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Power length-biased Suja distribution: properties and application

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In this paper, a new extension of length-biased Suja distribution (LBSD) called power length-biased Suja distribution (PLBSD) is suggested. The PLBSD is a two parameters distribution. The shape of the distribution is discussed and the moments of the new model are derived. Also, the harmonic mean and the Rényi entropy are presented and proved. The reliability, hazard, hazard rate, and odds functions of the PLBSD random variable are provided. Bonferroni and Lorenz curves and the Gini index as well as the stochastic ordering of the PLBSD are presented. The maximum likelihood method is used to estimate the parameters involved. The distributions of order statistics from the PLBSD are provided. The usefulness of the PLBSD for modeling reliability data is illustrated based on real data to show the performance of the new model.

keywords: Suja distribution; Length-biased Suja distribution; Bonferroni curve; Lorenz curve; Gini index; Rényi entropy; Stochastic ordering; Reliability analysis.

1 Introduction

Many authors are interested in proposing new flexible statistical models, including weighted distribution in various fields, such as sciences, ecology, biostatistics, medicine, pharmacy and environment. Shanker (2017) suggested a new continuous distribution of

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one parameter known as a Suja distribution (SD) with a probability density function (pdf) defined as

$$f_{SD}(x;\alpha) = \frac{\alpha^5}{\alpha^4+24}\left(1+x^4\right)e^{-\alpha x}; x>0,\ \alpha>0, \tag{1}$$

and a cumulative distribution function (cdf) as

$$F_{SD}(x;\alpha) = 1 - \frac{\left(24+24x\alpha+12x^2\alpha^2+4x^3\alpha^3+\alpha^4+x^4\alpha^4\right)}{24+\alpha^4}\mathrm{e}^{-x\alpha};\ \ x>0,\ \alpha>0. \tag{2}$$

The concept of weighted distributions has been firstly proposed by Fisher (1934), which studied how verification methods can affect the form of distribution of recorded observations.

Suppose that the random variable X has the probability distribution function $f(x)$. Then, the weighted distribution of X is given by

$$f_w(x) = \frac{w(x)f(x)}{E(w(x))} = \frac{w(x)f(x)}{\mu_w},\ \ x>0, \tag{3}$$

where $w(x)$ is a non-negative weight function.
If the weight function have the form $w(x) = X^\kappa$, the resulting PDF

$$f_\kappa(x) = \frac{x^\kappa f(x)}{E(X^\kappa)}, \tag{4}$$

is called a size biased version of X. If $\kappa = 1, 2$, we obtain the length and area biased distributions, respectively.

Many weighted distributions are suggested in the literature, as an example Al-Omari et al. (2019b) suggested a size biased Ishita distribution as a generalization of the Ishita distribution. Haeeeq et al. (2019) proposed Marshall-Olkin length-biased exponential distribution with its applications. Kilany (2016) suggested weighted Lomax distribution. Other types of distributions are also suggested as Al-Omari et al. (2019a) for the exponentiated new Weibull-Pareto distribution. Al-Omari and Gharaibeh (2018) for Topp-Leone Mukherjee-Islam distribution.

Al-Omari and Alsmairan (2019) suggested length-biased Suja distribution (LBSD) as a new modification of the SD based on the idea of weighted distribution. The pdf and cdf of the LBSD, respectively, are

$$f_{LBSD}(x;\alpha) = \frac{\alpha^6}{\alpha^4+120}\left(x+x^5\right)e^{-\alpha x}; x>0, \alpha>0, \tag{5}$$

$$F_{LBSD}(x;\alpha) = 1 - \frac{120+\alpha\left\{\begin{array}{l}120x+60x^2\alpha+20x^3\alpha^2\\ +\left(1+5x^4\right)\alpha^3+x\alpha^4\left(1+x^4\right)\end{array}\right\}}{120+\alpha^4}\mathrm{e}^{-\alpha x} \tag{6}$$

Let Y be a random variable with pdf $f(y)$, if we used the power transformation $X = Y^{\frac{1}{\beta}}$, the resulting distribution is called a power distribution.

Shukla and Shanker (2018) introduced the power Ishita distribution. Ghitany et al. (2013) proposed the power Lindely distribution as an extension of Lindley distribution. Dey et al. (2019) studied alpha-power transformed Lindley distribution. In the current work, we used the power transformation with the LBSD and proposed the power length-biased Suja distribution.

This paper is organized as follows: in Section 2, the pdf and cdf of the suggested power length-biased Suja distribution are presented and the shape of the distribution is discussed. In Section 3, The moments of the new model as the rth moment, mean, variance, coefficients of variation, skewness and kurtosis are provided with some calculations. In Section 4, the Bonferroni and Lorenz curves and the Gini index for the PLBSD are provided. Section 5 is devoted to the mean and median deviations as well as the reliability analysis of the PLBSD with some illustrations of the shapes of reliability functions. Rényi entropy and the harmonic mean are presented in Section 6 with some tables for these properties. Investigation of the stochastic ordering of the PLBSD is studied in Section 7. The order statistics and maximum likelihood estimation for the model parameters are derived in Section 8. Application of real data is given in Section 9. Finally, the paper is concluded in Section 10.

2 The suggested distribution

This section defines the pdf and cdf of PLBSD and illustrates the shapes of these functions. With reference to the power transformation $X = Y^{1/\beta}$, the pdf of the PLBSD is given by

$$f_{PLBSD}(x;\alpha,\beta) = \frac{\alpha^6\beta}{\alpha^4+120}x^{2\beta-1}\left(x^{4\beta}+1\right)\mathrm{e}^{-\alpha x^\beta}; x \geqslant 0, \alpha>0, \beta>0. \tag{7}$$

It is easy to prove that $\int_0^\infty f_{PLBSD}(x;\alpha,\beta) = 1$, since

$$\begin{aligned}\int_0^\infty f_{PLBSD}(x;\alpha,\beta)dx &= \frac{\alpha^6\beta}{\alpha^4+120}\int_0^\infty x^{2\beta-1}\left(x^{4\beta}+1\right)\mathrm{e}^{-\alpha x^\beta}dx \\ &= \frac{\alpha^6\beta}{\alpha^4+120}\left(\int_0^\infty x^{6\beta-1}\, e^{-\alpha x^\beta}dx + \int_0^\infty x^{2\beta-1}e^{-\alpha x^\beta}dx\right) \\ &= \frac{\alpha^6\beta}{\alpha^4+120}\left(\frac{120}{\alpha^6\beta}+\frac{1}{\alpha^2\beta}\right) = \frac{\alpha^6}{\alpha^4+120}\left(\frac{120+\alpha^4}{\alpha^6}\right) = 1.\end{aligned}$$

Theorem 1: Let X be a random variable follows the PLBSD with parameters α and β. The cumulative distribution function of X is defined as

$$F_{PLBSD}(x;\alpha,\beta) = 1 - \frac{\alpha^4\Gamma\left(2, x^\beta\alpha\right)+\Gamma\left(6, x^\beta\alpha\right)}{\alpha^4+120}; x>0, \alpha>0, \beta>0. \tag{8}$$

The cdf in (8) can be written as

$$F_{PLBSD}(x;\alpha,\beta) = 1 - \frac{\begin{pmatrix} \alpha^4 + \alpha^5 x^{5\beta} + 5\alpha^4 x^{4\beta} + 20\alpha^3 x^{3\beta} \\ +60\alpha^2 x^{2\beta} + \alpha\left(\alpha^4+120\right) x^{\beta} + 120 \end{pmatrix}}{\alpha^4 + 120} e^{-\alpha x^{\beta}}. \tag{9}$$

The graphs of the pdf and cdf of PLBSD for varying values of the distribution parameters $\alpha = 2, 4, 6, 8, 10, 12$ and $\beta = 4$ are shown in Figures 1 and 2, respectively.

