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Generalized quasi Lindley distribution: theoretical properties, estimation methods and applications

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In this paper, a new continuous two parameters generalized quasi Lindley distribution (GQLD) is suggested. The GQLD is a sum of two independent quasi Lindley distributed random variables. Comprehensive statistical properties of the new model are provided in closed forms includes moments, reliability function, hazard function, reversed hazard function, stochastic ordering, stress-strength reliability, and distribution of order statistics. The unknown parameters of the new distribution are estimated by the maximum likelihood, maximum product of spacing, ordinary least squares, weighted least squares, Cramer-von-Mises, and Anderson-Darling methods. A detailed simulation study is conducted to investigate the efficiency of the proposed estimators in terms of mean square errors. The performance of the suggested model is illustrated using two real data sets. It turns out that the GQLD can provide better fits than the quasi Lindley, Pareto, two-parameter Sujatha, and log-normal distributions. MSC: 62D05; 60E05; 62F10

keywords: Quasi Lindley distribution, Independent random variables, Cramér–von Mises estimation, Methods of least squares, Methods of minimum distances; Anderson–Darling estimation.

1 Introduction

In the last decades, the researchers have derived various distributions which can be used to fit real data sets in different fields and that have useful reliability characteristics. One

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of such distributions is the Lindley distribution (LD) Lindley (1958) with probability density function (pdf) given by

$$f(x,\theta) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}; x, \theta \geq 0. \tag{1}$$

The LD has useful applications in several areas of research in lifetime data analysis. Various modifications of the LD are suggested in the literature, for examples Nedjar and Zeghdoudi (2016) introduced gamma Lindley distribution, Kumar and Jose (2018) suggested double Lindley distribution. Ramos and Louzada (2016) introduced generalized weighted Lindley distribution. Tomy (2018) presented an extensively study about the Lindley distribution and its generalizations. The power Lindley distribution is introduced by Ghitany et al (2013). Hassan (2014) considered a convolution of Lindley distribution. Al-khazaleh and Al-Omari (2016) suggested transmuted two-parameter Lindley distribution.

Shanker and Mirsha (2013) suggested a two parameters quasi Lindley distribution (QLD) as a modification of the LD with pdf given by

$$f(x,\theta,\alpha) = \frac{\theta}{\alpha+1}(\alpha+\theta x)e^{-\theta x}; x > 0, \theta > 0, \alpha > -1. \tag{2}$$

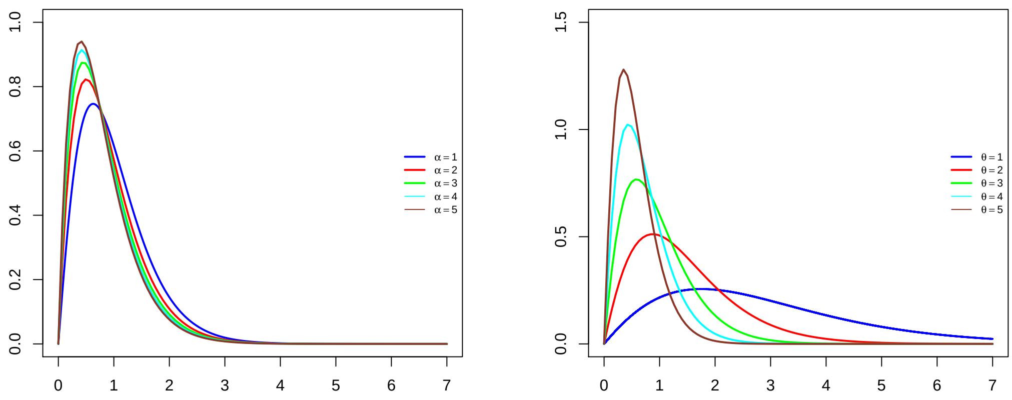
When $\alpha = \theta$, the QLD reduces to the LD and when $\alpha = 0$ it reduces to the gamma distribution with parameters $(2, \theta)$. The cumulative distribution function (cdf) of the QLD is defined as

$$F(x,\theta,\alpha) = 1 - \frac{1+\alpha+\theta x}{1+\alpha}e^{-\theta x}; x > 0, \theta > 0, \alpha > -1. \tag{3}$$

They showed that the QLD is more flexible than the LD in fitting some real data. As an application of the QLD, Al-Omari, Al-Nasser and Gogah (2018) investigated double acceptance sampling plan for the quasi Lindley distribution. Also,Al-Omari and Al-Nasser (2019) considered the two parameters quasi Lindley distribution in acceptance sampling plans from truncated life tests.

In light of the above importance of the QLD distribution, we are motivated to find a new modification of the QLD to be more flexible than it in fitting some real data. Hence, we considered the idea of sum of two independent quasi Lindley distributed random variables and constructed a new model called the generalized quasi Lindley distribution which is more useful fits to some real data sets than some well known competitive models including the QLD it self.

The layout of this paper is organized as follows. Section 2 concerns with the pdf and cdf of the GQLD and its shapes. In Section 3, the stochastic ordering and the reliability behavior of the new distribution are provided. The moments includes the rth moment, moment generating function, variance, the coefficients of skewness, kurtosis, and variation are presented in Section 4 theoretically and supported by simulations. Stress-strength reliability and the distribution of order statistics of the model are given in Section 5. In Section 6, different methods of estimation for the unknown distribution



(a) Different values of α when $\theta = 3$ (b) Different values of θ when $\alpha = 1.25$

Figure 1: Plots of the GQLD probability density function with different parameters values ((a) and (b)).

parameters, including maximum likelihood, maximum product of spacing's, ordinary least squares, weighted least squares, Cramer-von-Mises, and Anderson-Darling methods are considered. A simulation study in conducted to compare the performance of the estimators in Section 7. Illustrative examples of real data and some applications are given in Section 8. Finally, some concluding remarks and future works are presented in Section 9.

2 The GQLD distribution

This section introduces the pdf and cdf of the GQLD as well as some graphs for the GQLD pdf and cdf based on various distribution parameters. Shanker and Mirsha (2013) showed that the QLD of two parameters is more flexible than the base Lindley distribution while the new GQLD has two parameters that make it a flexible distribution than some existing distributions. A random variable X is said to have a GQLD distribution with parameters α and θ if its pdf is given by:

$$f_{GQLD}(x;\theta,\alpha) = \frac{\theta^2\left(\frac{\theta^2 x^3}{6} + \alpha\theta x^2 + \alpha^2 x\right)e^{-\theta x}}{(\alpha+1)^2};\ x \geq 0,\ \alpha > -1,\ \theta \geq 0. \tag{4}$$

Based on Figure (1) it can be noted that the distribution is positively skewed and the degree of the skewness depends on the values of the parameters. With $\alpha = 1.25$, as the value of θ is decreasing from $\theta = 5 \longrightarrow \theta = 1$, the shape of the distribution is going to be more flatting and the model is peak for large values of θ.

The corresponding cdf of the GQLD is given by

$$F_{GQLD}(x;\alpha,\ \theta)=1-\frac{\left(\theta^3x^3+3\left(2\alpha+1\right)\theta^2x^2+6\left(\alpha+1\right)^2\left(\theta x+1\right)\right)\mathrm{e}^{-\theta x}}{6\left(\alpha+1\right)^2}. \tag{5}$$

3 Stochastic Ordering and Reliability Analysis

3.1 Stochastic ordering

The stochastic ordering can be used to compare two positive continuous distributions. A random variable X is smaller than a random variable Y in

1. Mean residual life order denoted by $X \leq_{MRLO} Y$, if $m_X(x) \leq m_Y(x)$ for all x,
2. Hazard rate order denoted by $X \leq_{HRO} Y$, if $\overline{F}_X(x)/\overline{F}_Y(x)$ is decreasing in $x \geq 0$,
3. Stochastic order denoted by $X \leq_{SO} Y$, if $\overline{F}_X(x) \leq\leq \overline{F}_Y(x)$ for all x,
4. Likelihood ratio order denoted by $X \leq_{LRO} Y$, if $\dfrac{fx(x)}{f_Y(x)}$ is decreasing in $x \geq 0$.

Shaked and Shanthikumar (1994) showed that all these stochastic orders defined above are related to each other and the following relation is hold.

$$\begin{array}{ccccc} X \leq_{LRO} Y \Rightarrow & X \leq_{HRO} Y \Rightarrow & X \leq_{MRLO} Y. \\ & \Downarrow & \\ & X \leq_{SO} Y. & \end{array}$$

Theorem 1: Let the random variables X and Y be independent follow the pdf $f_X(x,\theta,\alpha)$ and $f_Y(x,\beta,\eta)$, respectively. If $(\theta \geq \beta, \alpha \leq \eta)$, then $X \leq_{LRO} Y, X \leq_{HRO} Y, X \leq_{MRLO} Y$ and $X \leq_{SO} Y$.

