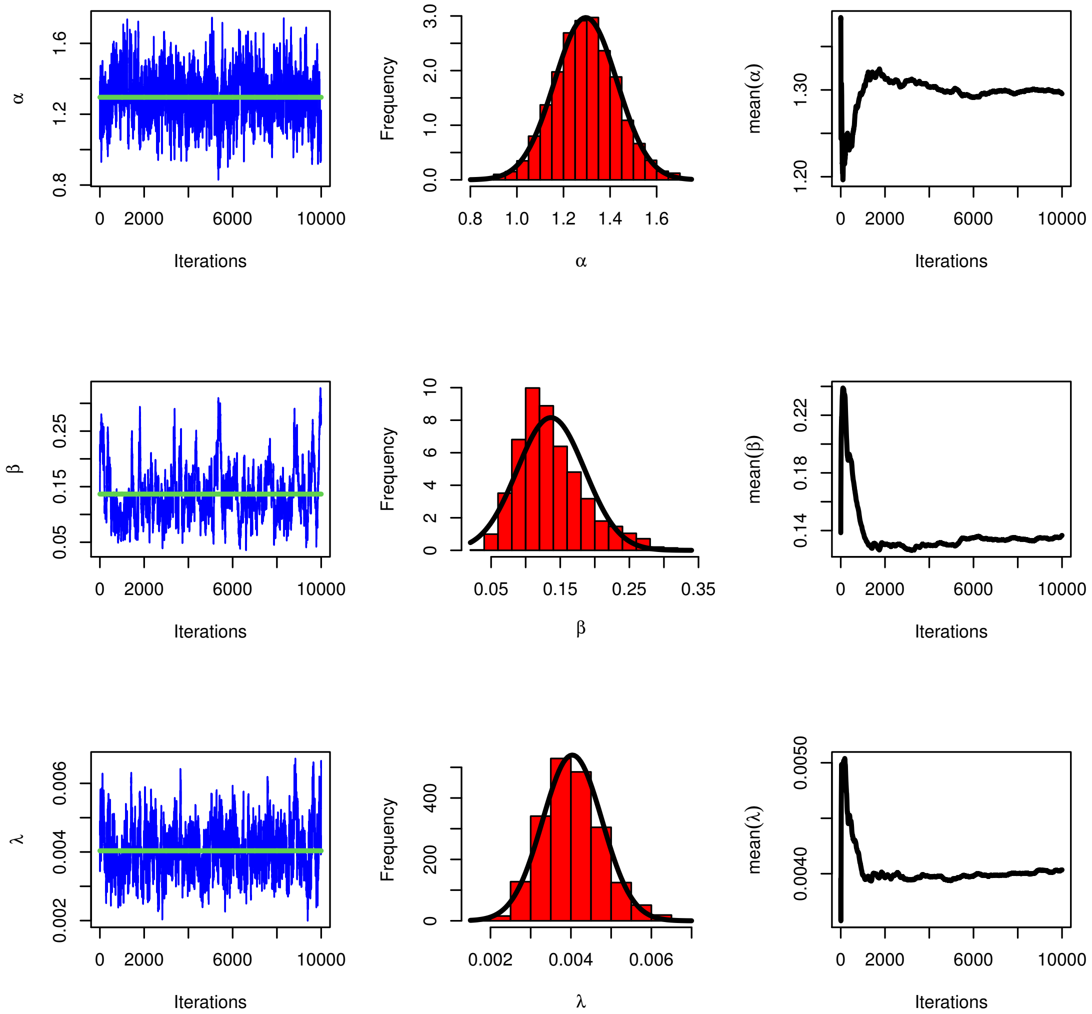

Figure 7: Convergence of MCMC estimation of OWIR for COVID-19 data of United States of America under complete sample.

Table 8: MLE, MPS, and Bayesian under progressive censored sample for COVID-19 data of United States of America

m	P		MLE		MPS		Bayesian	
			coef	SE	coef	SE	coef	SE
32	0.3	α	1.3189	0.1806	1.3050	0.1787	1.2989	0.1646
		β	0.1698	0.0920	0.1293	0.0639	0.2088	0.0805
		λ	0.0049	0.0012	0.0041	0.0008	0.0055	0.0012
	0.8	α	1.1140	0.1854	1.0409	0.1775	1.0911	0.1646
		β	0.4956	0.3235	0.4704	0.3099	0.6544	0.3141
		λ	0.0076	0.0028	0.0072	0.0028	0.0086	0.0031
42	0.3	α	1.3160	0.1618	1.3050	0.1631	1.3147	0.1545
		β	0.1587	0.0737	0.1293	0.0576	0.1823	0.0735
		λ	0.0048	0.0010	0.0042	0.0008	0.0051	0.0011
	0.8	α	1.3050	0.1603	1.3050	0.1681	1.2801	0.1458
		β	0.1293	0.0553	0.1293	0.0598	0.1556	0.0563
		λ	0.0040	0.0007	0.0041	0.0007	0.0042	0.0009
49 complete sample		α	1.3050	0.1514	1.3050	0.1565	1.2956	0.1343
		β	0.1293	0.0514	0.1293	0.0545	0.1366	0.0489
		λ	0.0040	0.0006	0.0041	0.0006	0.0040	0.0006

method is best estimation method to estimate parameter of OWIR distribution. We use binomial removal in progressive type-II censoring scheme for OWIR distribution. In Application of COVID-19 spread in the United Kingdom and United States of America, The OWIR distribution is best model to fit this data.

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