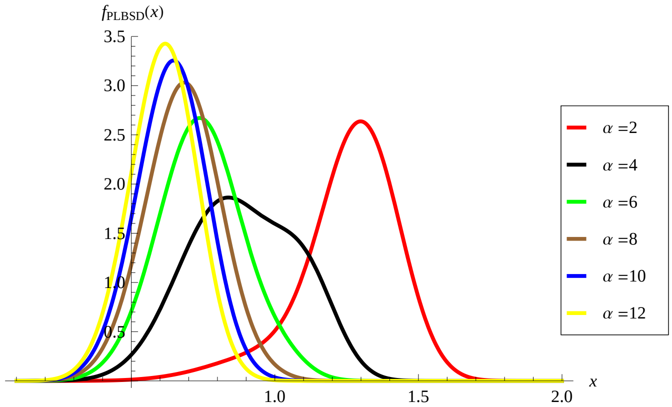

Figure 1: The pdf of PLBSD for $\alpha = 2, 4, 6, 8, 10, 12$ and $\beta = 4$

Based on Figure 1, it is clear that PLBSD is asymmetric and skewed to the left when $\alpha = 2$ and approximately is symmetric for other values of α given in the figure.

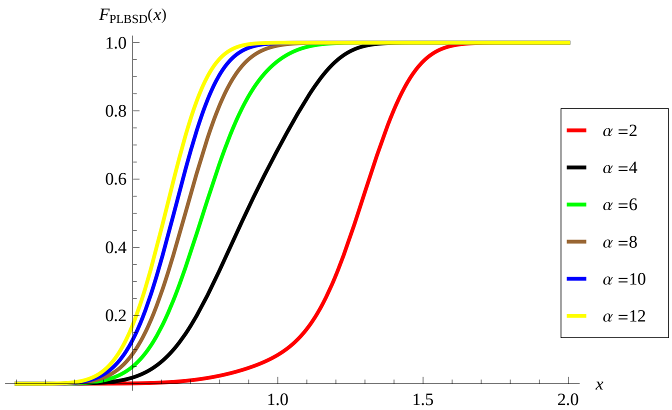

Figure 2: The cdf of PLBSD for $\alpha = 2, 4, 6, 8, 10, 12$ and $\beta = 4$

However, the shape of the pdf of the PLBSD can be described analytically. The critical points of the pdf of PLBSD are the roots of the equation

$$\frac{d\ln f_{PLBSD}(x;\alpha,\beta)}{dx}=\frac{\left(\alpha^4+120\right)x^{1-2\beta}e^{\alpha x^\beta}}{\alpha^6\beta\left(x^{4\beta}+1\right)}\begin{pmatrix}-\frac{\alpha^7\beta^2\left(x^{4\beta}+1\right)x^{3\beta-2}e^{-\alpha x^\beta}}{\alpha^4+120}\\+\frac{4\alpha^6\beta^2x^{6\beta-2}e^{-\alpha x^\beta}}{\alpha^4+120}\\+\frac{\alpha^6\beta(2\beta-1)\left(x^{4\beta}+1\right)x^{2\beta-2}e^{-\alpha x^\beta}}{\alpha^4+120}\end{pmatrix}. \tag{10}$$

Sometimes there are more than one root to Equation (10). If $x = x_0$ is the root of Equation (10), then for $\Upsilon(x) = \frac{d^2\ln f_{PLBSD}(x;\alpha,\beta)}{dx^2}$, $\Upsilon(x) < 0$, $\Upsilon(x > 0$, or $\Upsilon(x) = 0$ correspond to a local maximum, local minimum or a point of inflection, respectively.

3 The moments of PLBSD

This section presents the rth moment, mean, variance, coefficients of variation, skewness and kurtosis for the PLBSD. Also, some simulations for these measures are provided.

Theorem 2: Let $X \sim f_{PLBSD}(x;\alpha,\beta)$. Then, the rth moment of X is

$$E\left(X^r_{PLBSD}\right)=\frac{\alpha^{-\frac{r}{\beta}}\left(\alpha^4\Gamma\left(\frac{r}{\beta}+2\right)+\Gamma\left(\frac{r}{\beta}+6\right)\right)}{\alpha^4+120}, \beta>0, 2\beta+r>0, \alpha>0, r=1,2,... \tag{11}$$

Proof: The rth moment of the PLBSD can be obtained as follows:

$$\begin{aligned}
E\left(X^r_{PLBSD}\right)&=\int_0^\infty x^r f_{PLBSD}(x;\alpha,\beta)dx\\
&=\int_0^\infty \frac{\alpha^6\beta}{\alpha^4+120}\left[x^{6\beta+r-1}e^{-\alpha x^\beta}+x^{2\beta+r-1}e^{-\alpha x^\beta}\right]dx\\
&=\frac{\alpha^6\beta}{\alpha^4+120}\left[\int_0^\infty x^{6\beta+r-1}e^{-\alpha x^\beta}dx+\int_0^\infty x^{2\beta+r-1}e^{-\alpha x^\beta}dx\right]\\
&=\frac{\alpha^6\beta}{\alpha^4+120}\left[\frac{\alpha^{-\frac{r}{\beta}-6}\Gamma\left(\frac{r}{\beta}+6\right)}{\beta}+\frac{\alpha^{-\frac{r}{\beta}-2}\Gamma\left(\frac{r}{\beta}+2\right)}{\beta}\right]\\
&=\frac{1}{\alpha^4+120}\left[\alpha^{-\frac{r}{\beta}}\Gamma\left(\frac{r}{\beta}+6\right)+\alpha^{-\frac{r}{\beta}+4}\Gamma\left(\frac{r}{\beta}+2\right)\right]\\
&=\frac{\alpha^{-\frac{r}{\beta}}}{\alpha^4+120}\left[\Gamma\left(\frac{r}{\beta}+6\right)+\alpha^4\Gamma\left(\frac{r}{\beta}+2\right)\right].
\end{aligned}$$