Proof:

To prove this result, it is sufficient to show that $\dfrac{fx(x;\theta,\alpha)}{f_Y(x;\beta,\eta)}$ is a deceasing function of x, by taking the natural logarithm as follow:

$$
\begin{aligned}
\log \frac{f_X(x;\theta,\alpha)}{f_Y(x;\beta,\eta)} &= \log \left[\frac{\dfrac{\theta^2\left(\frac{\theta^2x^3}{6}+\alpha\theta x^2+\alpha^2 x\right)\mathrm{e}^{-\theta x}}{(\alpha+1)^2}}{\dfrac{\beta^2\left(\frac{\beta^2x^3}{6}+\eta\beta x^2+\eta^2 x\right)\mathrm{e}^{-\beta x}}{(\eta+1)^2}} \right] \\
&= 2\log(\frac{\theta}{\beta})+2\log(\frac{\eta+1}{\alpha+1})+(\beta-\theta)x+\log\left(\frac{\frac{\theta^2x^3}{6}+\alpha\theta x^2+\alpha^2 x}{\frac{\beta^2x^3}{6}+\eta\beta x^2+\eta^2 x}\right) \\
&= 2\log(\frac{\theta}{\beta})+2\log(\frac{\eta+1}{\alpha+1})+(\beta-\theta)x+\log\left(\frac{\theta^2x^3}{6}+\alpha\theta x^2+\alpha^2 x\right) \\
&- \log\left(\frac{\beta^2x^3}{6}+\eta\beta x^2+\eta^2 x\right).
\end{aligned}
$$

Taking the derivative of the last equation with respect to x yields

$$
\frac{d}{dx}\log \frac{f_X(x;\theta,\alpha)}{f_Y(x;\beta,\eta)} = (\beta-\theta)+\frac{\frac{\theta^2x^2}{2}+2\alpha\theta x+\alpha^2}{\frac{\theta^2x^3}{6}+\alpha\theta x^2+\alpha^2 x}-\frac{\frac{\beta^2x^2}{2}+2\beta\eta x+\eta^2}{\frac{\beta^2x^3}{6}+\beta\eta x^2+\eta^2 x}.
$$

Therefore, if $(\theta \geq \beta, \alpha \leq \eta)$, then $\frac{d}{dx}\log\frac{f_X(x;\theta,\alpha)}{f_Y(x;\beta,\eta)} < 0$, and hence the result is proved.

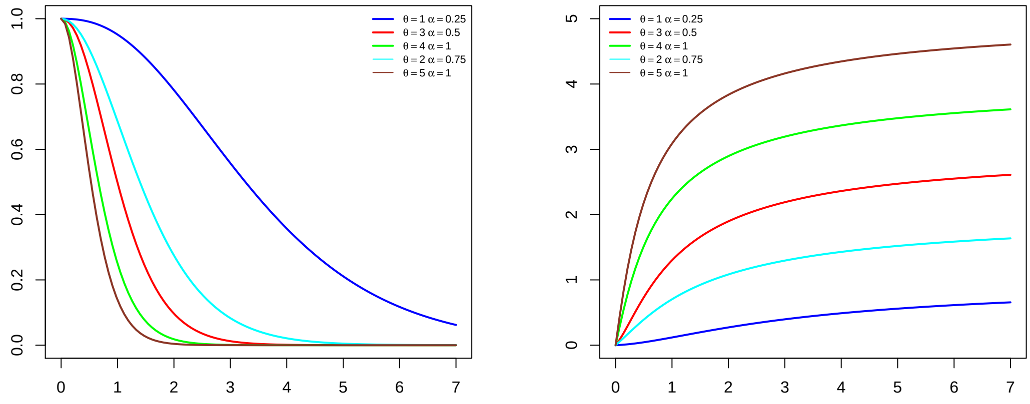
3.2 Reliability analysis

The corresponding reliability and hazard functions of the GQLD distribution are given, respectively by:

$$
R_{GQLD}(x;\theta,\ \alpha) = \frac{e^{-\theta x}\left(6(\alpha+1)^2+\theta^3x^3+3(2\alpha+1)\theta^2x^2+6(\alpha+1)^2\theta x\right)}{6(\alpha+1)^2}; \alpha > -1, \tag{6}
$$

and

$$
H_{GQLD}(x;\theta,\alpha) = \frac{\theta^2 x\left(6\alpha^2+\theta^2x^2+6\alpha\theta x\right)}{6(\alpha+1)^2+\theta^3x^3+3(2\alpha+1)\theta^2x^2+6(\alpha+1)^2\theta x}; x>0,\ \alpha>-1,\ \theta>0. \tag{7}
$$



(a) (b)

Figure 2: Reliability function (a) and hazard rate function (b) of the GQLD for various values of the parameters.

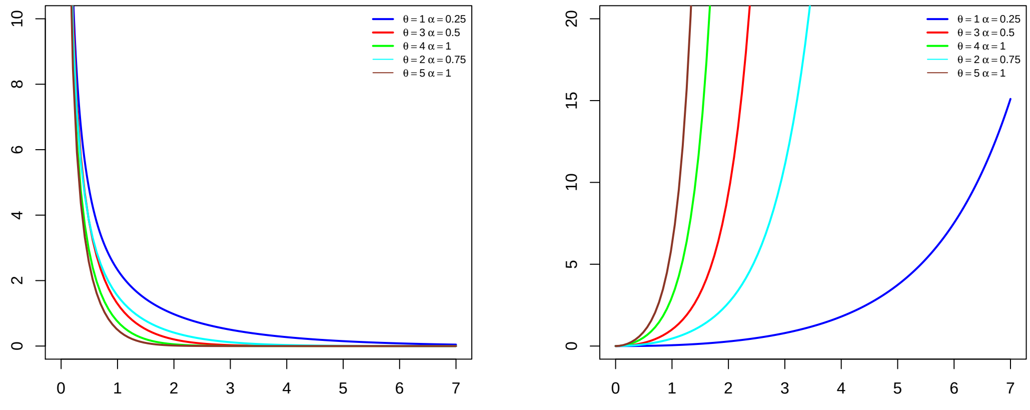
The reversed hazard rate and odds functions for the GQLD distribution, respectively, are defined as:

$$RH_{GQLD}(x;\theta,\alpha)=\frac{-\theta^2 x\left(6\alpha^2+\theta^2x^2+6\alpha\theta x\right)}{6(\alpha+1)^2+\theta^3x^3+3(2\alpha+1)\theta^2x^2+6(\alpha+1)^2\theta x-6(\alpha+1)^2e^{\theta x}}; \tag{8}$$

for $x>0,\ \alpha>-1,\ \theta>0$, and

$$O_{GQLD}(x;\theta,\alpha)=\frac{-6(\alpha+1)^2+\theta^3x^3+3(2\alpha+1)\theta^2x^2+6(\alpha+1)^2\theta x-6(\alpha+1)^2e^{\theta x}}{6(\alpha+1)^2+\theta^3x^3+3(2\alpha+1)\theta^2x^2+6(\alpha+1)^2\theta x}; \tag{9}$$

where $\alpha>-1,\ \theta>0$.



(a) (b)

Figure 3: The reversed hazard rate (a) and the odds functions (b) of the GQLD for various values of the parameters.

From Figure (3), it can be seen that the reliability function is an decreasing trend while the hazard function is an increasing function for all parameters values given in the plots. Also, Figure (4) shows that the reversed hazard rate function is decreasing while the odds function is increasing.

4 Moments and some related measures

In this section, we present the moment generating function, the rth moment, variance, coefficient of variation, coefficient of skewness, and the coefficient of kurtosis. The moment generating function (MGF) of the GQLD distribution is given by:

$$M_{GQLD}(t) = \frac{\theta^2(\theta + \alpha\theta - t\alpha \log(e))^2}{(1+\alpha)^2(\theta - t\log(e))^4};\ t \geq 0,\ \ \alpha > -1,\ \ \theta \geq 0. \tag{10}$$

The rth moment of the GQLD is defined as:

$$E(X^r) = \frac{\left(r^2 + 6(1+\alpha)^2 + r(5+6\alpha)\right)\Gamma(2+r)}{6\theta^r(1+\alpha)^2}, \alpha > -1,\ \ \theta \geq 0,\ \ r = 1, 2, 3... \tag{11}$$

Hence, the first four moments of the GQLD(θ,α) distributed random variable can be found by substituting $r = 1, 2, 3, 4$, respectively, in Equation (11)

$$E(X) = \frac{6\left(\alpha^2+1\right) + 6\alpha + 6}{3\left(\alpha+1\right)^2\theta}, \tag{12}$$

$$E(X^2) = \frac{6\,(\alpha+1)^2 + 2\,(6\alpha+5) + 4}{(\alpha+1)^2\,\theta^2}, \tag{13}$$

$$E(X^3) = \frac{4(24+18\alpha+6(1+\alpha)^2)}{\theta^3(1+\alpha)^2}, \tag{14}$$

$$E(X^4) = \frac{20(36+24\alpha+6(1+\alpha)^2)}{\theta^4(1+\alpha)^2}. \tag{15}$$

Then, the variance of the GQLD distribution can be obtained as

$$V(X) = E(X^2) - (E(X))^2 = \frac{2\left(\alpha^2+4\alpha+2\right)}{(\alpha+1)^2\,\theta^2}. \tag{16}$$

The coefficient of skewness, coefficient of kurtosis, and coefficient of variation of the GQLD distribution, respectively, are given by:

$$SK_{GQLD} = \frac{\sqrt{2}(\alpha(\alpha(\alpha+6)+6)+2)}{(\alpha(\alpha+4)+2)^{3/2}}, \tag{17}$$

$$Ku_{GQLD} = \frac{6(\alpha+1)^2(\alpha(\alpha+6)+3)}{(\alpha(\alpha+4)+2)^2}, \tag{18}$$

$$CV_{GQLD} = \frac{\sqrt{\alpha(\alpha+4)+2}}{\sqrt{2}(\alpha+2)}. \tag{19}$$

It can be noted that the coefficients of variation, skewness and kurtosis are free of the parameter θ. Tables (1) and 2, summarize some values of the mean, standard deviation, coefficients of variation, skewness and kurtosis for the GQLD with different values of the parameters α and θ.

