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# Extended asymmetry model based on logit transformation and decomposition of symmetry for square contingency tables with ordered categories

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Issues of symmetry (or asymmetry) arises naturally in the analysis of square contingency tables. Because many existing asymmetry models do not constrain the main diagonal cells, observations on these cells do not contribute to the likelihood ratio chi-squared test statistics. Herein we propose a model that indicates the asymmetry for the log odds. It can utilize the information in the main diagonal cells. Additionally, the symmetry model can be separated into different models, including the proposed model.

**keywords:** conditional symmetry model, contingency table analysis, symmetry model.

## 1 Introduction

This paper describes a method to handle matched-pairs data. Matched-pairs data occur in longitudinal studies, case-control studies, etc. For example, a square contingency table may arise when a sample of pairs of matched individuals (e.g., husbands and wives) are classified according to a categorical variable. Typically, responses in matched-pairs data are statistically dependent. That is, the independence model does not hold in a square contingency table formed from matched-pairs data.

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Contingency tables are analyzed to estimate an unknown probability distribution from the observed frequencies. Our research aims to devise a model that employs fewer parameters to describe the data, but provides an estimate with a high confidence level. To this end, we must consider a statistical model that fits the given dataset and specifies the structure of association between two categorical responses. For square contingency tables, the symmetry and asymmetry between two variables are representative patterns of association. Therefore, we are interested in determining whether the structure in a square contingency table is symmetric or asymmetric instead of its independence.

Table 1 is a square contingency table, which shows data constructed from a database on a website (Seiyama, 1995). These data describe the cross-classification of individual's income and spouse's income in Japan, which were examined in 1995. Income was divided into the following categories: (1) less than 700,000 yen, (2) about 1 million yen, (3) 2 to 4 million yen, and (4) over 5 million yen.

In this study, we are interested in determining whether an individual's income is equally likely to the spouse's income for the data in Table 1. Many statisticians have proposed various symmetry and asymmetry models (e.g., Bowker, 1948; McCullagh, 1978; Agresti, 1983). However, the parameters on the main diagonal cells are saturated in these models because these cells are not constrained. Because Fujisawa and Tahata (2018) were interested in utilizing the information on the main diagonal cells, they proposed models in which the parameters on the main diagonal cells are not saturated using the logit transformation. Their models provide an asymmetric structure for cell probabilities. Here, we consider a new model, which is an extension of Fujisawa and Tahata's models. The proposed model may be useful to visualize the structure where the one cell probability is a location shift of the symmetric cell probability on a logistic scale.

The symmetry model, which shows a symmetric structure of cell probabilities, can be decomposed into the symmetry of the odds ratio and the homogeneity of the marginal distribution (Caussinus, 1965). This may be useful to elucidate the origin when the symmetry model provides a poor fit for a real dataset. Additionally, Fujisawa and Tahata (2018) proposed the decomposition of the symmetry model. Similarly, we are interested in considering the decomposition of symmetry using the proposed model.

This paper is organized as follows. Section 2 describes the proposed model. Section 3 decomposes the symmetry model. Section 4 evaluates the goodness-of-fit of the proposed model. Section 5 provides a numerical example. Section 6 discusses the result, while Section 7 concludes the paper.

## 2 Models

Consider an $r \times r$ square contingency table with the same ordinal row and column classifications. Let $\pi_{ij}$ denote the probability that an observation will fall in the $i$th row and the $j$th column of the table ($i = 1, \ldots, r; j = 1, \ldots, r$). Consider the following asymmetric structure

$$\pi_{ij} = \begin{cases} \delta_{ji}\psi_{ij} & (i < j), \\ \psi_{ij} & (i \geq j), \end{cases} \tag{1}$$

where $\psi_{ij} = \psi_{ji}$ (Tahata and Tomizawa, 2015). $\{\delta_{ji}\}$ indicates the ratio between two symmetric cells. Note that Eq. (1) with $\{\delta_{ji} = 1\}$, $\{\delta_{ji} = \delta\}$, and $\{\delta_{ji} = \delta^{j-i}\}$ reduces to the symmetry (S) model (Bowker, 1948), the conditional symmetry model (McCullagh, 1978), and the linear diagonals-parameter symmetry model (Agresti, 1983), respectively. In these models, the parameters on the main diagonal cells are saturated.