Consequently, we can obtain the first four moments of the PLBSD distributed random variable by substituting $r = 1, 2, 3, 4$, respectively, in Equation (6) as

$$E\left(X_{PLBSD}\right) = \frac{\alpha^{-1/\beta}\left[\alpha^4\Gamma\left(2+\frac{1}{\beta}\right)+\Gamma\left(6+\frac{1}{\beta}\right)\right]}{\alpha^4+120}, \alpha>0, \beta>0,$$

$$E\left(X^2_{PLBSD}\right) = \frac{\alpha^{-2/\beta}\left[\alpha^4\Gamma\left(2+\frac{2}{\beta}\right)+\Gamma\left(6+\frac{2}{\beta}\right)\right]}{\alpha^4+120}, \alpha>0, \beta>0,$$

$$E\left(X^3_{PLBSD}\right) = \frac{\alpha^{-3/\beta}\left[\alpha^4\Gamma\left(2+\frac{3}{\beta}\right)+\Gamma\left(6+\frac{3}{\beta}\right)\right]}{\alpha^4+120}, \alpha>0, \beta>0,$$

$$E\left(X^4_{PLBSD}\right) = \frac{\alpha^{-4/\beta}\left[\alpha^4\Gamma\left(2+\frac{4}{\beta}\right)+\Gamma\left(6+\frac{4}{\beta}\right)\right]}{\alpha^4+120}, \alpha>0, \beta>0.$$

Hence, the variance of PLBSD is given by

$$\begin{aligned} Var_{PLBSD}(X_{PLBSD}) &= E\left(X^2_{PLBSD}\right) - \left[E(X_{PLBSD})\right]^2 \\ &= \frac{\alpha^{-2/\beta}}{\left(\alpha^4+120\right)^2}\left\{\begin{array}{l} \left(\alpha^4+120\right)\left[\alpha^4\Gamma\left(2+\dfrac{2}{\beta}\right)+\Gamma\left(6+\dfrac{2}{\beta}\right)\right] \\ -\left[\alpha^4\Gamma\left(2+\dfrac{1}{\beta}\right)+\Gamma\left(6+\dfrac{1}{\beta}\right)\right]^2 \end{array}\right\}. \end{aligned} \tag{12}$$

The coefficient of skewness determines the degree of skewness of PLBSD. It is given by

$$Sk_{PLBSD} = \frac{E\left(X^3\right) - 3\mu\sigma^2 - \mu^3}{\sigma^3}. \tag{13}$$

The coefficient of kurtosis measures the flatness of the distribution and for PLBSD it is defined as

$$Ku_{PLBSD} = \frac{E\left(X^4\right) - 4\mu E\left(X^3\right) + 6E\left(X^2\right)\sigma^2 + 3E\left(X^4\right)}{\sigma^8}. \tag{14}$$

The coefficient of variation of the PLBSD is defined as

$$Cv_{PLBSD} = \frac{\sigma_{PLBSD}}{\mu_{PLBSD}}. \tag{15}$$

Tables (1-3) present some values of the mean, standard deviation, coefficient of variation, coefficient skewness and coefficient kurtosis of PLBSD for various values of the parameters β and α.

The results in Tables (1-3) show that:

Table 1: The mean, standard deviation, coefficients of variation, skewness and kurtosis for the PLBSD with different values of the parameter β when $\alpha = 4$

β	μ_{PLBSD}	σ_{PLBSD}	Cv_{PLBSD}	Sk_{PLBSD}	Ku_{PLBSD}
1	0.81915	0.64968	0.79311	1.25434	4.56088
2	0.83537	0.34830	0.41694	0.42516	2.59253
3	0.86925	0.24807	0.28538	0.12224	2.43847
4	0.89312	0.19419	0.21743	-0.04276	2.48377
5	0.90997	0.15994	0.17577	-0.14799	2.56195
6	0.92235	0.13610	0.14756	-0.22134	2.63984
7	0.93179	0.11850	0.12718	-0.27552	2.71003
8	0.93920	0.10496	0.11176	-0.31723	2.77154
9	0.94518	0.09421	0.09968	-0.35036	2.82510
10	0.95009	0.08547	0.08996	-0.37732	2.87182
11	0.95419	0.07822	0.08197	-0.39970	2.91274
12	0.95768	0.07210	0.07529	-0.41857	2.94878
13	0.96068	0.06687	0.06961	-0.43471	2.98072
14	0.96328	0.06236	0.06473	-0.44867	3.00919
15	0.96555	0.05841	0.06049	-0.46086	3.03469
16	0.96757	0.05494	0.05678	-0.47160	3.05765
17	0.96936	0.05185	0.05349	-0.48114	3.07844
18	0.97096	0.04909	0.05056	-0.48966	3.09733
19	0.97241	0.04662	0.04794	-0.49732	3.11457
20	0.97371	0.04438	0.04558	-0.50425	3.13036
21	0.97490	0.04234	0.04343	-0.51054	3.14488
22	0.97599	0.04049	0.04148	-0.51629	3.15827
23	0.97699	0.03879	0.03970	-0.52155	3.17066
24	0.97790	0.03723	0.03807	-0.52638	3.18215
25	0.97875	0.03578	0.03656	-0.53085	3.19284
26	0.97954	0.03445	0.03517	-0.53498	3.20281
27	0.98026	0.03321	0.03388	-0.53881	3.21213
28	0.98094	0.03206	0.03268	-0.54238	3.22085
29	0.98158	0.03098	0.03157	-0.54571	3.22904
30	0.98217	0.02998	0.03052	-0.54883	3.23675
31	0.98272	0.02904	0.02955	-0.55174	3.24400
32	0.98325	0.02815	0.02863	-0.55449	3.25085
33	0.98374	0.02732	0.02777	-0.55706	3.25733
34	0.98420	0.02654	0.02696	-0.55950	3.26345
35	0.98464	0.02580	0.02620	-0.56179	3.26926
36	0.98505	0.02510	0.02548	-0.56396	3.27478
37	0.98545	0.02443	0.02479	-0.56602	3.28002
38	0.98582	0.02380	0.02415	-0.56797	3.28501
39	0.98617	0.02321	0.02353	-0.56982	3.28976
40	0.98651	0.02264	0.02295	-0.57159	3.29429

Table 2: The mean, standard deviation, coefficients of variation, skewness and kurtosis for the PLBSD with different values of the parameter β when $\alpha = 7$

