Electronic Journal of Applied Statistical Analysis
EJASA, Electron. J. App. Stat. Anal.
http://siba-ese.unisalento.it/index.php/ejasa/index
e-ISSN: 2070-5948
DOI: 10.1285/i20705948v14n2p305

Wrapped Akash Distribution

By Al-Khazaleh, Al-Meanazel

Published: 20 November 2021

Electronic Journal of Applied Statistical Analysis
Vol. 14, Issue 02, November 2021, 305-317
DOI: 10.1285/i20705948v14n2p305

Wrapped Akash Distribution

Ahmad M. H. Al-Khazaleh*and Ayat T. R. Al-Meanazel

Department of Mathematics, Al Al-bayt University
P.O.BOX 130040, Mafraq 25113, Jordan

Published: 20 November 2021

In this paper, we generate the wrapped Akash distribution which offers a more flexible model for analyzing some types of circular data set. Some properties of the wrapped Akash distribution are studied such as means, skewness, and kurtosis. We investigate the invariance properties of wrapped Akash distribution, invariance under change of initial direction, and the reference system orientation. The maximum likelihood methods is studied and some simulation studies are conducted to investigate the performance of the maximum likelihood estimator. A real life data set is used to test the goodness of fit of wrapped Akash distribution and examine its performance in comparison with some other distributions.

keywords: Akash distribution, Wrapped distribution, Invariance, Trigonometric moments, Invariant distribution..

1 Introduction

Lifetime data are used in various applied sciences such as medicine, and insurance. Many continuous distributions were proposed for analyzing lifetime data, one of the most used distribution is Lindley distribution due to its mathematical properties (Nadarajah S. and R., 2011). Shanker (2015) proposed new distribution which is more flexible than Lindley distribution. There are many wrapped distributions in literature, for example Al-khazaleh and Alkhazaleh (2020) proposed new wrapped Quasi Lindley distribution. The new distribution "Akash distribution" is a mixture of an exponential distribution with scale parameter λ and a gamma distribution with shape parameter 3 and scale

*Corresponding author: ahmed_2005kh@yahoo.com

ISSN: 2070-5948
http://siba-ese.unisalento.it/index.php/ejasa/index

parameter λ, the mixing proportion is $\frac{\lambda^2}{\lambda^2+2}$. The probability density function of Akash distribution is given by

$$f(x;\lambda) = \frac{\lambda^3}{\lambda^2+2}(1+x^2)e^{-x\lambda} \qquad ; x > 0, \quad \lambda > 0, \tag{1}$$

with cumulative distribution function

$$F(x;\lambda) = 1 - \left[1 + \frac{x\lambda(x\lambda+2)}{\lambda^2+2}\right] e^{-x\lambda} \qquad ; x > 0, \quad \lambda > 0. \tag{2}$$

The characteristic function for Akash distribution is defined as

$$\varphi_X(t) = \frac{\lambda^3}{\lambda^2+2}\left[\frac{1}{\lambda - it} + \frac{2}{(\lambda - it)^3}\right], \qquad ; i = \sqrt{-1}. \tag{3}$$

Circular distributions are used to analyze and model directional data, where the measurements are directions (two or three dimensions). Circular distributions are generated from known probability distribution in many different methods, such as, wrapping a known distribution around the untie circle (see, for example, Mardia and Jupp, 2000; Jammalamadaka and SenGupta, 2001). We introduce a new circular distribution by wrapping Akash distribution around the unite circle.

Consider a circular random variable θ defined by $\theta = X(mod\, 2\pi)$, where X is a random variable with probability density function $f(x)$. The probability density function and cumulative distribution function for $\theta \in [0, 2\pi)$ are given respectively by

$$g(\theta) = \sum_{m=-\infty}^{\infty} f(\theta + 2m\pi),$$

and

$$G(\theta;\lambda) = \sum_{m=0}^{\infty}\Big[F(\theta + 2m\pi) - F(2m\pi)\Big].$$