Let $X$ and $Y$ denote the row and column variables, respectively. Fujisawa and Tahata (2018) considered an asymmetric structure for cell probabilities defined by

$$L_{ij} = \Delta_{ji} + L_{ji} \quad (i < j), \tag{2}$$

where

$$L_{ij} = \log \frac{\pi_{ij}}{1 - \pi_{ij}} \quad \text{and} \quad L_{ji} = \log \frac{\pi_{ji}}{1 - \pi_{ji}}.$$

We note that $L_{ij}$ is the log odds where $(X, Y) = (i, j)$ instead of $(X, Y) \neq (i, j)$, and $L_{ji}$ is the log odds where $(X, Y) = (j, i)$ instead of $(X, Y) \neq (j, i)$. $\{\Delta_{ji}\}$ indicates the log odds ratios between two symmetric cells. The S model is a special case of this model with $\{\Delta_{ji} = 0\}$. Various models have been proposed by changing the structure of $\{\Delta_{ji}\}$. In the logit conditional symmetry (LoCS) model, Eq. (2) is replaced by $\{\Delta_{ji} = \Delta\}$. That is

$$L_{ij} = \Delta + L_{ji} \quad (i < j).$$

The LoCS model indicates that the log odds ratios are constant for $i < j$. Similarly, the logit linear diagonals-parameter symmetry (LoLDPS) model refers to a model where Eq. (2) is replaced by $\{\Delta_{ji} = (j - i)\Delta\}$. That is

$$L_{ij} = (j - i)\Delta + L_{ji} \quad (i < j).$$

This model indicates that the log odds ratio is proportional to the difference $j - i$ between the value of $X$ and the value of $Y$. For more details of both LoCS and LoLDPS models, please see Fujisawa and Tahata (2018).

Here, we propose a new model, which is an extended LoLDPS (or LoCS) model. The extended logit linear diagonals-parameter symmetry (ELoLDPS) model is defined as

$$L_{ij} = \Delta + (j - i)\Theta + L_{ji} \quad (i < j).$$

This model implies that the log odds ratios are expressed by the sum of constant $\Delta$ and $(j - i)\Theta$, which depend on the difference $j - i$ between the value of $X$ and the value of $Y$.

The ELoLDPS model may appear similar to the two-ratios parameter symmetry model (Tomizawa, 1987), which is given as

$$\pi_{ij} = \delta\theta^{j-i}\pi_{ji} \quad (i < j).$$

However, the log odds in the ELoLDPS model shows an asymmetric structure. Under this model

$$\pi_{ij} = \frac{\exp(\Delta + (j-i)\Theta + L_{ji})}{1 + \exp(\Delta + (j-i)\Theta + L_{ji})}, \quad \pi_{ji} = \frac{\exp(L_{ji})}{1 + \exp(L_{ji})} \quad (i < j).$$

Namely, this model indicates that the one cell probability is a location shift of the symmetric cell probability on the logistic scale. The shift depends on the difference between the values of $X$ and $Y$.

In addition, the ELoLDPS model with $\Delta = \Theta = 0$, $\Delta = 0$, and $\Theta = 0$ is reduced to the S model, the LoLDPS model, and the LoCS model, respectively. Therefore, the proposed model is an extension of these models.

## 3 Decomposition of the symmetry model

If the S model holds, the ELoLDPS model holds. However, the reverse does not hold generally. This section considers an additional condition to obtain the S model assuming that the ELoLDPS model holds.