β	μ_{PLBSD}	σ_{PLBSD}	Cv_{PLBSD}	Sk_{PLBSD}	Ku_{PLBSD}
1	0.312914	0.243937	0.779564	1.844220	8.33090
2	0.521689	0.201879	0.386973	0.633727	3.56284
3	0.637169	0.166673	0.261584	0.244578	3.07211
4	0.708482	0.140505	0.198319	0.042045	3.02074
5	0.756590	0.120985	0.159909	-0.084190	3.06186
6	0.791153	0.106047	0.134042	-0.170980	3.12369
7	0.817159	0.094310	0.115412	-0.234510	3.18672
8	0.837424	0.084870	0.101346	-0.283100	3.24528
9	0.853656	0.077124	0.090346	-0.321520	3.29800
10	0.866947	0.070660	0.081505	-0.352660	3.34499
11	0.878028	0.065187	0.074243	-0.378430	3.38677
12	0.887408	0.060496	0.068171	-0.400120	3.42398
13	0.895450	0.056430	0.063019	-0.418620	3.45722
14	0.902421	0.052874	0.058592	-0.434600	3.48704
15	0.908521	0.049738	0.054746	-0.448530	3.51390
16	0.913904	0.046952	0.051375	-0.460790	3.53819
17	0.918689	0.044460	0.048395	-0.471660	3.56026
18	0.922970	0.042219	0.045743	-0.481370	3.58037
19	0.926823	0.040192	0.043366	-0.490100	3.59878
20	0.930310	0.038351	0.041224	-0.497980	3.61568
21	0.933479	0.036670	0.039284	-0.505130	3.63125
22	0.936373	0.035131	0.037518	-0.511650	3.64563
23	0.939025	0.033715	0.035904	-0.517620	3.65896
24	0.941466	0.032409	0.034424	-0.523110	3.67134
25	0.943718	0.031200	0.033060	-0.528170	3.68287
26	0.945804	0.030078	0.031801	-0.532860	3.69363
27	0.947741	0.029033	0.030634	-0.537200	3.70371
28	0.949544	0.028059	0.029550	-0.541250	3.71315
29	0.951227	0.027148	0.028540	-0.545020	3.72202
30	0.952801	0.026294	0.027597	-0.548540	3.73037
31	0.954277	0.025492	0.026714	-0.551840	3.73824
32	0.955663	0.024738	0.025885	-0.554950	3.74567
33	0.956968	0.024027	0.025107	-0.557860	3.75270
34	0.958198	0.023355	0.024374	-0.560610	3.75935
35	0.959360	0.022720	0.023683	-0.563210	3.76567
36	0.960459	0.022119	0.023030	-0.565660	3.77167
37	0.961500	0.021549	0.022412	-0.567990	3.77737
38	0.962487	0.021007	0.021826	-0.570190	3.78280
39	0.963426	0.020492	0.021270	-0.572280	3.78798
40	0.964318	0.020001	0.020742	-0.574270	3.79292

Table 3: The mean, standard deviation, coefficients of variation, skewness and kurtosis for the PLBSD with different values of the parameter α when $\beta = 2$

α	μ_{PLBSD}	σ_{PLBSD}	Cv_{PLBSD}	Sk_{PLBSD}	Ku_{PLBSD}
1	2.390200	0.503855	0.210800	0.129554	3.14339
2	1.607390	0.425435	0.264674	0.298360	3.28147
3	1.136210	0.414390	0.364713	0.038420	2.38930
4	0.835368	0.348296	0.416937	0.425157	2.59253
5	0.671555	0.279058	0.415541	0.657020	3.21321
6	0.579710	0.231877	0.399988	0.685973	3.53787
7	0.521689	0.201879	0.386973	0.633727	3.56284
8	0.480757	0.181944	0.378454	0.572247	3.47229
9	0.449518	0.167746	0.373169	0.523460	3.36971
10	0.424385	0.156972	0.369880	0.488887	3.28638
11	0.403433	0.148378	0.367788	0.465176	3.22514
12	0.385524	0.141264	0.366422	0.448946	3.18152
13	0.369934	0.135212	0.365504	0.437717	3.15060
14	0.356171	0.129957	0.364872	0.429823	3.12851
15	0.343887	0.125321	0.364425	0.424177	3.11254
16	0.332824	0.121182	0.364104	0.420068	3.10083
17	0.322785	0.117451	0.363868	0.417029	3.09212
18	0.313616	0.114060	0.363691	0.414746	3.08555
19	0.305197	0.110957	0.363557	0.413006	3.08053
20	0.297429	0.108102	0.363454	0.411663	3.07664
21	0.290230	0.105462	0.363374	0.410613	3.07360
22	0.283533	0.103011	0.363310	0.409783	3.07120
23	0.277282	0.100726	0.363260	0.409121	3.06927
24	0.271429	0.098588	0.363219	0.408587	3.06772
25	0.265934	0.096583	0.363186	0.408153	3.06646
26	0.260760	0.094697	0.363159	0.407797	3.06542
27	0.255878	0.092919	0.363137	0.407504	3.06457
28	0.251261	0.091238	0.363118	0.407259	3.06386
29	0.246886	0.089645	0.363103	0.407055	3.06326
30	0.242732	0.088134	0.363089	0.406883	3.06276
31	0.238782	0.086696	0.363078	0.406737	3.06234
32	0.235018	0.085328	0.363069	0.406613	3.06197
33	0.231427	0.084022	0.363061	0.406507	3.06166
34	0.227996	0.082775	0.363054	0.406416	3.06140
35	0.224714	0.081582	0.363048	0.406337	3.06117
36	0.221569	0.080439	0.363043	0.406268	3.06097
37	0.218553	0.079343	0.363038	0.406209	3.06079
38	0.215657	0.078291	0.363034	0.406157	3.06064
39	0.212874	0.077280	0.363031	0.406111	3.06051
40	0.210195	0.076307	0.363028	0.406071	3.06039

- The mean values of the PLBSD are increasing as β is increasing for fixed values of $\alpha = 4, 7$. As an example, when $\alpha = 4$, the mean values are 0.81915 and 0.97371 for $\beta = 1$ and 20, respectively. For $\beta = 2$, the mean values are decreasing as the values of α are increasing.
- The PLBSD is asymmetric distribution as the values of the skewness are not zero. However, the skewness value is about 0.5 for all cases considered in the tables.
- The values of the standard deviation, coefficient of variation and the skewness are less than one except the skewness when $\beta = 1$ for $\alpha = 4, 7$ in Tables 1 and 2.
- The kurtosis values are about 3 for all given values of the distribution parameters in the tables.