2 Probability Densities Functions

Let X be an Akash random variable, then the circular random variable generated by X is $\theta = X(mod\ 2\pi)$ with probability density function $g(\theta;\lambda)$ given by

$$\begin{aligned} g(\theta;\lambda) &= \sum_{m=0}^{\infty} f(\theta + 2m\pi) \\ &= \sum_{m=0}^{\infty} \frac{\lambda^3}{\lambda^2+2}\big(1 + (\theta + 2m\pi)^2\big)e^{-(\theta+2m\pi)\lambda} \\ &= \frac{\lambda^3\ e^{-\theta\lambda}\Big[(1+\theta^2)(1-e^{-2\pi\lambda})^2 + 4\pi e^{-2\pi\lambda}(\theta + \pi + \pi e^{-2\pi\lambda} - \theta e^{-2\pi\lambda})\Big]}{(\lambda^2+2)(1-e^{-2\pi\lambda})^3}. \end{aligned} \tag{4}$$

The cumulative distribution function $G(\theta;\lambda)$ for $\theta \in [0, 2\pi)$ and $\lambda > 0$ is given by

$$\begin{aligned} G(\theta;\lambda) &= \sum_{m=0}^{\infty} [F(\theta + 2m\pi) - F(2m\pi)] \\ &= \sum_{m=0}^{\infty} \Bigg[1 - \left(1 + \frac{(\theta + 2m\pi)\lambda([\theta + 2m\pi]\lambda + 2)}{\lambda^2 + 2}\right) e^{-(\theta+2m\pi)\lambda} \\ &\qquad - 1 + \left(1 + \frac{2m\pi\lambda(2m\pi\lambda + 2)}{\lambda^2 + 2}\right) e^{-(2m\pi)\lambda} \Bigg] \qquad (5) \\ &= \frac{\begin{bmatrix} 4\pi\lambda e^{-2\pi\lambda}\big[(1 - [\theta\lambda + 1]e^{-\theta\lambda})(1 - e^{-2\pi\lambda}) + \pi\lambda(1 - e^{-\theta\lambda})(1 + e^{-2\pi\lambda})\big] \\ +\big[(1 - e^{-\theta\lambda})(\lambda^2 + 2) - \theta\lambda(\theta\lambda + 2)e^{-\theta\lambda}\big](1 - e^{-2\pi\lambda})^2 \end{bmatrix}}{(\lambda^2 + 2)(1 - e^{-2\pi\lambda})^3}. \end{aligned}$$

Figure 1 shows the probability density function (PDF) of the wrapped Akash distribution with different values of λ. While, Figure 2 shows the cumulative distribution function of the wrapped Akash distribution with different values of λ .

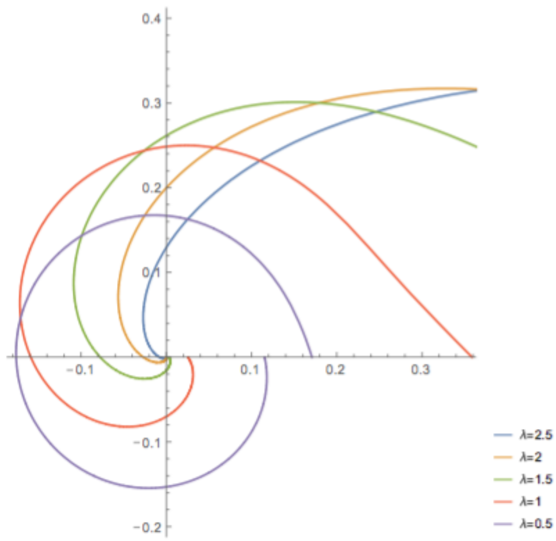

Figure 1: The PDF of wrapped Akash distribution for $\lambda = 0.5, 1, 1.5, 2, 2.5$.

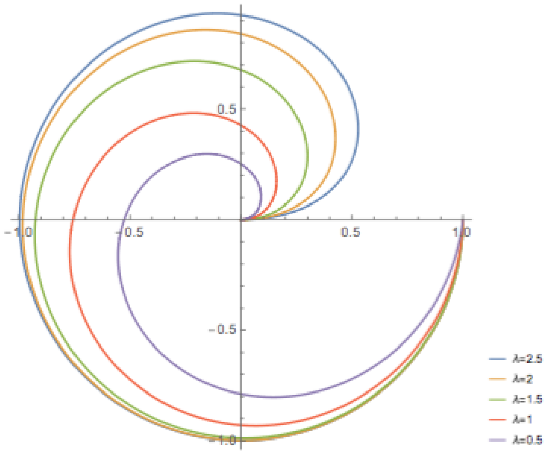

Figure 2: The CDF of wrapped Akash distribution for $\lambda = 0.5, 1, 1.5, 2, 2.5$.