The global symmetry (GS) model is defined by

$$\delta_U = \delta_L,$$

where

$$\delta_U = \sum\sum_{i<j} \pi_{ij} \quad \text{and} \quad \delta_L = \sum\sum_{i<j} \pi_{ji},$$

(Read, 1977). In addition, the mean equality (ME) model is defined by

$$E(X) = E(Y),$$

where

$$E(X) = \sum_{i=1}^{r}\sum_{j=1}^{r} i\pi_{ij} \quad \text{and} \quad E(Y) = \sum_{i=1}^{r}\sum_{j=1}^{r} j\pi_{ij}.$$

This leads to the following theorem:

**Theorem 1** *The S model holds if and only if the ELoLDPS model, the GS model, and the ME model all hold.*

*Proof.* If the S model holds, then

$$\delta_U = \sum\sum_{i<j} \pi_{ij} = \sum\sum_{i<j} \pi_{ji} = \delta_L$$

and

$$\sum_{i=1}^{r}\sum_{j=1}^{r} i\pi_{ij} = \sum_{i=1}^{r}\sum_{j=1}^{r} i\pi_{ji} = \sum_{i=1}^{r}\sum_{j=1}^{r} j\pi_{ij}.$$

Namely, the GS model and the ME model hold. As mentioned above, the ELoLDPS model with $\Delta = \Theta = 0$ is the S model. The necessity is proved.

Next, we show the sufficiency. Assume that the ELoLDPS model, the GS model, and the ME model all hold. Since the ELoLDPS model holds,

$$L_{ij} - L_{ji} = \Delta + (j-i)\Theta \quad (i < j).$$

Namely

$$\log \frac{\pi_{ij}}{1-\pi_{ij}} - \log \frac{\pi_{ji}}{1-\pi_{ji}} = \Delta + (j-i)\Theta. \tag{3}$$

The sum of the left-hand side of Eq. (3) multiplied by $\pi_{ij} - \pi_{ji}$ gives

$$\sum\sum_{i<j}(\pi_{ij} - \pi_{ji}) \left\{ \log \frac{\pi_{ij}}{1-\pi_{ij}} - \log \frac{\pi_{ji}}{1-\pi_{ji}} \right\}.$$

Similarly, for the right-hand side of Eq. (3),

$$\Delta \sum\sum_{i<j}(\pi_{ij} - \pi_{ji}) + \Theta \sum\sum_{i<j}(j-i)(\pi_{ij} - \pi_{ji}). \tag{4}$$

From the GS and ME models, we obtain

$$\sum\sum_{i<j}(\pi_{ij} - \pi_{ji}) = 0,$$

and

$$\sum\sum_{i<j}(j-i)(\pi_{ij} - \pi_{ji}) = 0.$$

Consequently, Eq. (4) is equal to zero. Hence,

$$\sum\sum_{i<j}(\pi_{ij} - \pi_{ji}) \left\{ \log \frac{\pi_{ij}}{1-\pi_{ij}} - \log \frac{\pi_{ji}}{1-\pi_{ji}} \right\} = 0. \tag{5}$$

In the case of $\pi_{ij} > \pi_{ji}$, the left-hand side of Eq. (5) is positive since both the difference $\pi_{ij} - \pi_{ji}$ and the difference $\log(\pi_{ij}/(1-\pi_{ij})) - \log(\pi_{ji}/(1-\pi_{ji}))$ are positive. In the case of $\pi_{ij} < \pi_{ji}$, the left-hand side of Eq. (5) is positive since both the difference $\pi_{ij} - \pi_{ji}$ and the difference $\log(\pi_{ij}/(1-\pi_{ij})) - \log(\pi_{ji}/(1-\pi_{ji}))$ are negative. Accordingly, Eq. (5) with $\pi_{ij} = \pi_{ji}$ only holds. That is, the S model holds.

We note that Theorem 1 includes the results of Fujisawa and Tahata (2018).

## 4 Goodness-of-fit test

For the $r \times r$ contingency tables, let $n_{ij}$ denote the observed frequency in the $(i,j)$th cell of the table with $n = \sum\sum n_{ij}$ and $m_{ij}$ denote the corresponding expected frequency. Assuming that $\{n_{ij}\}$ has a multinomial distribution, $\hat{m}_{ij}$ denotes the maximum likelihood estimate (MLE) of $m_{ij}$. The MLE for a model is obtained by the Newton-Raphson method for log-likelihood equations.