From Tables (1) and (2), we can note that the mean and standard deviation values are deceasing as the values of α and θ, respectively, are increasing. Also, from Table (1), the values of coefficients of variation, skewness and kurtosis are increasing as the values of α are increasing when $\theta = 1.5$. However, from Table (2), we have the same values of $SK_{GQLD} = 1.241883$, $Ku_{GQLD} = 5.23469$, and $CV_{GQLD} = 0.661438$ for all values of α.

5 The Stress-Strength Reliability and Order Statistics

5.1 Stress-strength reliability

Let the random variables X and Y be independent follow the pdf $f(x)$. The stress-strength reliability demonstrates the life of a component that has random strength X that is subjected to random stress Y. When the stress is applied to the component exceeds the strength $(X < Y)$, the component fails instantly and hence it will function satisfactorily until $(X > Y)$. For more about the stress-strength see Mahdizadeh and

Table 1: The mean, standard deviation, coefficients of variation, skewness and kurtosis for the GQLD(θ,α) with different values of the parameter α when $\theta = 1.5$

α	μ_{GQLD}	σ_{GQLD}	SK_{GQLD}	Ku_{GQLD}	CV_{GQLD}
0.1	2.545454	1.330575	1.005852	4.51242	0.522726
0.2	2.444444	1.324041	1.018837	4.54196	0.541653
0.3	2.358974	1.315462	1.034914	4.58094	0.557641
0.4	2.285714	1.305839	1.052090	4.62494	0.571304
0.5	2.222222	1.295767	1.069344	4.67128	0.583095
0.6	2.166666	1.285604	1.086149	4.71831	0.593355
0.7	2.117647	1.275565	1.102242	4.76501	0.602350
0.8	2.074071	1.265778	1.117502	4.81075	0.610286
0.9	2.035087	1.256316	1.131888	4.85514	0.617328
1.0	2.000000	1.247219	1.145405	4.89796	0.623609
1.1	1.968254	1.238502	1.158082	4.93908	0.629239
1.2	1.939393	1.230168	1.169958	4.97846	0.634305
1.3	1.913043	1.222212	1.181084	5.01610	0.638883
1.4	1.888888	1.214622	1.191508	5.05201	0.643035
1.5	1.866666	1.207384	1.201279	5.08626	0.646813
1.6	1.846153	1.200482	1.210444	5.11889	0.650261
1.7	1.827160	1.193898	1.219046	5.14998	0.653417
1.8	1.809523	1.187615	1.227128	5.17960	0.656313
1.9	1.793103	1.181618	1.234729	5.20781	0.658979
2.0	1.777777	1.175889	1.241883	5.23469	0.661437
2.1	1.763440	1.170414	1.248624	5.26031	0.663710
2.2	1.750000	1.165177	1.254981	5.28474	0.665815
2.3	1.737374	1.160166	1.260983	5.30803	0.667770
2.4	1.725490	1.155366	1.266654	5.33026	0.669587
2.5	1.714285	1.150766	1.272018	5.35147	0.671280
2.6	1.703703	1.146354	1.277097	5.37173	0.672860
2.7	1.693693	1.142120	1.281910	5.39109	0.674337
2.8	1.684210	1.138054	1.286475	5.40960	0.675719
2.9	1.675214	1.134146	1.290808	5.42730	0.677016
3.0	1.666667	1.130388	1.294925	5.44423	0.678233
3.1	1.658536	1.126771	1.298839	5.46045	0.679377
3.2	1.650793	1.123289	1.302565	5.47598	0.680454
3.3	1.643410	1.119933	1.306113	5.49080	0.681469
3.4	1.636363	1.116698	1.309494	5.50514	0.682426
3.5	1.629629	1.113577	1.312720	5.51883	0.683331
3.6	1.623188	1.110564	1.315799	5.53198	0.684187
3.7	1.617021	1.107655	1.318739	5.54460	0.684997
3.8	1.611111	1.104843	1.321550	5.55672	0.685764
3.9	1.605442	1.102125	1.324239	5.56838	0.686493
4.0	1.600000	1.099494	1.326812	5.57958	0.687184

Table 2: The mean, standard deviation, coefficients of variation, skewness and kurtosis for the GQLD(θ,α) with different values of the parameter θ when $\alpha = 2$

θ	μ_{GQLD}	σ_{GQLD}	SK_{GQLD}	Ku_{GQLD}	CV_{GQLD}
0.1	26.666667	17.638342	1.241883	5.23469	0.661438
0.2	13.333333	8.819171	↑	↑	↑
0.3	8.888889	5.879447	↑	↑	↑
0.4	6.666667	4.409586	↑	↑	↑
0.5	5.333333	3.527668	↑	↑	↑
0.6	4.444444	2.939724	↑	↑	↑
0.7	3.809524	2.519763	↑	↑	↑
0.8	3.333333	2.204793	↑	↑	↑
0.9	2.962963	1.959816	↑	↑	↑
1.0	2.666667	1.763834	↑	↑	↑
1.1	2.424242	1.603486	↑	↑	↑
1.2	2.222222	1.469862	↑	↑	↑
1.3	2.051282	1.356796	↑	↑	↑
1.4	1.904762	1.259882	↑	↑	↑
1.5	1.777778	1.175889	↑	↑	↑
1.6	1.666667	1.102396	↑	↑	↑
1.7	1.568627	1.037550	↑	↑	↑
1.8	1.481481	0.979908	↑	↑	↑
1.9	1.403509	0.928334	↑	↑	↑
2.0	1.333333	0.881917	↑	↑	↑
2.1	1.269841	0.839921	↑	↑	↑
2.2	1.212121	0.801743	↑	↑	↑
2.3	1.159420	0.766884	↑	↑	↑
2.4	1.111111	0.734931	↑	↑	↑
2.5	1.066667	0.705534	↑	↑	↑
2.6	1.025641	0.678398	↑	↑	↑
2.7	0.987654	0.653272	↑	↑	↑
2.8	0.952381	0.629941	↑	↑	↑
2.9	0.919540	0.608219	↑	↑	↑
3	0.888889	0.587945	↑	↑	↑
3.1	0.860215	0.568979	↑	↑	↑
3.2	0.833333	0.551198	↑	↑	↑
3.3	0.808081	0.534495	↑	↑	↑
3.4	0.784314	0.518775	↑	↑	↑
3.5	0.761905	0.503953	↑	↑	↑
3.6	0.740741	0.489954	↑	↑	↑
3.7	0.720721	0.476712	↑	↑	↑
3.8	0.701754	0.464167	↑	↑	↑
3.9	0.683761	0.452265	↑	↑	↑
4.0	0.666667	0.440959	↑	↑	↑

Zamanzade (2017, 2018a, 2018b, 2018c, 2019). The stress-strength reliability is defined as

$$R = P(Y < X) = \int_0^{\infty} P(Y < X|X = x) f(x) dx = \int_0^{\infty} f(x; \theta, \alpha) F(x; \omega, \delta) dx. \quad (20)$$

Let the random variables X and Y be independent and observed from the GQLD. Then the stress-strength reliability $R_{GQLD} = P(Y_{GQLD} < X_{GQLD})$, is given by

$$R_{GQLD} = \frac{\psi}{\delta^{-2}(\theta+1)^2(\omega+1)^2(\alpha+\delta)^7}, \quad (21)$$

where

$$\psi = \left(\begin{array}{c} \alpha^5(3\theta(\theta+4)+10)\omega^2 + \alpha^4\delta\omega(\theta(13\theta+44)\omega + 8\theta(\theta+5) + 30\omega + 40) \\ +\alpha^3\delta^2(\theta^2(22\omega^2+26\omega+5) + 2\theta(\omega(31\omega+50)+15) + 35(\omega+1)^2) \\ +\alpha^2\delta^3(\theta^2(6\omega(3\omega+5)+11) + 42\theta(\omega+1)^2 + 21(\omega+1)^2)+ \\ 7\alpha\delta^4(\theta+1)^2(\omega+1)^2 + \delta^5(\theta+1)^2(\omega+1)^2 \end{array} \right).$$

5.2 Order statistics

Assume that $X_1, X_2, \ldots, X_m$ is a random sample from the GQLD with pdf and cdf $f(x)$ and $F(x)$, respectively. David and Nagaraja (2003) defined the pdf of the ith order statistic $X_{(i:m)}$ for $i = 1, 2, \ldots m$ as

$$\begin{aligned} f_{(i:m)}(x) &= \frac{m!}{(i-1)!(m-i)!}[F(x)]^{i-1}[1-F(x)]^{m-i} f(x) \\ &= \frac{m!}{(i-1)!(m-i)!}\sum_{k=0}^{m-i}(-1)^k \binom{m-i}{k} F^{k+i-1}(x) f(x), \end{aligned} \quad (22)$$

and the corresponding cdf is given by

$$F_{(i:m)}(x) = \sum_{j=t}^{m} \binom{m}{j} F(x)]^j [1-F(x)]^{m-j} = \sum_{j=t}^{m}\sum_{k=0}^{m-i}(-1)^k \binom{m}{j}\binom{m-i}{k} F^{j+k}(x). \quad (23)$$

Based on Equation (24) and (23), the pdf and cdf of the ith order statistic, $X_{(i:m)}$, from the GQLD, respectively are

$$\begin{aligned} f_{(i:m)}^{GQLD}(x) = & \frac{m!\theta^2 e^{-\theta x}}{(i-1)!(m-i)!6^{k+i-1}(\alpha+1)^{2(k+i)}} \sum_{k=0}^{m-i}(-1)^k \binom{m-i}{k} \left(\frac{\theta^2 x^3}{6} + \alpha\theta x^2 + \alpha^2 x \right) \\ & \times \left(6(\alpha+1)^2 - \left(\theta^3 x^3 + 3(2\alpha+1)\theta^2 x^2 + 6(\alpha+1)^2(\theta x+1) \right) e^{-\theta x} \right)^{k+i-1}, \end{aligned} \quad (24)$$

and

$$F_{(i:m)}^{GQLD}(x) = \sum_{j=t}^{m}\sum_{k=0}^{m-i}(-1)^k\binom{m}{j}\binom{m-i}{k}\times \left(1-\frac{\left(\theta^3x^3+3\left(2\alpha+1\right)\theta^2x^2+6\left(\alpha+1\right)^2\left(\theta x+1\right)\right)\mathrm{e}^{-\theta x}}{6\left(\alpha+1\right)^2}\right)^{j+k}. \quad (25)$$

6 Methods of estimation

In this section, we consider six methods of estimation for estimating the unknowns parameters α and θ of the GQLD distribution. These methods include the maximum likelihood method, method of maximum product of spacings, ordinary least square method, weight least square method, method of Cramer-Von-Mises and Anderson-Darling method. Some authors used these methods in estimating distribution parameters, see for example Afify, Gemeay and Ibrahim (2020), Afify et al (2016), Aldahlan and Afify (2020) and Al-Mofleh, Afify and Ibrahim (2020).