3 Hazard Function

The reliability function of wrapped Akash distribution is given by

$$\begin{aligned} R(\theta;\lambda) &= 1 - G(\theta;\lambda) \\ &= 1 - \frac{\begin{bmatrix} 4\pi\lambda e^{-2\pi\lambda}\big[(1-[\theta\lambda+1]e^{-\theta\lambda})(1-e^{-2\pi\lambda}) + \pi\lambda(1-e^{-\theta\lambda})(1+e^{-2\pi\lambda})\big] \\ +\big[(1-e^{-\theta\lambda})(\lambda^2+2) - \theta\lambda(\theta\lambda+2)e^{-\theta\lambda}\big](1-e^{-2\pi\lambda})^2 \end{bmatrix}}{(\lambda^2+2)(1-e^{-2\pi\lambda})^3} \\ &= \frac{\begin{bmatrix} \big[(\lambda^2+2)(1-[1-e^{-2\pi\lambda}]^3) + e^{-\theta\lambda}(\lambda^2+2-\theta\lambda[\theta\lambda+2])\big](1-e^{-2\pi\lambda})^2 \\ -4\pi\lambda e^{-2\pi\lambda}\big[(1-[\theta\lambda+1]e^{-\theta\lambda})(1-e^{-2\pi\lambda}) + \pi\lambda(1-e^{-\theta\lambda})(1+e^{-2\pi\lambda})\big] \end{bmatrix}}{(\lambda^2+2)(1-e^{-2\pi\lambda})^3}. \end{aligned} \tag{6}$$

The hazard function of wrapped Akash distribution is given by

$$\begin{aligned} H(\theta;\lambda) &= \frac{g(\theta;\lambda)}{R(\theta;\lambda)} \\ &= \frac{\lambda^3\, e^{-\theta\lambda}\big[(1+\theta^2)(1-e^{-2\pi\lambda})^2 + 4\pi e^{-2\pi\lambda}(\theta+\pi+\pi e^{-2\pi\lambda} - \theta e^{-2\pi\lambda})\big]}{\begin{bmatrix} \big[(\lambda^2+2)(1-[1-e^{-2\pi\lambda}]^3) + e^{-\theta\lambda}(\lambda^2+2-\theta\lambda[\theta\lambda+2])\big](1-e^{-2\pi\lambda})^2 \\ -4\pi\lambda e^{-2\pi\lambda}\big[(1-[\theta\lambda+1]e^{-\theta\lambda})(1-e^{-2\pi\lambda}) + \pi\lambda(1-e^{-\theta\lambda})(1+e^{-2\pi\lambda})\big] \end{bmatrix}}. \end{aligned} \tag{7}$$

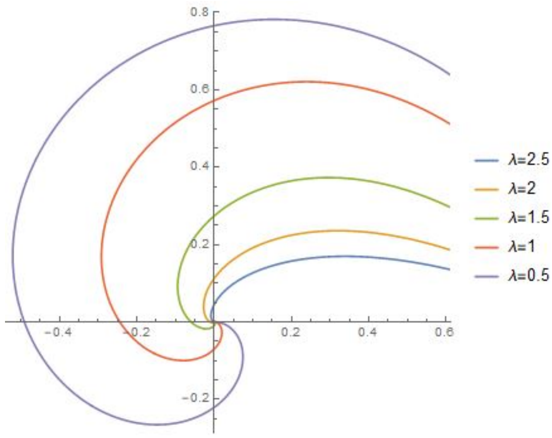

Figure 3: The hazard function of wrapped Akash distribution for $\lambda = 0.5, 1, 1.5, 2, 2.5$.

4 Maximum Likelihood Estimate

Let $\theta_1, \theta_2, \ldots, \theta_n$ be a random sample from wrapped Akash distribution, then the likelihood function is given by

$$
\begin{aligned}
L(\lambda/\theta) &= \prod_{i=1}^{n} g(\theta_i;\lambda) \\
&= \prod_{i=1}^{n} \frac{\lambda^3\ e^{-\theta_i\lambda}\left[(1+{\theta_i}^2)(1-e^{-2\pi\lambda})^2 + 4\pi e^{-2\pi\lambda}(\theta_i+\pi+\pi e^{-2\pi\lambda}-\theta_i e^{-2\pi\lambda})\right]}{(\lambda^2+2)(1-e^{-2\pi\lambda})^3} \\
&= \left(\frac{\lambda^3}{(\lambda^2+2)(1-e^{-2\pi\lambda})^3}\right)^n \left(e^{-\lambda\sum_{i=1}^{n}\theta_i}\right) \\
&\qquad \left(\prod_{i=1}^{n}\left[(1+{\theta_i}^2)(1-e^{-2\pi\lambda})^2 + 4\pi e^{-2\pi\lambda}(\theta_i+\pi+\pi e^{-2\pi\lambda}-\theta_i e^{-2\pi\lambda})\right]\right).
\end{aligned}
$$