4 Bonferroni and Lorenz curves and Gini Index

Assume that the random variable X is a non-negative with a continuous and twice differentiable cumulative distribution function. The Bonferroni curve of the random variable X is defined as

$$B(p) = \frac{1}{p\mu}\int_0^q xf(x)dx = \frac{1}{p\mu}\left(\int_0^\infty xf(x)dx - \int_q^\infty xf(x)dx\right) = \frac{1}{p\mu}\left(\mu - \int_q^\infty xf(x)dx\right),$$

where $q = F^{-1}(p)$ and $p \in (0, 1]$. The Lorenz curve is defined as

$$L(p) = \frac{1}{\mu}\int_0^q xf(x)dx = \frac{1}{\mu}\left(\int_0^\infty xf(x)dx - \int_q^\infty xf(x)dx\right) = \frac{1}{\mu}\left(\mu - \int_q^\infty xf(x)dx\right).$$

The Gini index is given by

$$G = 1 - \frac{1}{\mu}\int_0^\infty (1 - F(x))^2 dx = \frac{1}{\mu}\int_0^\infty F(x)(1 - F(x))dx.$$

The following theorem involves the Bonferroni curve, Lorenz curve and Gini index for the PLBSD.

Theorem 4: The Bonferroni curve, Lorenz curve and Gini index for PLBSD, respectively, are:

$$B(p) = \frac{\beta^4 q\alpha^{1/\beta}\left(\alpha q^\beta\right)^{-1/\beta}\left(\alpha^4\Gamma\left(2+\frac{1}{\beta}\right)+\Gamma\left(6+\frac{1}{\beta}\right)-\alpha^4\Gamma\left(2+\frac{1}{\beta},q^\beta\alpha\right)-\Gamma\left(6+\frac{1}{\beta},q^\beta\alpha\right)\right)}{p\left(\beta\left(\beta\left(\beta\left(\left(\alpha^4+120\right)\beta+154\right)+71\right)+14\right)+1\right)\Gamma\left(2+\frac{1}{\beta}\right)}, \quad (16)$$

Table 4: Gini index of the PLBSD distribution for some values of α and β

α	$G(\alpha,3)$	α	$G(\alpha,3)$	β	$G(\alpha,3)$	β	$G(5,\beta)$	β	$G(5,\beta)$	β	$G(5,\beta)$
1	0.079990	16	0.140459	31	0.140179	1	0.425291	16	0.030830	31	0.015970
2	0.101904	17	0.140394	32	0.140177	2	0.232001	17	0.029030	32	0.015473
3	0.145325	18	0.140346	33	0.140174	3	0.158606	18	0.027428	33	0.015006
4	0.162996	19	0.140310	34	0.140173	4	0.120360	19	0.025993	34	0.014566
5	0.158606	20	0.140281	35	0.140171	5	0.096943	20	0.024702	35	0.014151
6	0.151820	21	0.140260	36	0.140170	6	0.081142	21	0.023532	36	0.013760
7	0.147272	22	0.140242	37	0.140168	7	0.069765	22	0.022468	37	0.013389
8	0.144590	23	0.140229	38	0.140167	8	0.061184	23	0.021496	38	0.013038
9	0.143018	24	0.140217	39	0.140166	9	0.054481	24	0.020605	39	0.012704
10	0.142070	25	0.140208	40	0.140166	10	0.049101	25	0.019785	40	0.012388
11	0.141479	26	0.140201	41	0.140165	11	0.044688	26	0.019028	41	0.012086
12	0.141098	27	0.140195	42	0.140164	12	0.041002	27	0.018326	42	0.011799
13	0.140843	28	0.140190	43	0.140164	13	0.037878	28	0.017674	43	0.011526
14	0.140669	29	0.140186	44	0.140163	14	0.035196	29	0.017067	44	0.011265
15	0.140547	30	0.140182	45	0.140163	15	0.032869	30	0.016501	45	0.011015

$$L(p) = \frac{\beta^4 q\alpha^{1/\beta}\left(\alpha q^\beta\right)^{-1/\beta}\left(\alpha^4\Gamma\left(2+\frac{1}{\beta}\right)+\Gamma\left(6+\frac{1}{\beta}\right)-\alpha^4\Gamma\left(2+\frac{1}{\beta},q^\beta\alpha\right)-\Gamma\left(6+\frac{1}{\beta},q^\beta\alpha\right)\right)}{\left(\beta\left(\beta\left(\beta\left(\left(\alpha^4+120\right)\beta+154\right)+71\right)+14\right)+1\right)\Gamma\left(2+\frac{1}{\beta}\right)}, \quad (17)$$

and the Gini index is

$$G = 1-\frac{\left(\alpha^4+120\right)\beta^4\alpha^{1/\beta}\int_0^\infty\left(1-\frac{\alpha^4+\alpha^4\left(-\Gamma\left(2,x^\beta\alpha\right)\right)-\Gamma\left(6,x^\beta\alpha\right)+120}{\alpha^4+120}\right)^2 dx}{\left(\beta\left(\beta\left(\beta\left(\left(\alpha^4+120\right)\beta+154\right)+71\right)+14\right)+1\right)\Gamma\left(2+\frac{1}{\beta}\right)}. \quad (18)$$

To study the behavior of the Gini index for the PLBSD distributed random variable, Table (4) contains some values of the Gini index of the new model for various values of the distribution parameters α and β.

It is clear that all values in Table (4) are less than one. However, for all values of α when $\beta = 3$ the average of the Gini index is 0.139748 and the average is 0.046069 when $\alpha = 5$ for all values of β .

5 Mean and median deviations and reliability analysis

To measure the scatter in the population, the mean deviation about the mean ($D_\mu(x)$), and the mean deviation about the median ($D_M(x)$) can be used, where

$$D_\mu(x) = \int_0^\infty |x-\mu| f(x)dx = \int_0^\mu (\mu - x) f(x)dx + \int_\mu^\infty (x-\mu) f(x)dx = 2\mu F(\mu) - 2\int_0^\mu x f(x)dx,$$

and

$$D_M(x) = \int_0^\infty |x-M| f(x)dx = \int_0^M (M - x) f(x)dx + \int_M^\infty (x-M) f(x)dx = \mu - 2\int_0^M x f(x)dx,$$

where μ and M are the population mean and median, respectively. The mean deviation about the mean and the mean deviation about the median of the PLBSD, respectively, are

$$\begin{aligned} D_\mu(x) = &\frac{2\mu\left(\alpha^4\left(-\Gamma\left(2,\alpha\mu^\beta\right)\right)+\alpha^4-\Gamma\left(6,\alpha\mu^\beta\right)+120\right)}{\alpha^4+120} \\ &- \frac{2\mu\left(\alpha\mu^\beta\right)^{-1/\beta}\left(\alpha^4\left(\Gamma\left(2+\frac{1}{\beta}\right)-\Gamma\left(2+\frac{1}{\beta},\alpha\mu^\beta\right)\right)-\Gamma\left(6+\frac{1}{\beta},\alpha\mu^\beta\right)+\Gamma\left(6+\frac{1}{\beta}\right)\right)}{\alpha^4+120}, \end{aligned} \tag{19}$$

and

$$\begin{aligned} D_M(x) = &\frac{\alpha^{-1/\beta}\left[\alpha^4\Gamma\left(2+\frac{1}{\beta}\right)+\Gamma\left(6+\frac{1}{\beta}\right)\right]}{\alpha^4+120} \\ &- \frac{2M\left(\alpha M^\beta\right)^{-1/\beta}\left[\alpha^4\Gamma\left(2+\frac{1}{\beta}\right)+\Gamma\left(6+\frac{1}{\beta}\right)-\alpha^4\Gamma\left(2+\frac{1}{\beta},M^\beta\alpha\right)-\Gamma\left(6+\frac{1}{\beta},M^\beta\alpha\right)\right]}{\alpha^4+120}. \end{aligned} \tag{20}$$

Also, this section provides the reliability function, hazard function, reversed hazard function and the odds function for the PLBSD with their shapes for different values of the distribution parameters.