6.1 Maximum likelihood estimation

The method of maximum likelihood (MLE) is the most commonly used in statistical inference due its good properties such as the consistency and asymptotic normality. Let $x_1, x_2 \ldots, x_n$ be a random sample of size n selected from the GQLD. The likelihood function is given by:

$$L(x;\theta,\alpha)=\prod_{i=1}^{n}f(x_i,\theta,\alpha)=\left(\frac{\theta}{\alpha+1}\right)^{2n}\prod_{i=1}^{n}\left(\alpha^2x_i+\alpha\theta x_i^2+\theta^2\frac{x_i^3}{6}\right)e^{-\theta x_i}, \quad (26)$$

and the log-likelihood function is

$$\ln L\ (x;\theta,\ \alpha)=2n\ln\left(\frac{\theta}{\alpha+1}\right)+\sum_{i=1}^{n}\ln\left(\alpha^2x_i+\alpha\theta x_i^2+\theta^2\frac{x_i^3}{6}\right)-\theta\sum_{i=1}^{n}x_i. \quad (27)$$

Take the derivatives of Equation (27) with respect to θ and α as

$$\frac{d\ln}{d\theta}=\frac{2n}{\theta}+\sum_{i=1}^{n}\frac{2x_i\left(x_i\theta+3\alpha\right)}{x_i^2\theta^2+6\alpha x_i\theta+6\alpha^2}-\sum_{i=1}^{n}x_i, \quad (28)$$

$$\frac{d\ln}{d\alpha}=-\frac{2n}{\alpha+1}+\sum_{i=1}^{n}\frac{12\alpha+6\theta x_i}{6\alpha^2+6\theta x_i\alpha+\theta^2x_i^2}. \quad (29)$$

Since there is no closed form for these equations, then the MLEs $\hat{\theta}$ and $\hat{\alpha}$ of θ and α, respectively, can be solved simultaneously using a numerical method.

6.2 Method of maximum product of spacings

The maximum product of spacing (MPS) method is suggested by Cheng and Amin (1979,1983) as an alternative to the maximum likelihood method and they proved the consistency and the efficiency of the MPS estimators as the MLE estimators. The MPS method requires a maximization of the geometric mean of the spacings in the data with respect to the parameters. Consider a random sample of size n, $x_1, x_2, \ldots, x_n$ selected from the GQLD distribution, then the uniform spacings are given as:

$$D_i(\alpha,\theta) = F(x_{i:n}|\alpha,\theta) - F(x_{i-1:n}|\alpha,\theta), i = 1, 2, \ldots, n,$$

where $F(x_{0:n}|\alpha,\theta) = 0$ and $F(x_{n+1:n}|\alpha,\theta) = 1$. Clearly $\sum_{i=1}^{n+1} D_i(\alpha,\theta) = 1$.

The MPSs, estimators $\hat{\alpha}_{MPS}$ and $\hat{\theta}_{MPS}$, are the values of α and θ, which maximize the geometric mean of the spacing:

$$H(\alpha,\theta|\mathrm{x}) = \left[\prod_{i=1}^{n+1} D_i(\alpha,\theta)\right]^{\frac{1}{n+1}}. \tag{30}$$

The natural logarithm of (30) is:

$$M(\alpha,\theta|\mathrm{x}) = \frac{1}{n+1}\sum_{i=1}^{n+1} \log D_i(\alpha,\theta).$$

The MPSs estimators $\hat{\alpha}_{MPS}$ and $\hat{\theta}_{MPS}$ of the parameters α and θ, respectively, can also be obtained by solving the nonlinear equations:

$$\frac{\partial}{\partial\alpha}M(\alpha,\ \theta) = \frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i(\alpha,\theta)}\left[\Psi_1(x_{i:n}|\alpha,\ \theta) - \Psi_1(x_{i-1:n}|\alpha,\ \theta)\right] = 0,$$

$$\frac{\partial}{\partial\theta}M(\alpha,\theta) = \frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i(\alpha,\theta)}\left[\Psi_2(x_{i:n}|\alpha,\theta) - \Psi_2(x_{i-1:n}|\alpha,\theta)\right] = 0,$$

where

$$\Psi_1(x_{i:n}|\alpha,\theta) = \frac{\partial}{\partial\alpha}F(x_{i:n}|\alpha,\theta) = \frac{\theta^2 x_{i:n}^2 \mathrm{e}^{-\theta x_{i:n}}\left(3\alpha + \theta x_{i:n}\right)}{3\left(\alpha+1\right)^3}, \tag{31}$$

and

$$\Psi_2(x_{i:n}|\alpha,\theta) = \frac{\partial}{\partial\theta}F(x_{i:n}|\alpha,\theta) = \frac{x_{i:n}^2\theta\left(x_{i:n}^2\theta^2 + 6\alpha x_{i:n}\theta + 6\alpha^2\right)\mathrm{e}^{-x_{i:n}\theta}}{6\left(\alpha+1\right)^2}, \tag{32}$$

which can be obtained numerically.

6.3 Methods of least squares

The least square methods are introduced by Swain, Venkatraman and Wilson (1988) to estimate the parameters of beta distribution. Let $X_{i:n}$ be the ith order statistic of the random sample $(X_1, X_2, \ldots, X_n)$ with distribution function $F(x)$, then a main result in probability theory indicates that $F(X_{(i)}) \sim Beta(i, n-i+1)$.

Moreover, we have
$\mathbb{E}[F(X_{(i)})] = \dfrac{i}{n+1}$ and $\mathrm{Var}[F(X_{(i)})] = \dfrac{i(n-i+1)}{(n+1)^2(n+2)}$.

Using the expectations and variances, we obtain two variants of the least squares methods.

6.3.1 Ordinary least squares

In case of GQLD distribution, the ordinary least square estimators $\hat{\alpha}_{OLS}$ and $\hat{\theta}_{OLS}$ of the parameters α and θ, respectively can be obtained by minimizing the function:

$$\begin{aligned}\Omega(\alpha,\theta|\mathrm{x}) &= \sum_{i=1}^{n}\left[F(x_{i:n}|\alpha,\theta) - \frac{i}{n+1}\right]^2\\ &= \sum_{i=1}^{n}\left[1 - \frac{\left(x_{i:n}^3\theta^3 + 3\,(2\alpha+1)\,x_{i:n}^2\theta^2 + 6\,(\alpha+1)^2\,(x_{i:n}\theta+1)\right)\mathrm{e}^{-x_{i:n}\theta}}{6\,(\alpha+1)^2} - \frac{i}{n+1}\right]^2,\end{aligned}$$

with respect to α and θ. Alternatively, these estimates can also be obtained by solving the following nonlinear equations:

$$\sum_{i=1}^{n}\left[F(x_{i:n}|\alpha,\theta) - \frac{i}{n+1}\right]\Psi_1(x_{i:n}|\alpha,\theta) = 0,$$

$$\sum_{i=1}^{n}\left[F(x_{i:n}|\alpha,\theta) - \frac{i}{n+1}\right]\Psi_2(x_{i:n}|\alpha,\theta) = 0,$$

where $\Psi_1(x_{i:n}|\alpha,\theta)$ and $\Psi_2(x_{i:n}|\alpha,\theta)$ are defined as in Equations 31 and 32, respectively.

6.3.2 Weighted least squares

For the GQLD distribution, the weighted least square estimators of α and θ say, $\hat{\alpha}_{WLS}$ and $\hat{\theta}_{WLS}$, respectively can be obtained by minimizing the function:

$$\begin{aligned}W(\alpha,\theta|\mathrm{x}) &= \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_{i:n}|\alpha,\theta) - \frac{i}{n+1}\right]^2\\ &= \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\\ &\quad \left[1 - \frac{\left(x_{i:n}^3\theta^3 + 3\,(2\alpha+1)\,x_{i:n}^2\theta^2 + 6\,(\alpha+1)^2\,x_{i:n}\theta + 6\,(\alpha+1)^2\right)\mathrm{e}^{-x_{i:n}\theta}}{6\,(\alpha+1)^2} - \frac{i}{n+1}\right]^2,\end{aligned}$$

with respect to α and θ. Equivalently, these estimators are the solution of the following nonlinear equations:

$$\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_{i:n}|\alpha,\theta) - \frac{i}{n+1}\right] \Psi_1(x_{i:n}|\alpha,\theta) = 0,$$

$$\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_{i:n}|\alpha,\theta) - \frac{i}{n+1}\right] \Psi_2(x_{i:n}|\alpha,\theta) = 0,$$

where $\Psi_1(x_{i:n}|\alpha,\theta)$ and $\Psi_2(x_{i:n}|\alpha,\theta)$ are specified as in 31 and 32, respectively.