Hence, the log-likelihood function is given by

$$
\begin{aligned}
lnL(\lambda/\theta) &= 3n\ ln\lambda - n\ ln(\lambda^2+2) - 3n\ ln(1-e^{-2\pi\lambda}) - \lambda\sum_{i=1}^{n}\theta_i \\
&\quad + \sum_{i=1}^{n} ln\left[(1+{\theta_i}^2)(1-e^{-2\pi\lambda})^2 + 4\pi e^{-2\pi\lambda}(\theta_i+\pi+\pi e^{-2\pi\lambda}-\theta_i e^{-2\pi\lambda})\right].
\end{aligned}
$$

The derivative of the log-likelihood function with respect to the parameter λ is

$$\frac{\partial\ lnL(\lambda/\theta)}{\partial\ \lambda} = \frac{3n}{\lambda} - \frac{2n\lambda}{\lambda^2+2} - \frac{6n\pi e^{-2\pi\lambda}}{1-e^{-2\pi\lambda}} - \sum_{i=1}^{n}\theta_i + \sum_{i=1}^{n}\frac{4\pi e^{-2\pi\lambda}\left[1+\theta_i{}^2-2\pi\theta_i-2\pi^2\right]-4\pi e^{-4\pi\lambda}\left[1+\theta_i{}^2-4\pi\theta_i+4\pi^2\right]}{(1+\theta_i{}^2)(1-e^{-2\pi\lambda})^2+4\pi e^{-2\pi\lambda}(\theta_i+\pi+\pi e^{-2\pi\lambda}-\theta_i e^{-2\pi\lambda})}. \tag{8}$$

By equating the derivative in (8) to zero and using numerical methods to solve the resulting equation we can found the maximum likelihood estimator for the distribution parameter λ.

5 Characteristic Function and Trigonometric Moments

The p^{th} trigonometric moment for wrapped distribution is equal to the value of the characteristic function of the distribution which generate the wrapped distribution. Hence, the p^{th} trigonometric moment for wrapped Akash distribution is

$$\begin{aligned}\varphi_\theta(p) &= \varphi_X(p)\\ &= \frac{\lambda^3}{\lambda^2+2}\left[\frac{1}{\lambda-ip}+\frac{2}{(\lambda-ip)^3}\right]\\ &= \frac{\lambda^3}{\lambda^2+2}(\lambda^2-p^2+2-2\ ip\lambda)(\lambda-ip)^{-3}\ ; i=\sqrt{-1},\ p=\mp1,\mp2,\ldots\end{aligned} \tag{9}$$

Since $(a-ib)^{-r} = (a^2+b^2)^{\frac{-r}{2}}e^{ir\ arctan(b/a)}$ for $a, b, r \in \Re$, we have

$$(\lambda^2-p^2+2-2\ ip\lambda) = ((\lambda^2-p^2+2)^2+4p^2\lambda^2)^{\frac{1}{2}}e^{-i\ arctan\left(\frac{2p\lambda}{\lambda^2-p^2+2}\right)},$$

and

$$(\lambda-ip)^{-3} = (\lambda^2+p^2)^{-3/2}e^{3i\ arctan\left(\frac{p}{\lambda}\right)}.$$

Then we can write the p^{th} trigonometric moment for wrapped Akash distribution $\varphi_\theta(p)$ as

$$\varphi_\theta(p) = \rho_p e^{i\mu_p};\quad i=\sqrt{-1},\ p=\mp1,\mp2,\ldots, \tag{10}$$

where

$$\rho_p = \frac{\lambda^3}{\lambda^2+2}\sqrt{\frac{(\lambda^2-p^2+2)^2+4p^2\lambda^2}{(\lambda^2+p^2)^3}},$$

and

$$\mu_p = 3\ arctan\left(\frac{p}{\lambda}\right) - arctan\left(\frac{2p\lambda}{\lambda^2-p^2+2}\right).$$