For the ELoLDPS model, the Lagrangian must be maximized with respect to $\{\pi_{ij}\}$, $\lambda$, $\{\lambda_{ij}\}$, $\Delta$, and $\Theta$, which is given as

$$L = \sum_{i=1}^{r}\sum_{j=1}^{r} n_{ij}\log\pi_{ij} + \lambda\left(\sum_{i=1}^{r}\sum_{j=1}^{r}\pi_{ij} - 1\right) + \sum\sum_{i<j}\lambda_{ij}\Bigg(\log\pi_{ij} - \log\sum\sum_{(k,l)\neq(i,j)}\pi_{kl} - \Delta - (j-i)\Theta - \log\pi_{ji} + \log\sum\sum_{(k,l)\neq(i,j)}\pi_{lk}\Bigg).$$

Setting the partial derivatives of $L$ to zero gives

$$\begin{cases} \dfrac{n_{st}}{\pi_{st}} + \lambda + \dfrac{\lambda_{st}}{\pi_{st}} - \sum\sum_{\substack{i<j \\ (i,j)\neq(s,t)}} \dfrac{\lambda_{ij}}{\sum\sum_{(k,l)\neq(i,j)}\pi_{kl}} + \sum\sum_{i<j}\dfrac{\lambda_{ij}}{\sum\sum_{(k,l)\neq(i,j)}\pi_{lk}} = 0 \quad (s<t), \\ \dfrac{n_{ss}}{\pi_{ss}} + \lambda - \sum\sum_{i<j}\lambda_{ij}\left(\dfrac{1}{\sum\sum_{(k,l)\neq(i,j)}\pi_{kl}} - \dfrac{1}{\sum\sum_{(k,l)\neq(i,j)}\pi_{lk}}\right) = 0 \quad (s = 1, ..., r), \\ \dfrac{n_{ts}}{\pi_{ts}} + \lambda - \dfrac{\lambda_{st}}{\pi_{ts}} + \sum\sum_{\substack{i<j \\ (i,j)\neq(s,t)}} \dfrac{\lambda_{ij}}{\sum\sum_{(k,l)\neq(i,j)}\pi_{lk}} - \sum\sum_{i<j}\dfrac{\lambda_{ij}}{\sum\sum_{(k,l)\neq(i,j)}\pi_{kl}} = 0 \quad (s<t), \end{cases}$$

as well as

$$\sum_i\sum_j \pi_{ij} = 1, \quad \sum\sum_{i<j}\lambda_{ij} = 0, \quad \sum\sum_{i<j}(j-i)\lambda_{ij} = 0,$$

and

$$\log\pi_{ij} - \log\sum\sum_{(k,l)\neq(i,j)}\pi_{kl} - \Delta - (j-i)\Theta - \log\pi_{ji} + \log\sum\sum_{(k,l)\neq(i,j)}\pi_{lk} = 0 \quad (i<j).$$

The Newton-Raphson method can be used to solve these equations.

For the goodness-of-fit test, we can use test statistics such as the likelihood ratio chi-squared statistic, which is defined by

$$G^2 = 2\sum_{i=1}^{r}\sum_{j=1}^{r} n_{ij}\log\frac{n_{ij}}{\hat{m}_{ij}}.$$

Under the ELoLDPS model, $G^2$ has a chi-square distribution with $r(r-1)/2-2$ degrees of freedom.

## 5 Numerical example

The models described herein are used to analyze the data in Table 1. Table 2 shows the value of $G^2$ for each model applied to the data in Table 1.

The S model has a poor fit. We infer that the cell probabilities lack a symmetry structure. Furthermore, the LoCS and LoLDPS models also fit poorly. On the other hand, the ELoLDPS model fits well. The MLE of $\Delta$ is $\hat{\Delta} = 0.48$ and the MLE of $\Theta$ is $\hat{\Theta} = -0.20$ in the ELoLDPS model. Therefore, we infer that the log odds ratio $L_{ij} - L_{ji}$ is $0.48 - 0.20(j - i)$ for all $i < j$. Namely, (1) spouse's income tends to be higher than individual's income for $j - i = 1$, (2) spouse's income tends to be slightly higher than individual's income for $j - i = 2$, and (3) spouse's income tends to be lower than individual's income for $j - i = 3$.