6.4 Methods of minimum distances

Here, we use two popular methods based on the minimization of test statistics between the theoretical and empirical cumulative distribution functions. These methods are Cramer-von-Mises method and Anderson-Darling method, for more details see D'Agostino and Stephens (1986) and Luceno (2006).

6.4.1 Cramer-von-Mises method

The Cramer-von-Mises estimators (CVEs) $\hat{\alpha}$ and $\hat{\theta}$ of α and θ, respectively are obtained by minimizing the following function:

$$\begin{aligned} CV(\alpha,\theta) &= \frac{1}{12n} + \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_{(i:n)};\alpha,\theta) - \frac{2i-1}{2n}\right]^2 \\ &= \frac{1}{12n} + \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \\ &\left[1 - \frac{\left(\theta^3 x_{(i:n)}^3 + 3\,(2\alpha+1)\,\theta^2 x_{(i:n)}^2 + 6\,(\alpha+1)^2\,\theta x_{(i:n)} + 6\,(\alpha+1)^2\right) \mathrm{e}^{-\theta x_{(i:n)}}}{6\,(\alpha+1)^2} - \frac{2i-1}{2n}\right]^2, \end{aligned}$$

with respect to α and θ. Equivalently, these estimators are the solution of the following nonlinear equations:

$$\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_{i:n}|\alpha,\theta) - \frac{2i-1}{2n}\right] \Psi_1(x_{i:n}|\alpha,\theta) = 0,$$

$$\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_{i:n}|\alpha,\theta) - \frac{2i-1}{2n}\right] \Psi_2(x_{i:n}|\alpha,\theta) = 0,$$

where $\Psi_1(x_{i:n}|\alpha,\theta)$ and $\Psi_2(x_{i:n}|\alpha,\theta)$ are given in Equations 31 and 32, respectively.

6.4.2 Anderson-Darling method

The Anderson-Darling (AD) estimates of the GQLD distribution parameters α and θ denoted by $\hat{\alpha}_{AD}$ and $\hat{\theta}_{AD}$ can be obtained by minimizing the following function:

$$A(\alpha,\ \theta) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\{\log\ F(x_{i:n}|\alpha,\ \theta) + \log \overline{F}(x_{n-i+1:n}|\alpha,\ \theta)\},$$

with respect to α and θ, or by solving the following two equations

$$\frac{\partial A(\alpha,\theta)}{\partial\alpha} = \sum_{i=1}^{n}(2i-1)\left\{\frac{\Psi_1(x_{i:n}|\alpha,\theta)}{F(x_{i:n}|\alpha,\theta)} - \frac{\Psi_1(x_{n-i+1:n}|\alpha,\theta)}{\overline{F}(x_{n-i+1:n}|\alpha,\theta)}\right\} = 0,$$

and

$$\frac{\partial A(\alpha,\theta)}{\partial\theta} = \sum_{i=1}^{n}(2i-1)\left\{\frac{\Psi_2(x_{i:n}|\alpha,\theta)}{F(x_{i:n}|\alpha,\theta)} - \frac{\Psi_2(x_{n-i+1:n}|\alpha,\theta)}{\overline{F}(x_{n-i+1:n}|\alpha,\theta)}\right\} = 0,$$

where $\Psi_1(x_{i:n}|\alpha,\theta)$ and $\Psi_2(x_{i:n}|\alpha,\theta)$ are specified in Equations 31 and 32, respectively.

7 Simulation

This section involves Monte Carlo simulation study to evaluate the performances of the proposed estimators of the GQLD parameters α and θ. This comparison was carried out by taking random samples of different sizes ($n = 20, 40, 60, 80, 100$ and 200) with various pairs of parameters values $(\alpha,\theta) = (0.25, 1), (0.5, 1.5), (1, 5), (1, 3)$. The estimators are compared in terms of there mean squared errors (MSE). The results of the estimates (Es) and MSE are summarized in the Tables (2-5) as well as the Figures (3-8) explain the results.

n	MLEs		MPS		OLS		WLS		CVEs		AD	
	Es	**MSE**	**Es**	**MSE**	**Es**	**MSE**	**Es**	**MSE**	**Es**	**MSE**	**Es**	**MSE**
20	0.346	0.1380	0.309	0.1345	0.381	0.1775	0.375	0.1627	0.364	0.1556	0.370	0.1565
	1.012	0.0163	1.035	0.0187	1.005	0.0194	1.005	0.0184	1.010	0.0184	1.005	0.0176
40	0.295	0.0467	0.265	0.0415	0.313	0.0571	0.309	0.0529	0.302	0.0511	0.308	0.0524
	1.007	0.0077	1.021	0.0085	1.004	0.0090	1.004	0.0085	1.007	0.0085	1.004	0.0083
60	0.274	0.0278	0.250	0.0255	0.291	0.0353	0.287	0.0322	0.281	0.0312	0.286	0.0320
	1.005	0.0050	1.016	0.0054	1.003	0.0059	1.003	0.0055	1.005	0.0055	1.002	0.0054
80	0.264	0.0196	0.244	0.0183	0.275	0.0231	0.273	0.0216	0.269	0.0211	0.273	0.0213
	1.003	0.0035	1.012	0.0037	1.001	0.0041	1.001	0.0038	1.003	0.0038	1.001	0.0038
100	0.260	0.0164	0.242	0.0156	0.267	0.0196	0.265	0.0182	0.261	0.0179	0.265	0.0181
	1.002	0.0029	1.009	0.0031	0.999	0.0034	0.999	0.0032	1.001	0.0032	0.999	0.0032
200	0.258	0.0084	0.247	0.0083	0.262	0.0106	0.261	0.0097	0.259	0.0096	0.261	0.0097
	1.001	0.0015	1.006	0.0015	1.002	0.0017	1.002	0.0016	1.002	0.0016	1.001	0.0016

Table 3: Estimates and MSEs using the ML, MPS, OLS, WLS, AD and CV methods for the GQLD parameters with $\alpha = 0.25$ and $\theta = 1$.

n	MLEs		MPS		OLS		WLS		CVEs		AD	
	Es	**MSE**	**Es**	**MSE**	**Es**	**MSE**	**Es**	**MSE**	**Es**	**MSE**	**Es**	**MSE**
20	0.633	0.4060	0.590	0.4631	0.671	0.4718	0.664	0.4496	0.651	0.4456	0.659	0.4440
	1.521	0.0387	1.561	0.0452	1.511	0.0456	1.509	0.0431	1.518	0.0435	1.509	0.0411
40	0.575	0.1441	0.538	0.1449	0.593	0.1653	0.589	0.1537	0.5811	0.1517	0.588	0.1518
	1.517	0.0204	1.542	0.0226	1.513	0.0234	1.512	0.0219	1.518	0.0221	1.512	0.0214
60	0.541	0.0819	0.512	0.0819	0.555	0.0945	0.552	0.0878	0.545	0.0869	0.552	0.0879
	1.510	0.0138	1.529	0.0150	1.507	0.0160	1.506	0.0149	1.510	0.0150	1.506	0.0148
80	0.523	0.0561	0.500	0.0562	0.531	0.0646	0.530	0.0602	0.525	0.0597	0.530	0.0598
	1.505	0.0095	1.520	0.0101	1.501	0.0115	1.502	0.0107	1.505	0.0107	1.501	0.0105
100	0.510	0.0415	0.490	0.0421	0.517	0.0485	0.514	0.0450	0.510	0.0448	0.515	0.0446
	1.499	0.0075	1.511	0.0079	1.497	0.0092	1.497	0.0085	1.499	0.0085	1.497	0.0084
200	0.507	0.0183	0.496	0.0184	0.508	0.0212	0.508	0.0197	0.506	0.0196	0.508	0.0196
	1.500	0.0038	1.508	0.0039	1.499	0.0043	1.499	0.0040	1.500	0.0040	1.499	0.0040

Table 4: Estimates and MSEs using the ML, MPS, OLS, WLS, AD and CV methods for the GQLD parameters with $\alpha = 0.5$ and $\theta = 1.5$.

n	MLEs		MPS		OLS		WLS		CVEs		AD	
	Es	**MSE**	**Es**	**MSE**	**Es**	**MSE**	**Es**	**MSE**	**Es**	**MSE**	**Es**	**MSE**
20	1.386	1.8851	1.379	2.1664	1.409	1.9160	1.402	1.8619	1.395	1.8935	1.390	1.7993
	5.097	0.5239	5.246	0.6081	5.050	0.5968	5.049	0.5675	5.083	0.5728	5.049	0.5476
40	1.167	0.6219	1.147	0.6977	1.185	0.6728	1.178	0.6333	1.172	0.6403	1.175	0.6234
	5.046	0.2483	5.139	0.2782	5.026	0.2973	5.025	0.2775	5.044	0.2799	5.024	0.2698
60	1.116	0.3656	1.097	0.3826	1.130	0.3923	1.123	0.3667	1.118	0.3683	1.123	0.3652
	5.043	0.1722	5.112	0.1879	5.029	0.2035	5.028	0.1896	5.041	0.1908	5.027	0.1856
80	1.080	0.1983	1.064	0.2036	1.089	0.2171	1.085	0.2022	1.081	0.2020	1.085	0.2019
	5.036	0.1244	5.091	0.1345	5.022	0.1466	5.023	0.1354	5.033	0.1361	5.022	0.1339
100	1.050	0.1462	1.036	0.1497	1.054	0.1566	1.053	0.1491	1.049	0.1490	1.052	0.1485
	5.014	0.0963	5.060	0.1019	5.002	0.1162	5.004	0.1073	5.013	0.1076	5.003	0.1063
200	1.031	0.0665	1.023	0.0672	1.031	0.0716	1.031	0.0683	1.029	0.0682	1.031	0.0681
	5.011	0.0487	5.039	0.0507	5.006	0.0579	5.007	0.0536	5.012	0.0537	5.006	0.0533