Another way to express the p^{th} trigonometric moment of wrapped Akash distribution as $\varphi_\theta(p) = \alpha_p + i\beta_p$ where $\alpha_p = E(cos\ p\theta)$ and $\beta_p = E(sin\ p\theta)$

The non-central trigonometric moments for wrapped Akash distribution are given by

$$\alpha_p = \rho_p cos\ \mu_p = \frac{\lambda^3}{\lambda^2+2}\sqrt{\frac{(\lambda^2-p^2+2)^2+4p^2\lambda^2}{(\lambda^2+p^2)^3}} cos\left[3\ arctan\left(\frac{p}{\lambda}\right) - arctan\left(\frac{2p\lambda}{\lambda^2-p^2+2}\right)\right], \tag{11}$$

and

$$\beta_p = \rho_p sin\ \mu_p = \frac{\lambda^3}{\lambda^2+2}\sqrt{\frac{(\lambda^2-p^2+2)^2+4p^2\lambda^2}{(\lambda^2+p^2)^3}} sin\left[3\ arctan\left(\frac{p}{\lambda}\right) - arctan\left(\frac{2p\lambda}{\lambda^2-p^2+2}\right)\right]. \tag{12}$$

While the central trigonometric moments of wrapped Akash distribution are given by

$$\bar{\alpha}_p = \rho_p cos(\mu_p - p\mu_1) = \frac{\lambda^3}{\lambda^2+2}\sqrt{\frac{(\lambda^2-p^2+2)^2+4p^2\lambda^2}{(\lambda^2+p^2)^3}} cos\left[3\ arctan\left(\frac{p}{\lambda}\right) - arctan\left(\frac{2p\lambda}{\lambda^2-p^2+2}\right) - 3p\ arctan\left(\frac{1}{\lambda}\right) + p\ arctan\left(\frac{2\lambda}{\lambda^2+1}\right)\right], \tag{13}$$

and

$$\bar{\beta}_p = \rho_p sin(\mu_p - p\mu_1) = \frac{\lambda^3}{\lambda^2+2}\sqrt{\frac{(\lambda^2-p^2+2)^2+4p^2\lambda^2}{(\lambda^2+p^2)^3}} sin\left[3\ arctan\left(\frac{p}{\lambda}\right) - arctan\left(\frac{2p\lambda}{\lambda^2-p^2+2}\right) - 3p\ arctan\left(\frac{1}{\lambda}\right) + p\ arctan\left(\frac{2\lambda}{\lambda^2+1}\right)\right]. \tag{14}$$

Table 1 shows the values of non-central and central trigonometric moments for $p = 1, 2$ and $\lambda = 0.5, 1, 1.5, 2, 4$.

Parameter	Non-central Trigonometric Moments				Central Trigonometric Moments			
λ	α_1	α_2	β_1	β_2	$\bar{\alpha}_1$	$\bar{\alpha}_2$	$\bar{\beta}_1$	$\bar{\beta}_2$
0.5	-0.056	0.002	0.030	-0.017	0.064	0.015	0	-0.008
1	0	0	0.333	-0.123	0.333	0.008	0	0.123
1.5	0.314	0.095	0.510	0.290	0.599	0.216	0	-0.216
2	0.576	0.250	0.501	0.417	0.764	0.447	0	-0.190
4	0.912	0.725	0.277	0.434	0.953	0.844	0	-0.043

Table 1: Trigonometric moments for wrapped Akash distribution for $p = 1, 2$ and $\lambda = 0.5, 1, 1.5, 2, 4$.

6 Invariance Properties of Wrapped Akash Distribution

To avoid conflicting or misleading statistical inference the distribution used to study circular variables must be invariant with respect to changes of initial direction and changes of orientation (see, for example, Mastrantonio et al., 2015, 2019).

Any circular distribution with probability density function $g(\theta;\psi)$ is $ICID$ and ICO iff for $\delta = \{-1, 1\}$, $\xi \in \{2\pi j\}_{j=0}^{l-1}$, and $\theta^* = \delta(\theta + \xi)$ the probability densities functions $g(\theta;\psi)$ and $g(\theta^*;\psi^*)$ belong to the same parametric family (i.e., their characteristic function must be of the same functional form).