The reliability function of PLBSD is given by

$$\begin{aligned} R_{P_{LBSD}}(x;\alpha,\beta) &= 1 - F_{PLBSD}(x;\alpha,\beta) \\ &= \frac{\alpha^4\Gamma\left(2,x^\beta\alpha\right)+\Gamma\left(6,x^\beta\alpha\right)}{\alpha^4+120}. \end{aligned} \tag{21}$$

The hazard rate function of PLBSD is

$$\begin{aligned} H_{PLBSD}(x;\alpha,\beta) &= \frac{f_{PLBSD}(x;\alpha,\beta)}{1-F_{PLBSD}(x;\alpha,\beta)} \\ &= \frac{\alpha^6\beta x^{2\beta-1}\left(x^{4\beta}+1\right)\mathrm{e}^{-\alpha x^\beta}}{\alpha^4\Gamma\left(2,x^\beta\alpha\right)+\Gamma\left(6,x^\beta\alpha\right)}. \end{aligned} \tag{22}$$

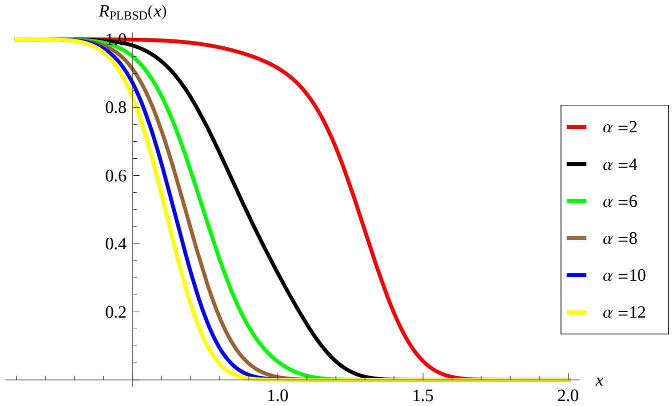

Figure 3: The reliability function of PLBSD for $\alpha = 2, 4, 6, 8, 10, 12$ and $\beta = 4$

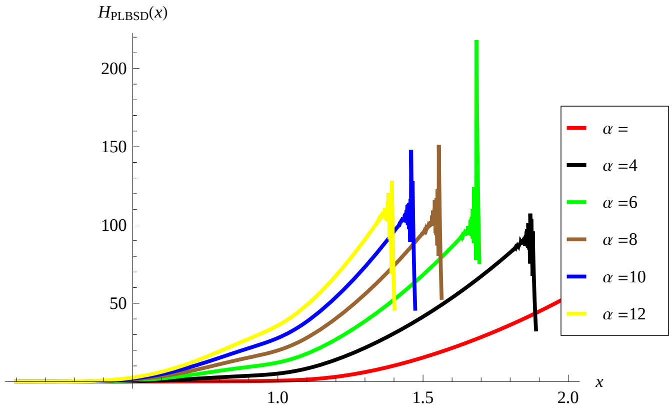

Figure 4: The hazard function of PLBSD for $\alpha = 2, 4, 6, 8, 10, 12$ and $\beta = 4$

The reversed hazard rate and odds functions for PLBSD, respectively, are defined as

$$\begin{aligned} RH_{PLBSD}(x;\alpha,\beta) &= \frac{f_{PLBSD}(x;\alpha,\beta)}{F_{PLBSD}(x;\alpha,\beta)} \\ &= \frac{\alpha^6 \beta x^{2\beta-1}\left(x^{4\beta}+1\right) \mathrm{e}^{-\alpha x^{\beta}}}{\alpha^4 + 120 - \alpha^4 \Gamma\left(2, x^{\beta}\alpha\right) - \Gamma\left(6, x^{\beta}\alpha\right)}, \end{aligned} \tag{23}$$

and

$$\begin{aligned} O_{PLBSD}(x;\alpha,\beta) &= \frac{F_{PLBSD}(x;\alpha,\beta)}{1 - F_{PLBSD}(x;\alpha,\beta)} \\ &= \frac{\alpha^4 + 120 - \alpha^4 \Gamma\left(2, x^{\beta}\alpha\right) - \Gamma\left(6, x^{\beta}\alpha\right)}{\alpha^4 \Gamma\left(2, x^{\beta}\alpha\right) + \Gamma\left(6, x^{\beta}\alpha\right)}. \end{aligned} \tag{24}$$

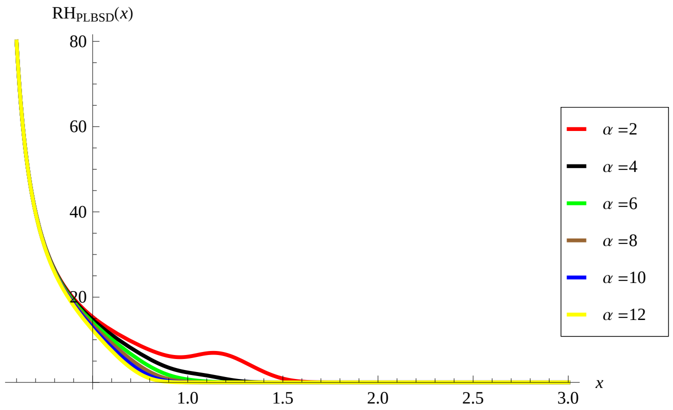

Figure 5: The reversed hazard rate functions of PLBSD for $\alpha = 2, 4, 6, 8, 10, 12$ and $\beta = 4$

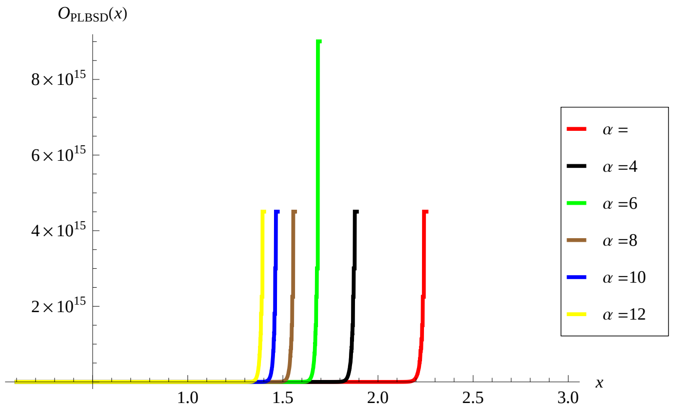

Figure 6: The odds functions of PLBSD for $\alpha = 2, 4, 6, 8, 10, 12$ and $\beta = 4$