Table 5: Estimates and MSEs using the ML, MPS, OLS, WLS, AD and CV methods for the GQLD parameters with $\alpha = 1$ and $\theta = 5$.

n	MLEs		MPS		OLS		WLS		CVEs		AD	
	Es	**MSE**	**Es**	**MSE**	**Es**	**MSE**	**Es**	**MSE**	**Es**	**MSE**	**Es**	**MSE**
20	1.400	2.0004	1.386	2.2400	1.429	2.0480	1.412	1.9422	1.404	1.9696	1.409	1.9358
	3.067	0.1900	3.158	0.2296	3.041	0.2278	3.037	0.2128	3.058	0.2153	3.040	0.2045
40	1.185	0.6085	1.163	0.6670	1.195	0.6428	1.189	0.6070	1.183	0.6109	1.189	0.5997
	3.038	0.0887	3.093	0.1012	3.024	0.1052	3.023	0.0984	3.035	0.0994	3.024	0.0960
60	1.107	0.2874	1.087	0.2980	1.116	0.3107	1.114	0.2969	1.110	0.2975	1.111	0.2899
	3.024	0.0597	3.064	0.0653	3.013	0.0707	3.015	0.0658	3.023	0.0662	3.013	0.0644
80	1.089	0.2007	1.073	0.2071	1.099	0.2185	1.096	0.2066	1.092	0.2065	1.096	0.2057
	3.029	0.0442	3.063	0.0488	3.020	0.0525	3.021	0.0486	3.027	0.0490	3.020	0.0480
100	1.067	0.1619	1.053	0.1658	1.073	0.1770	1.070	0.1682	1.067	0.1681	1.070	0.1659
	3.010	0.0364	3.039	0.0387	3.009	0.0444	3.008	0.0411	3.014	0.0413	3.008	0.0405
200	1.016	0.0625	1.008	0.0633	1.019	0.0664	1.018	0.0637	1.016	0.0637	1.018	0.0636
	3.000	0.0173	3.017	0.0178	2.996	0.0201	2.997	0.0188	3.000	0.0188	2.997	0.0188

Table 6: Estimates and MSEs using the ML, MPS, OLS, WLS, AD and CV methods for the GQLD parameters with $\alpha = 1$ and $\theta = 3$.

The following observations can be drawn from the Tables 2-5:

- The estimators considered in this study are consistent, where the MSE values of the proposed methods of estimation are decreasing as the sample size increasing for all values of the distribution parameters.

- For fixed sample size, the MSE of the MLEs is smaller than all other methods considered in this study.

- The bias values of the suggested estimators are decreasing as the sample sizes are increasing, and approach zero for all cases for large n.

Also, we complete this section with some inferences as the interval estimation. More specifically, we use the ML method to calculate lower bound (LB), upper bound (UB) and average length (AL) of the (two-sided asymptotic) confidence interval estimation of the distribution parameters at the levels 90% and 95%. The obtained result are listed in Tables (6-9). From Tables (6-9), it is clear that the width of the intervals is narrowering as the sample sizes are increasing.

n	90%			95%		
	LB	**UB**	**AL**	**LB**	**UB**	**AL**
20	-0.3214	0.7260	1.0475	-0.4218	0.8263	1.2482
	0.8129	1.2240	0.4504	0.7735	1.2634	0.4898
40	-0.1570	0.4923	0.6493	-0.2192	0.5545	0.7738
	0.8617	1.1487	0.3144	0.8342	1.1762	0.3419
60	-0.0983	0.4154	0.5138	-0.1475	0.4646	0.6122
	0.8892	1.1238	0.2571	0.8667	1.1463	0.2795
80	-0.0696	0.3600	0.4297	-0.1108	0.4012	0.5120
	0.9024	1.1050	0.2220	0.8830	1.1244	0.2414
100	-0.0458	0.3372	0.3831	-0.0826	0.3739	0.4565
	0.9136	1.0949	0.1987	0.8962	1.1123	0.2160
200	0.0167	0.2847	0.2679	-0.0089	0.3103	0.3192
	0.9381	1.0661	0.1402	0.9258	1.0783	0.1524

Table 7: LBs, UBs and ALs of the confidence interval estimation for the ML method for the GQLD with $\alpha = 0.25$, and $\theta = 1$.

n	90%			95%		
	LB	**UB**	**AL**	**LB**	**UB**	**AL**
20	-0.2998	1.5110	1.8108	-0.4732	1.6845	2.1577
	1.1930	1.8418	0.7109	1.1308	1.9039	0.7730
40	-0.0165	1.1047	1.1212	-0.1239	1.2121	1.3360
	1.2828	1.7398	0.5007	1.2390	1.7836	0.5445
60	0.0870	0.9610	0.8739	0.0033	1.0447	1.0414
	1.3228	1.6954	0.4083	1.2871	1.7311	0.4440
80	0.1444	0.8797	0.7352	0.0740	0.9501	0.8761
	1.3450	1.6671	0.3529	1.3142	1.6980	0.3837
100	0.1796	0.8258	0.6462	0.1177	0.8877	0.7700
	1.3605	1.6482	0.3153	1.3329	1.6758	0.3428
200	0.2772	0.7297	0.4524	0.2339	0.7730	0.5390
	1.4025	1.6060	0.2229	1.3830	1.6255	0.2424

Table 8: LBs, UBs and ALs of the confidence interval estimation for the ML method for the GQLD with $\alpha = 0.5$, and $\theta = 1.5$.

n	90%			95%		
	LB	**UB**	**AL**	**LB**	**UB**	**AL**
20	-0.3075	2.5003	2.8078	-0.5764	2.7692	3.3457
	3.9274	6.2564	2.5521	3.7043	6.4795	2.7752
40	0.0868	2.0748	1.9879	-0.1035	2.2652	2.3688
	4.2318	5.8644	1.7890	4.0754	6.0208	1.9454
60	0.2665	1.8714	1.6049	0.1128	2.0252	1.9123
	4.3675	5.6965	1.4562	4.2402	5.8238	1.5835
80	0.3724	1.7465	1.3741	0.2407	1.8782	1.6374
	4.4517	5.6014	1.2598	4.3416	5.7116	1.3700
100	0.4389	1.6438	1.2049	0.3235	1.7593	1.4357
	4.4985	5.5237	1.1234	4.4003	5.6219	1.2216
200	0.6091	1.4380	0.8288	0.5297	1.5174	0.9876
	4.6453	5.3697	0.7938	4.5759	5.4391	0.8632

Table 9: LBs, UBs and ALs of the confidence interval estimation for the ML method for the GQLD with $\alpha = 1$, and $\theta = 5$.

n	90%			95%		
	LB	**UB**	**AL**	**LB**	**UB**	**AL**
20	-0.3032	2.4380	2.7413	-0.5658	2.7006	3.2665
	2.3488	3.7413	1.5259	2.2154	3.8747	1.6593
40	0.0865	2.0637	1.9772	-0.1028	2.2531	2.3559
	2.5425	3.5234	1.0748	2.4485	3.6173	1.1687
60	0.2646	1.8687	1.6040	0.1110	2.0223	1.9113
	2.6223	3.4202	0.8743	2.5458	3.4966	0.9507
80	0.3655	1.7293	1.36380	0.2349	1.8599	1.6250
	2.6694	3.3587	0.7554	2.6033	3.4248	0.8214
100	0.4436	1.6655	1.2218	0.3266	1.7826	1.4559
	2.7053	3.3219	0.6756	2.6462	3.3810	0.7347
200	0.6072	1.4344	0.8272	0.5279	1.5137	0.9857
	2.7857	3.2202	0.4760	2.7441	3.2618	0.5177

Table 10: LBs, UBs and ALs of the confidence interval estimation for the ML method for the GQLD with $\alpha = 1$, and $\theta = 3$.

8 Two real data examples

This section empirically shows that the suggested GQLD can be considered as an alternative to some well known distributions of two paramters as quasi Lindley distribution, the Pareto distribution, two-parameter Sujatha distribution, and the log-normal distribution.

Two real data sets are considered for illustration. The first data set represent the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal (1960), previously studied by Afify et al (2016). The data points are

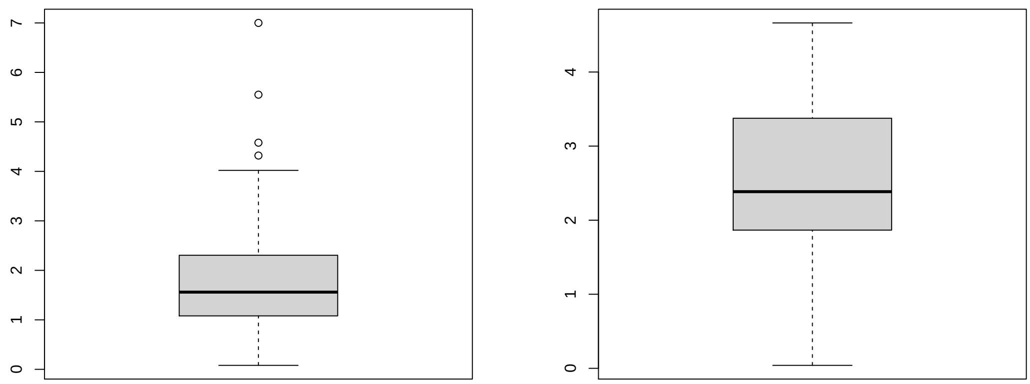
Data set 1: 0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 07, .08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55.