For wrapped Akash distribution the characteristics function of θ^* is given by

$$\begin{aligned}\varphi_{\theta^*}(p) &= e^{ip\delta\xi}\varphi_\theta(p\delta)\\ &= \frac{\lambda^3\big(cos(p\delta\xi) - i\ sin(p\delta\xi)\big)}{\lambda^2+2}\ (\lambda^2 - p^2 + 2 - 2\ ip\delta\lambda)(\lambda - ip\delta)^{-3}.\end{aligned} \tag{15}$$

The real and imaginary parts of (9) and (15) are the same only if $\delta = 1$ and $\xi = 0$, therefore, wrapped Akash distribution is not $ICID$ and ICO. The invariant version of wrapped Akash distribution is given by

$$g(\theta^*;\lambda,\delta,\xi) = \frac{\lambda^3\ e^{-(\delta\theta^*-\xi)\lambda}\begin{bmatrix}(1+\theta^{*2} - 2\delta\theta^*\xi + \xi^2)(1-e^{-2\pi\lambda})^2\\ +4\pi e^{-2\pi\lambda}\big(\delta\theta^* - \xi + \pi + \pi e^{-2\pi\lambda} - (\delta\theta^* - \xi)e^{-2\pi\lambda}\big)\end{bmatrix}}{(\lambda^2+2)(1-e^{-2\pi\lambda})^3}.$$

7 Means and Related Measures

The mean direction and resultant length for wrapped Akash distribution are given respectively by

$$\mu = \mu_1 = 3\ arctan\left(\frac{1}{\lambda}\right) - arctan\left(\frac{2\lambda}{\lambda^2+1}\right), \tag{16}$$

and

$$\rho = \rho_1 = \frac{\lambda^3}{\lambda^2+2}\sqrt{\frac{(\lambda^2+1)^2+4\lambda^2}{(\lambda^2+1)^3}}. \tag{17}$$

Table 2 shows the values of mean direction and mean resultant for $\lambda = 0.5, 1, 1.5, 2, 4$.

Means	**Parameter λ**				
	0.5	1	1.5	2	4
Direction μ	2.647	1.571	1.019	0.716	0.295
Resultant Length ρ	0.080	0.667	1.948	3.818	16.202

Table 2: Means for wrapped Akash distribution for $\lambda = 0.5, 1, 1.5, 2, 4$.

While, the circular variance and standard deviation for wrapped Akash distribution are given respectively by

$$\begin{aligned} V &= 1-\rho \\ &= 1-\frac{\lambda^3}{\lambda^2+2}\sqrt{\frac{(\lambda^2+1)^2+4\lambda^2}{(\lambda^2+1)^3}}, \end{aligned} \tag{18}$$

and

$$\begin{aligned} \sigma &= \sqrt{-2ln\rho} \\ &= \sqrt{-2ln\left(\frac{\lambda^3}{\lambda^2+2}\sqrt{\frac{(\lambda^2+1)^2+4\lambda^2}{(\lambda^2+1)^3}}\right)}. \end{aligned} \tag{19}$$

Table 3 shows the values of circular variance and circular standard deviation for $\lambda = 0.5, 1, 1.5, 2, 4$.

Measures of Variation	**Parameter λ**				
	0.5	1	1.5	2	4
Circular Variance V	0.936	0.667	0.401	0.236	0.047
Circular Standard Deviation σ	6.621	2.197	0.348	-0.330	-0.680

Table 3: Measures of variation for wrapped Akash distribution for $\lambda = 0.5, 1, 1.5, 2, 4$.

8 Skewness and Kurtosis

For circular distributions the skewness coefficient is calculated by $\zeta_1 = \bar{\beta}_2 V^{-3/2}$, and the kurtosis coefficient is calculated by $\zeta_2 = \left(\bar{\alpha}_2 - (1-V)^4\right) V^{-2}$. For wrapped Akash distribution the skewness and the kurtosis coefficients are given respectively by