## 6 Discussion

We propose the ELoLDPS model and decompose the S model using the ELoLDPS model. In particular, we are interested in applying a more generalized model such that for a fixed $k$ $(k = 1, \dots, r-1)$,

$$L_{ij} = \Delta + \sum_{l=1}^{k} (j^l - i^l)\Theta_l + L_{ji} \quad (i < j).$$

Similarly, we decompose the S model using this model. Assuming that the generalized ELoLDPS model holds, we can consider an additional condition to obtain the S model.

For a fixed $k$ $(k = 1, \dots, r-1)$, the ME$_k$ model is defined by

$$E(X^l) = E(Y^l) \quad (l = 1, ..., k),$$

where

$$E(X^l) = \sum_{i=1}^{r}\sum_{j=1}^{r} i^l \pi_{ij} \quad \text{and} \quad E(Y^l) = \sum_{i=1}^{r}\sum_{j=1}^{r} j^l \pi_{ij}.$$

This leads to the following theorem:

**Theorem 2** *The S model holds if and only if the generalized ELoLDPS model, the GS model, and the ME$_k$ model all hold.*

*Proof.* If the S model holds, then

$$\delta_U = \sum_{i<j}\sum \pi_{ij} = \sum_{i<j}\sum \pi_{ji} = \delta_L$$

and

$$\sum_{i=1}^{r}\sum_{j=1}^{r} i^l \pi_{ij} = \sum_{i=1}^{r}\sum_{j=1}^{r} i^l \pi_{ji} = \sum_{i=1}^{r}\sum_{j=1}^{r} j^l \pi_{ij}.$$

Namely, the GS model and the $\mathrm{ME}_k$ model hold. Additionally, the generalized ELoLDPS model with $\Delta = \Theta_1 = \cdots = \Theta_k = 0$ is the S model. The necessity is proved.

Next, we demonstrate the sufficiency. Assume that the generalized ELoLDPS model, the GS model, and the $\mathrm{ME}_k$ model all hold. Since the generalized ELoLDPS model holds, then

$$L_{ij} - L_{ji} = \Delta + \sum_{l=1}^{k} (j^l - i^l)\Theta_l \quad (i < j).$$

Namely

$$\log \frac{\pi_{ij}}{1-\pi_{ij}} - \log \frac{\pi_{ji}}{1-\pi_{ji}} = \Delta + \sum_{l=1}^{k} (j^l - i^l)\Theta_l. \tag{6}$$

Multiplying the sum of the left-hand side of Eq. (6) by $\pi_{ij} - \pi_{ji}$ gives

$$\sum\sum_{i<j} (\pi_{ij} - \pi_{ji}) \left\{ \log \frac{\pi_{ij}}{1-\pi_{ij}} - \log \frac{\pi_{ji}}{1-\pi_{ji}} \right\}.$$

Similarly, for the right-hand side of Eq. (6),

$$\Delta \sum\sum_{i<j} (\pi_{ij} - \pi_{ji}) + \sum_{l=1}^{k} \Theta_l \sum\sum_{i<j} (j^l - i^l)(\pi_{ij} - \pi_{ji}). \tag{7}$$

From the GS and $\mathrm{ME}_k$ models, we can obtain

$$\sum\sum_{i<j} (\pi_{ij} - \pi_{ji}) = 0,$$

and

$$\sum\sum_{i<j} (j^l - i^l)(\pi_{ij} - \pi_{ji}) = 0 \quad (l = 1, ..., k).$$

Consequently, Eq. (7) is equal to zero. Hence,

$$\sum\sum_{i<j} (\pi_{ij} - \pi_{ji}) \left\{ \log \frac{\pi_{ij}}{1-\pi_{ij}} - \log \frac{\pi_{ji}}{1-\pi_{ji}} \right\} = 0. \tag{8}$$

Therefore, Eq. (8) with $\pi_{ij} = \pi_{ji}$ only holds. That is, the S model holds.