The second data set is presented in Murthy, Xie and Jiang (2004) and recently studied by Ramos et al (2013). This data set consists of 153 observations, of which 88 are classified as failed windshields, and the remaining 65 are service times of windshields that have not failed at the time of observation. The unit for measurement is 1000 h. The data are given by

Data set 2: 0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779,1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.82,3, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663.

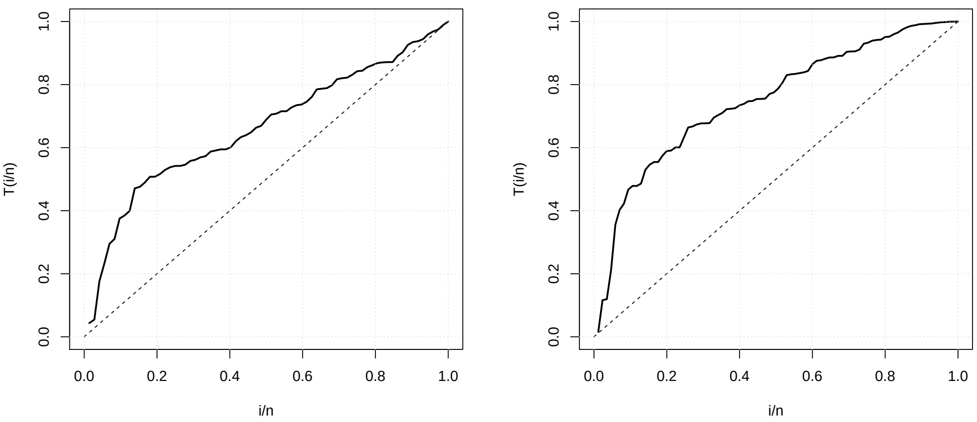
Table 11: Statistical properties of data sets 1 and 2

	n	min	max	mean	median	standard derivation	kurtosis	skweness
Data set 1:	72	0.08	7.00	1.836	1.560	1.215	3.954	1.718
Data set 2:	85	0.040	4.663	2.562	2.385	1.113	-0.689	0.085



(a) Box plot for data set 1 (b) Box plot for data set 2

Figure 4: Box plots for data sets 1 and 2.



(a) TTT plot for data set 1 (b) TTT plot for data set 2

Figure 5: TTT plots for data sets 1 and 2.

The GQLD distribution is fitted to these two real data sets and compared with the following models:

- Quasi Lindley distribution: $f(x) = \frac{\theta(\alpha+x\theta)}{\alpha+1}\mathrm{e}^{-\theta x}$.
- Pareto distribution: $f(x) = \frac{\alpha\theta^{\alpha}}{x^{\alpha+1}}$.
- Two-parameter Sujatha distribution Tesfay and Shanker (2018):
  $f(x) = \dfrac{\theta^3\left(x^2+\alpha x+1\right)\mathrm{e}^{-\theta x}}{\theta^2+\alpha\theta+2}$.
- Log-normal distribution: $f(x) = \frac{1}{x\sigma\sqrt{2\pi}}\exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right)$.

Table 12: The goodness of fit tests for data set 1.

Model	AIC	AICc	BIC	HQIC	K-S	p-value
GQLD	209.5955	209.7694	214.1488	211.4082	0.092806	0.564624
QLD	211.3587	211.5326	215.912	213.1714	0.136295	0.137765
PD	235.5451	235.7191	240.0985	237.3578	0.268966	0.000059
TSPD	211.4393	211.6132	215.9926	213.252	0.139958	0.119102
LND	220.6035	220.7774	225.1569	222.4162	0.135989	0.139429

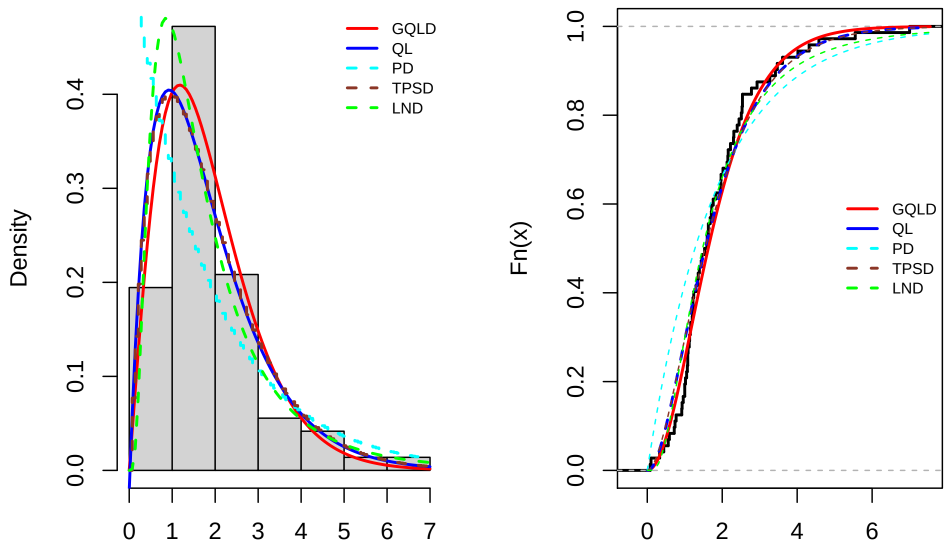


Figure 6: Plots of estimated probability density functions and cumulative distribution functions for data set 1

Table 13: The goodness of fit tests for data set 2.

Model	AIC	AICc	BIC	HQIC	K-S	p-value
GQLD	275.8206	275.967	280.7059	277.7856	0.103619	0.321002
QLD	293.5131	293.6595	298.3984	295.4781	0.180696	0.007769
PD	333.9754	334.1217	338.8607	335.9404	0.303101	0.000003
TPSD	293.5074	293.6537	298.3927	295.4724	0.180964	0.007642
LND	315.3184	315.4648	320.2037	317.2834	0.155402	0.032963

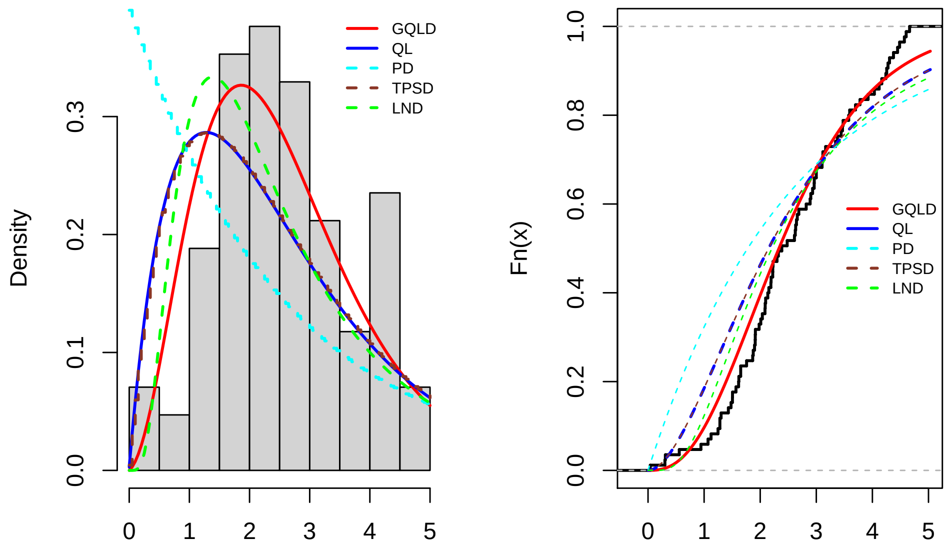


Figure 7: Plots of estimated probability density functions and cumulative distribution functions for data set 2

To assess the goodness of fit, we consider Akaike information criterion (AIC) introduced by Akaike (1974), Baysian information criterion (BIC) proposed by Schwarz (1978), Hannan-Quinn Information Criterion (HQIC) suggested by Hannan and Quinn (1979), Consistent Akaike Information Criterion (CAIC) by Bozdogan (1987), Camér-von Mises Criterion (C–M) proposed by Camer (1928), Anderson-Darling Criterion (A–D) by Stephens (1974) along with P-values. The measure with the lower values indicate better fit.

Also, the totla test time plots (TTT) Asrest (1987) and box plots of the two data sets are presented in Figures (4 and 5). The TTT plots are concave indicating an increasing failure rate. Based on the results reported in Tables 11 and 12, we observe that the GQLD provides the better fit with the smallest values of AIC, CAIC, BIC, HQIC, K-S and A-D with maximum P-values as compared to its competitive models. Figures 6 and 7 support this claim.