$$\begin{aligned}\zeta_1 &= \bar{\beta}_2 V^{-3/2} \\ &= \frac{\lambda^3}{\lambda^2+2}\sqrt{\frac{(\lambda^2-2)^2+16\lambda^2}{(\lambda^2+4)^3}}\left(1-\frac{\lambda^3}{\lambda^2+2}\sqrt{\frac{(\lambda^2+1)^2+4\lambda^2}{(\lambda^2+1)^3}}\right)^{\frac{-3}{2}} \\ &\quad \left(sin\left[3\ arctan\left(\frac{2}{\lambda}\right)-arctan\left(\frac{4\lambda}{\lambda^2-2}\right)-6\ arctan\left(\frac{1}{\lambda}\right)\right.\right. \\ &\quad \left.\left.+2\ arctan\left(\frac{2\lambda}{\lambda^2+1}\right)\right]\right),\end{aligned} \tag{20}$$

and

$$\begin{aligned}\zeta_2 &= \left(\bar{\alpha}_2-(1-V)^4\right)V^{-2} \\ &= \left(1-\frac{\lambda^3}{\lambda^2+2}\sqrt{\frac{(\lambda^2+1)^2+4\lambda^2}{(\lambda^2+1)^3}}\right)^{-2}\left(\frac{\lambda^3}{\lambda^2+2}\sqrt{\frac{(\lambda^2-2)^2+16\lambda^2}{(\lambda^2+4)^3}}\right. \\ &\quad cos\left[3\ arctan\left(\frac{2}{\lambda}\right)-arctan\left(\frac{4\lambda}{\lambda^2-2}\right)-6\ arctan\left(\frac{1}{\lambda}\right)\right. \\ &\quad \left.\left.+2\ arctan\left(\frac{2\lambda}{\lambda^2+1}\right)\right]-\left(\frac{\lambda^3}{\lambda^2+2}\sqrt{\frac{(\lambda^2+1)^2+4\lambda^2}{(\lambda^2+1)^3}}\right)^4\right).\end{aligned} \tag{21}$$

Table 4 shows the values of skewness and kurtosis coefficients for $\lambda = 0.5, 1, 1.5, 2, 4$.

Coefficients	**Parameter λ**				
	0.5	1	1.5	2	4
Skewness ζ_1	-0.008	0.225	-0.851	-1.654	-4.254
Kurtosis ζ_2	-0.171	-0.732	0.825	7.727	383.068

Table 4: Skewness and kurtosis coefficients for wrapped Akash distribution for $\lambda = 0.5, 1, 1.5, 2, 4$.

9 Numerical Studies

9.1 Simulation Study

A simulation study is carried out for samples of sizes 30, 80, 100, 200 and 350, drawn from wrapped Akash distribution. The samples have been drawn for $\lambda = 0.5, \lambda = 1, \lambda = 2$ and $\lambda = 4$ and maximum likelihood estimator for the parameter λ are obtained. The procedure has been repeated for 10000 and the mean square error (MSE), mean relative error (MRE) and estimator bias (Bias) for the estimate are computed. Table 5 and Table 6 shows that the maximum likelihood estimator is closed to the true value of the parameter and hence the estimation method is adequate.

Sample size	$\lambda = 0.5$				$\lambda = 1$			
	MLE	**MSE**	**MRE**	**Bias**	**MLE**	**MSE**	**MRE**	**Bias**
30	0.4025	0.2626	0.1059	0.5251	0.9801	0.1351	0.0532	0.1350
80	0.4014	0.2096	0.0816	0.4192	0.9996	0.0669	0.0088	0.0669
100	0.4009	0.1982	0.0767	0.3965	1.0016	0.0582	0.0057	0.0582
200	0.4149	0.1555	0.0582	0.3110	1.0007	0.0404	0.0026	0.0404
350	0.4302	0.1312	0.0472	0.2625	0.9994	0.0335	0.0018	0.0335

Table 5: Simulation results for wrapped Akash distribution for $\lambda = 0.5$ and $\lambda = 1$.

Sample size	$\lambda = 2$				$\lambda = 4$			
	MLE	**MSE**	**MRE**	**Bias**	**MLE**	**MSE**	**MRE**	**Bias**
30	2.0310	0.1903	0.0597	0.0952	4.1022	0.4832	0.3996	0.1208
80	2.0111	0.1134	0.0206	0.0567	4.0491	0.2833	0.1302	0.0708
100	2.0097	0.1023	0.0170	0.0511	4.0338	0.2541	0.1046	0.0635
200	2.0040	0.0715	0.0082	0.0357	4.0213	0.1769	0.0501	0.0442
350	2.0050	0.0573	0.0052	0.0287	4.0127	0.1474	0.0348	0.0369

Table 6: Simulation results for wrapped Akash distribution for $\lambda = 2$ and $\lambda = 4$.