Note that the $\mathrm{ME}_{r-1}$ model is the marginal homogeneity (MH) model (Stuart, 1955; Tahata and Tomizawa, 2015, 2008). We can obtain the following corollary:

**Corollary 1** *The S model holds if and only if the generalized ELoLDPS model with $k = r - 1$, the GS model, and the MH model all hold.*

## 7 Concluding remarks

We propose the ELoLDPS model, which is based on a logit transformation. This model indicates that a cell probability is a location shift of the symmetric cell probability on the logistic scale. To visualize the asymmetric structure of two symmetric cells on the logistic scale, the proposed model should be applied instead of the model in Eq. (1).

The proposed model constrains the main diagonal cells since many observations tend to concentrate along these cells. In this model, the observed frequencies on the main diagonal cells affect the likelihood ratio chi-squared statistic. That is, this model utilizes information in the main diagonal cells.

The proposed model can decompose the S model. When the S model has a poor fit, this decomposition may determine the reason.

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Table 1: Individual's income and spouse's income data in Japan (Seiyama, 1995)

| Individual's income | Spouse's income (1) | (2) | (3) | (4) | Total |
|---|---|---|---|---|---|
| (1) | 27 | 22 | 168 | 269 | 486 |
| | (27.000)[a] | (19.500) | (176.000) | (271.500) | (494.000) |
| | (26.997)[b] | (20.359) | (183.146) | (281.863) | (512.365) |
| | (26.999)[c] | (19.623) | (178.041) | (275.940) | (500.603) |
| | (26.998)[d] | (22.257) | (183.422) | (259.764) | (492.441) |
| (2) | 17 | 12 | 87 | 106 | 222 |
| | (19.500) | (12.000) | (77.000) | (103.500) | (212.000) |
| | (18.636) | (11.999) | (80.272) | (107.853) | (218.759) |
| | (19.376) | (12.000) | (77.467) | (104.745) | (213.588) |
| | (16.741) | (11.999) | (87.543) | (108.021) | (224.304) |
| (3) | 184 | 67 | 151 | 134 | 536 |
| | (176.000) | (77.000) | (151.000) | (110.000) | (514.000) |
| | (168.951) | (73.687) | (150.985) | (114.548) | (508.171) |
| | (173.976) | (76.524) | (150.997) | (110.649) | (512.146) |
| | (168.690) | (66.452) | (150.988) | (124.682) | (510.812) |
| (4) | 274 | 101 | 86 | 93 | 554 |
| | (271.500) | (103.500) | (110.000) | (93.000) | (578.000) |
| | (261.255) | (99.136) | (105.324) | (92.991) | (558.705) |
| | (267.088) | (102.248) | (109.328) | (92.998) | (571.662) |
| | (283.337) | (98.974) | (95.140) | (92.993) | (570.444) |
| Total | 502 | 202 | 492 | 602 | 1798 |
| | (494.000) | (212.000) | (514.000) | (578.000) | |
| | (475.839) | (205.180) | (519.726) | (597.255) | |
| | (487.440) | (210.395) | (515.833) | (584.333) | |
| | (495.766) | (199.682) | (517.093) | (585.459) | |

Notes: [a]Estimated expected frequencies by the S model.
[b]Estimated expected frequencies by the LoCS model.
[c]Estimated expected frequencies by the LoLDPS model.
[d]Estimated expected frequencies by the ELoLDPS model.

Table 2: Likelihood-ratio chi-square values $G^2$ for models to the data in Table 1

| Model | Degree of freedom | $G^2$ |
|---|---|---|
| S | 6 | 14.70* |
| LoCS | 5 | 12.23* |
| LoLDPS | 5 | 14.46* |
| ELoLDPS | 4 | 5.00 |
| ME | 1 | 0.17 |
| GS | 1 | 2.15 |

Note: *Significant at the 0.05 level