Finally, we apply the methods of estimation like the maximum likelihood (MLE), maximum product spacing (MPS), ordinary least square (OLS), weight least square (WLS), Cramer-von-Mises (CV) and Anderson-Darling (AD). As true parameters are unknown in real life data set, the mean squared error cannot be used to compare the estimators. Therefore, we have used Kolmogorov-Smirnov value (KS) and mean absolute error (MAE) which are defined as:

$$SMAE = \frac{\sum_{i=1}^{n} \left| S(x_i) - \hat{F}(x_i) \right|}{n} and KS = \max_{i} \left[\frac{1}{n} - \hat{F}(x_i), \hat{F}(x_i) - \frac{i-1}{n} \right],$$

where $S(x_i)$ is sample (observed) distribution function and $\hat{F}(x_i)$ is expected distribution function which are respectively, defined as:

$$S(x_i) = \frac{number\ elements\ in\ the\ sample\ \leq x}{n} = \frac{1}{n} \sum_{i=1}^{n} 1_{x_i \leq x},$$

and

$$\hat{F}(x_i) = 1 - \frac{\left(\hat{\theta}^3 x^3 + 3\,(2\hat{\alpha}+1)\,\hat{\theta}^2 x^2 + 6\,(\hat{\alpha}+1)^2\,\hat{\theta} x + 6\,(\hat{\alpha}+1)^2 \right) \mathrm{e}^{-\hat{\theta} x}}{6\,(\hat{\alpha}+1)^2}$$

with parameter estimates ($\hat{\hat{\alpha}}$ and $\hat{\hat{\theta}}$) form any particular method. The obtained numerical results are mentioned in Tables 13 and 14.

Method	ES($\hat{\theta}$)	ES($\hat{\alpha}$)	KS	MAE
MLE	1.6759	0.8546	0.0927	0.0352
MPS	1.4704	1.6806	0.1083	0.0404
OLS	2.0453	0.2804	0.0760	0.0226
WLS	1.9903	0.3188	0.0741	0.0252
CV	2.0627	0.2321	0.0677	0.0243
AD	1.8958	0.4527	0.0820	0.0274

Table 14: The ES, KS and MAE for data set 1.

Method	ES($\hat{\theta}$)	ES($\hat{\alpha}$)	KS	MAE
MLE	1.4468	0.1709	0.1037	0.0344
MPS	1.4106	0.2269	0.1059	0.0360
OLS	1.6020	-0.1148	0.0709	0.0166
WLS	1.6774	-0.1632	0.0609	0.0176
CV	1.7044	-0.1846	0.0582	0.0191
AD	1.5034	0.0090	0.0802	0.0211

Table 15: The ES, KS and MAE for data set 2.

9 Conclusions

In this article, we have developed the GQLD distribution along with some of its properties such as, some plots of the pdf and cdf, moments, hazard rate function, reliability function, residual and reverse residual life function, stress-strength reliability. The maximum likelihood estimates is computed as well as the maximum product of spacing's, ordinary least squares, weighted least squares, Cramer-von-Mises, and Anderson-Darling methods are obtained. Some confidence intervals are provided with some simulations. Applications of real data sets are studied for illustration. It is proved that the GQLD is empirically better than other competitors models considered in this study. Future work can look at the estimation of the GQLD parameters based on ranked set sampling methods, see for example Al-Omari (2011, 2012), Haq et al . (2014a, 2014b, 2015, 2016), and Al-Omari and Haq (2019).

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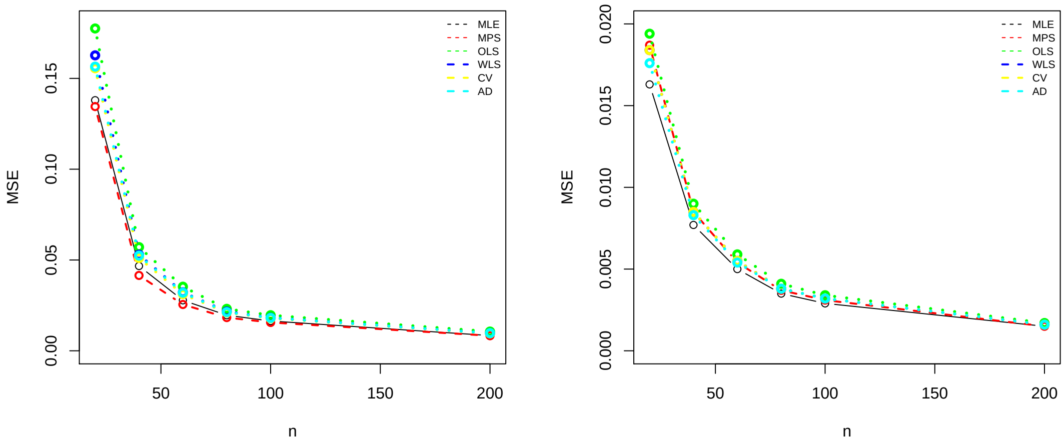
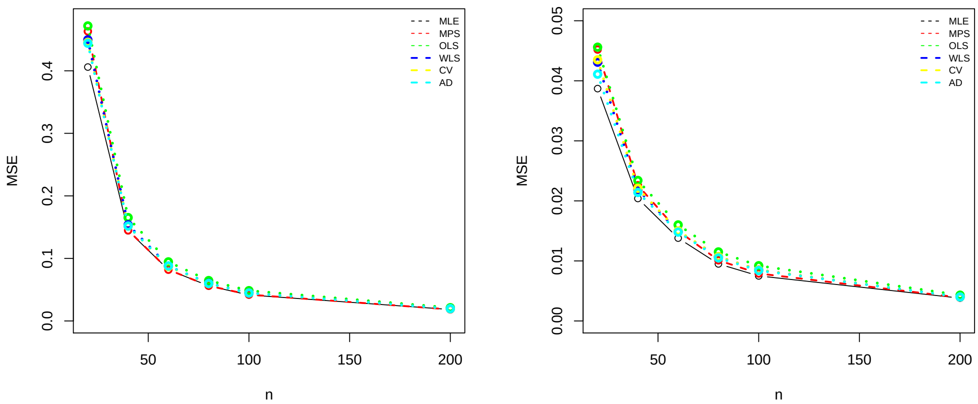


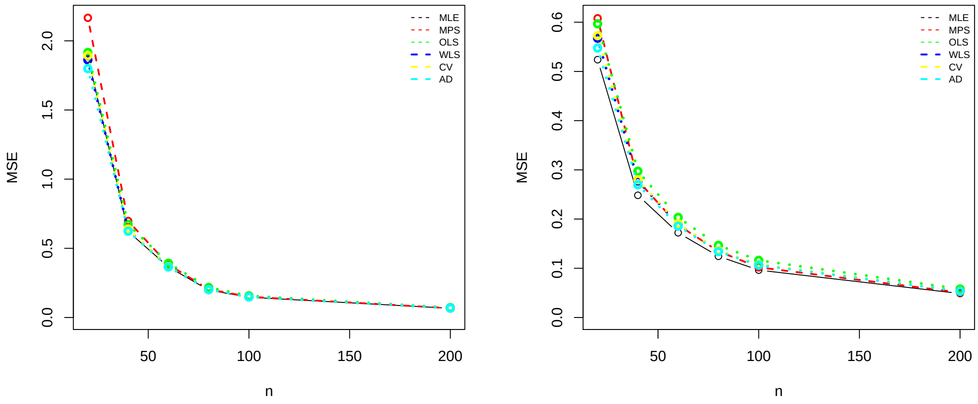
(a) MSEs of α when $\alpha = 0.25$ (b) MSEs of θ when θ=1

Figure 8: The MSE values of MLEs, MPSs, OLSs, WLSs, CVEs and ADs methods for $\alpha = 0.25$ (a) and $\theta = 1$ (b).



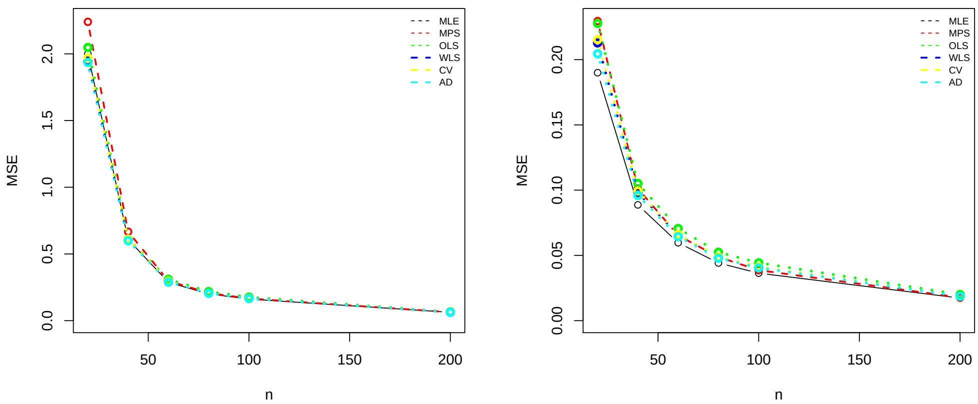
(a) MSEs of α when $\alpha = 0.5$ (b) MSEs of θ when θ=1

Figure 9: The MSE values of MLEs, MPSs, OLSs, WLSs, CVEs and ADs methods for $\alpha = 0.5$ (a) and $\theta = 1$ (b).



(a) MSEs of α when $\alpha = 1$ (b) MSEs of θ when θ=5

Figure 10: The MSE values of MLEs, MPSs, OLSs, WLSs, CVEs and ADs methods for $\alpha = 1$ (a) and $\theta = 5$ (b).



(a) MSEs of α when $\alpha = 1$ (b) MSEs of θ when θ=3

Figure 11: The MSE values of MLEs, MPSs, OLSs, WLSs, CVEs and ADs methods for $\alpha = 1$ (a) and $\theta = 3$ (b).