9.2 Application

In this subsection, we compare the performance of wrapped Akash distribution with the performance of wrapped xgamma distribution (Al-Mofleh and Sen, 2019), wrapped Lindley distribution (Joshi and Jose, 2018) and wrapped exponential distribution (Jammalamadaka and Kozubowski, 2004) by fitting these distributions to Fisher-B5 data set (Fisher, 1995). The data set is described as the 164 measurements of long-axis orientation of feldspar laths in basalt rock, it contains 60 orientated observations (in degrees).

The maximum likelihood estimates (MLEs) and corresponding standard errors (SE) of the parameter for Fisher-B5 data set are given in Table 7.

Distribution	**MLE**	**SE**
Wrapped Akash	1.4038	0.1097
Wrapped xgamma	1.3208	0.1301
Wrapped Lindley	1.0309	0.1096
Wrapped exponential	0.6640	0.1008

Table 7: MLEs and their SE for the fitted distributions.

The goodness of fit of wrapped Akash distribution to Fisher-B5 data set is also compared with wrapped xgamma distribution, wrapped Lindley distribution and wrapped exponential distribution based on -2 loglikelihood (-2L), Akaike information criterion (AIC), Bayesian information criterion (BIC) , Kolmogorov-Smirnov (K-S) and the corresponding p-values for the models. The results are presented in Table 8. Clearly, wrapped Akash distribution has the lowest values of the -2L, AIC, BIC , K-S(stat) and the largest value of K-S(p-value). Therefore, wrapped Akash distribution is the best distribution for fitting Fisher-B5 data set

Distribution	**-2L**	**AIC**	**BIC**	**K-S(stat)**	**K-S(p-value)**
Wrapped Akash	154.6774	156.6773	158.7717	0.1005	0.5526
Wrapped xgamma	156.0570	158.0570	160.1514	0.1042	0.5323
Wrapped Lindley	156.8536	158.8536	160.9480	0.1112	0.4482
Wrapped exponential	159.4301	161.4301	163.5244	0.1165	0.3900

Table 8: -2L, AIC, BIC, K-S(stat) and K-S(p-value) statistics for the fitted distributions.

10 Conclusions

A new circular distribution generated from wrapping Akash distribution is proposed. The explicit expressions for the probability density function and the cumulative density function for wrapped Akash distribution were obtained. Also, we have obtained the reliability function and hazard function. Simulation studies provides less bias and mean square error of the maximum likelihood estimator. Moreover, the trigonometric moments, measures of variation, and invariance properties of the new distribution were studied. The new distribution provides better fit and more flexible model in comparison to other competitive distributions.

References

Al-khazaleh, A. M. and Alkhazaleh, S. (2020). On wrapping of quasi lindley distribution. *MDPI Open Access Journals, Mathematics*, 10(7):930.

Al-Mofleh, H. and Sen, S. (2019). The wrapped xgamma distribution for modeling circular data appearing in geological context. arXiv:1903.00177v1 [stat.ME].

Fisher, N. (1995). Statistical analysis of circular data. *Cambridge University Press*.

Jammalamadaka, S. and Kozubowski, T. (2004). New families of wrapped distributions for modeling skew circular data. *Communications in Statistics - Theory and Methods*, 33(9):2059–2074.

Jammalamadaka, S. R. and SenGupta, A. (2001). *Topics in circular statistics.* World Scientific Publishing Co. Pte. Ltd.

Joshi, S. and Jose, K. K. (2018). Wrapped lindley distribution. *Communications in Statistics - Theory and Methods*, 47(5):1013–1021.

Mardia, K. V. and Jupp, P. E. (2000). *Directional Statistics.* John Wiley & Sons,Ltd.

Mastrantonio, G., Lasinio, G. J., Maruotti, A., and Calise, G. (2015). On initial direction, orientation and discreteness in the analysis of circular variables. arXiv:1509.08638 [stat.ME].

Mastrantonio, G., Lasinio, G. J., Maruotti, A., and Calise, G. (2019). Invariance properties and statistical inference for circular data. *Statistica Sinica*, 29(1):67–80.

Nadarajah S., B. H. and R., T. (2011). A generalized lindley distribution. *Sankhya Series B*, 73:331–359.

Shanker, R. (2015). Akash distribution and its applications. *International Journal of Probability and Statistics*, 4(3):65–75.