6 Rényi Entropy and Harmonic Mean

The Rényi entropy of a random variable X is a measure of variation of the uncertainty and it defined as

$$RE(\eta) = \frac{1}{1-\eta} \log \left(\int_0^\infty f(x)^\eta dx \right), \eta > 0, \eta \neq 1.$$

Theorem 4: If $X \sim f_{PLBSD}(x; \theta, \beta)$, the Rényi entropy of X is defined as

$$RE(\eta,\alpha,\beta) = \frac{1}{1-\eta}\,\log\left[\left(\frac{\alpha^6}{\alpha^4+120}\right)^{\eta}\beta^{\eta-1}\sum_{i=0}^{\eta}\binom{\eta}{i}(\alpha\eta)^{\frac{-6\beta\eta+\eta+4\beta i-1}{\beta}}\Gamma\left(\frac{-4i\beta+6\eta\beta-\eta+1}{\beta}\right)\right]. \tag{25}$$

Proof: The theorem can be proved as

$$\begin{aligned}
RE_{PLBSD}(\eta,\alpha,\beta) &= \frac{1}{1-\eta}\,\log\left[\int_0^{\infty}\left(\frac{\alpha^6\beta}{\alpha^4+120}x^{2\beta-1}\left(x^{4\beta}+1\right)\mathrm{e}^{-\alpha x^{\beta}}\right)^{\eta}dx\right] \\
&= \frac{1}{1-\eta}\,\log\left[\int_0^{\infty}\left(\frac{\alpha^6\beta}{\alpha^4+120}\left(x^{6\beta-1}+x^{2\beta-1}\right)\mathrm{e}^{-\alpha x^{\beta}}\right)^{\eta}dx\right] \\
&= \frac{1}{1-\eta}\,\log\left[\left(\frac{\alpha^6\beta}{\alpha^4+120}\right)^{\eta}\int_0^{\infty}\left(x^{6\beta-1}+x^{2\beta-1}\right)^{\eta}\mathrm{e}^{-\alpha\eta x^{\beta}}dx\right] \\
&= \frac{1}{1-\eta}\,\log\left[\left(\frac{\alpha^6\beta}{\alpha^4+120}\right)^{\eta}\int_0^{\infty}\sum_{i=0}^{\eta}\binom{\eta}{i}x^{2i\beta-i}x^{(6\beta-1)(\eta-i)}\,\mathrm{e}^{-\alpha\eta x^{\beta}}dx\right] \\
&= \frac{1}{1-\eta}\,\log\left[\left(\frac{\alpha^6\beta}{\alpha^4+120}\right)^{\eta}\int_0^{\infty}\sum_{i=0}^{\eta}\binom{\eta}{i}x^{6\beta\eta-4i\beta-\eta}\,\mathrm{e}^{-\alpha\eta x^{\beta}}dx\right] \\
&= \frac{1}{1-\eta}\,\log\left[\left(\frac{\alpha^6\beta}{\alpha^4+120}\right)^{\eta}\sum_{i=0}^{\eta}\binom{\eta}{i}\int_0^{\infty}x^{6\beta\eta-4i\beta-\eta}\,\mathrm{e}^{-\alpha\eta x^{\beta}}dx\right] \\
&= \frac{1}{1-\eta}\,\log\left[\left(\frac{\alpha^6\beta}{\alpha^4+120}\right)^{\eta}\sum_{i=0}^{\eta}\binom{\eta}{i}\frac{(\alpha\eta)^{\frac{-6\beta\eta+\eta+4\beta i-1}{\beta}}\Gamma\left(\frac{-4i\beta+6\eta\beta-\eta+1}{\beta}\right)}{\beta}\right] \\
&= \frac{1}{1-\eta}\,\log\left[\left(\frac{\alpha^6}{\alpha^4+120}\right)^{\eta}\beta^{\eta-1}\sum_{i=0}^{\eta}\binom{\eta}{i}(\alpha\eta)^{\frac{-6\beta\eta+\eta+4\beta i-1}{\beta}}\Gamma\left(\frac{-4i\beta+6\eta\beta-\eta+1}{\beta}\right)\right].
\end{aligned}$$

Table 5: Rényi entropy values for selected values of the PLBSD distribution parameters

$\beta = 0.7, \eta = 0.2$						$\alpha = 0.6, \beta = 0.9$					
α	RE	α	RE	α	RE	η	RE	η	RE	η	RE
0.1	23.0083	1.6	9.02125	3.1	4.50657	2	2.95312	3.5	2.85072	5	2.79880
0.2	19.5425	1.7	8.68520	3.2	4.21431	2.1	2.94308	3.6	2.84627	5.1	2.79620
0.3	17.5150	1.8	8.36073	3.3	3.92211	2.2	2.93372	3.7	2.84199	5.2	2.79366
0.4	16.0763	1.9	8.04551	3.4	3.63020	2.3	2.92495	3.8	2.83789	5.3	2.79121
0.5	14.9599	2	7.73756	3.5	3.33881	2.4	2.91673	3.9	2.83395	5.4	2.78882
0.6	14.0469	2.1	7.43518	3.6	3.04822	2.5	2.90900	4	2.83016	5.5	2.78650
0.7	13.2741	2.2	7.13696	3.7	2.75868	2.6	2.90171	4.1	2.82651	5.6	2.78424
0.8	12.6031	2.3	6.84170	3.8	2.47047	2.7	2.89482	4.2	2.82299	5.7	2.78205
0.9	12.0094	2.4	6.54841	3.9	2.18382	2.8	2.88830	4.3	2.81959	5.8	2.77991
1	11.4760	2.5	6.25634	4	1.89897	2.9	2.88211	4.4	2.81632	5.9	2.77784
1.1	10.9905	2.6	5.96488	4.1	1.61616	3	2.87624	4.5	2.81315	6	2.77581
1.2	10.5438	2.7	5.67361	4.2	1.33556	3.1	2.87065	4.6	2.81009	6.1	2.77384
1.3	10.1286	2.8	5.38224	4.3	1.05737	3.2	2.86532	4.7	2.80713	6.2	2.77191
1.4	9.7395	2.9	5.09062	4.4	0.78173	3.3	2.86023	4.8	2.80427	6.3	2.77003
1.5	9.3716	3	4.79871	4.5	0.50878	3.4	2.85537	4.9	2.80149	6.4	2.76820

Table (5) summarizes Rényi entropy values for various value of α, β and η. It can be noted that as the values of α are increasing the RE are decreasing. Also, as the values of η are increasing, the entropy values are decreasing and it is running from 2.95312 to 2.7682 for $\eta = 2$ to $\eta = 6.4$.

Theorem 5: The harmonic mean of the PLBSD distribution is given by

$$H.M_{PLBSD} = \frac{\left(\alpha^4 + 120\right) \alpha^{-\frac{1}{\beta}} \beta^4}{\left(\alpha^4\beta^4 + 120\beta^4 - 154\beta^3 + 71\beta^2 - 14\beta + 1\right) \Gamma\left(2 - \frac{1}{\beta}\right)}, \beta > \frac{1}{2}, \alpha > 0. \tag{26}$$

Proof: To prove the harmonic mean of the PLBSD, let

$$
\begin{aligned}
\frac{1}{H.M_{PLBSD}} &= \int_0^\infty \frac{1}{x}\frac{\alpha^6\beta}{\alpha^4+120}x^{2\beta-1}\left(x^{4\beta}+1\right)\mathrm{e}^{-\alpha x^\beta}dx \\
&= \int_0^\infty \frac{\alpha^6\beta}{\alpha^4+120}x^{2\beta-2}\left(x^{4\beta}+1\right)\mathrm{e}^{-\alpha x^\beta}dx \\
&= \frac{\alpha^6\beta}{\alpha^4+120}\left[\int_0^\infty x^{6\beta-2}\mathrm{e}^{-\alpha x^\beta}dx+\int_0^\infty x^{2\beta-2}\mathrm{e}^{-\alpha x^\beta}dx\right] \\
&= \frac{\alpha^6\beta}{\alpha^4+120}\left[\frac{\alpha^{\frac{1}{\beta}-6}\Gamma\left(6-\frac{1}{\beta}\right)}{\beta}+\frac{\alpha^{\frac{1}{\beta}-2}\Gamma\left(2-\frac{1}{\beta}\right)}{\beta}\right] \\
&= \frac{\alpha^6}{\alpha^4+120}\left[\alpha^{\frac{1}{\beta}-6}\Gamma\left(6-\frac{1}{\beta}\right)+\alpha^{\frac{1}{\beta}-2}\Gamma\left(2-\frac{1}{\beta}\right)\right] \\
&= \frac{\alpha^6}{\alpha^4+120}\left[\alpha^{\frac{1}{\beta}-6}\left(5-\frac{1}{\beta}\right)\left(4-\frac{1}{\beta}\right)\left(3-\frac{1}{\beta}\right)\left(2-\frac{1}{\beta}\right)\Gamma\left(2-\frac{1}{\beta}\right)+\alpha^{\frac{1}{\beta}-2}\Gamma\left(2-\frac{1}{\beta}\right)\right] \\
&= \frac{\alpha^6}{\alpha^4+120}\left[\alpha^{\frac{1}{\beta}-6}\left(\frac{5\beta-1}{\beta}\right)\left(\frac{4\beta-1}{\beta}\right)\left(\frac{3\beta-1}{\beta}\right)\left(\frac{2\beta-1}{\beta}\right)+\alpha^{\frac{1}{\beta}-2}\right]\Gamma\left(2-\frac{1}{\beta}\right) \\
&= \frac{\alpha^4\alpha^{\frac{1}{\beta}}}{\alpha^4+120}\left[\alpha^{-4}\left(\frac{5\beta-1}{\beta}\right)\left(\frac{4\beta-1}{\beta}\right)\left(\frac{3\beta-1}{\beta}\right)\left(\frac{2\beta-1}{\beta}\right)+1\right]\Gamma\left(2-\frac{1}{\beta}\right) \\
&= \frac{\alpha^4\alpha^{\frac{1}{\beta}}\beta^{-4}}{\alpha^4+120}\left[\alpha^{-4}\left(5\beta-1\right)\left(4\beta-1\right)\left(3\beta-1\right)\left(2\beta-1\right)+\beta^4\right]\Gamma\left(2-\frac{1}{\beta}\right) \\
&= \frac{\alpha^4\alpha^{\frac{1}{\beta}}\beta^{-4}}{\alpha^4+120}\left[\alpha^{-4}\left(120\beta^4-154\beta^3+71\beta^2-14\beta+1\right)+\beta^4\right]\Gamma\left(2-\frac{1}{\beta}\right) \\
&= \frac{\alpha^{\frac{1}{\beta}}\beta^{-4}}{\alpha^4+120}\left(\alpha^4\beta^4+120\beta^4-154\beta^3+71\beta^2-14\beta+1\right)\Gamma\left(2-\frac{1}{\beta}\right).
\end{aligned}
$$

Hence, the harmonic mean of the PLBSD is

$$
H.M_{PLBSD} = \frac{\left(\alpha^4+120\right)\alpha^{-\frac{1}{\beta}}\beta^4}{\left(120\beta^4-154\beta^3+71\beta^2-14\beta+1+\alpha^4\beta^4\right)\Gamma\left(2-\frac{1}{\beta}\right)}.
$$

Based on Tables (6) and (7), we can conclude that

- For $\beta = 3, 7$, the harmonic mean values are decreasing in α.
- For fixed value of $\alpha \geq 4$, the harmonic mean values for $\beta = 3$ are less than their counterparts when $\beta = 7$.
- The values in Table (6) are increasing in β for $\alpha = 5, 9$. However, for fixed β, the harmonic mean values are decreasing as α is increasing.

7 Stochastic ordering of the PLBSD

The stochastic ordering can be used to compare two positive continuous distributions. A random variable X is smaller than random variable Y in

Table 6: Harmonic mean of the PLBSD for selected values of α when $\beta = 3, 7$

α	$(\alpha, \beta = 3)$	α	$(\alpha, \beta = 3)$	α	$(\alpha, \beta = 3)$	α	$(\alpha, \beta = 7)$	α	$(\alpha, \beta = 7)$	α	$(\alpha, \beta = 7)$
1	1.739610	16	0.439898	31	0.352643	1	1.271450	16	0.710235	31	0.646013
2	1.299000	17	0.431036	32	0.348928	2	1.126110	17	0.704063	32	0.643088
3	0.983001	18	0.422856	33	0.345366	3	1.004760	18	0.698302	33	0.640266
4	0.790193	19	0.415269	34	0.341945	4	0.915661	19	0.692903	34	0.637540
5	0.688418	20	0.408204	35	0.338656	5	0.862190	20	0.687824	35	0.634904
6	0.629136	21	0.401599	36	0.335489	6	0.828859	21	0.683031	36	0.632353
7	0.589350	22	0.395405	37	0.332438	7	0.805595	22	0.678495	37	0.629882
8	0.559701	23	0.389577	38	0.329495	8	0.787765	23	0.674190	38	0.627486
9	0.536069	24	0.384080	39	0.326654	9	0.773221	24	0.670096	39	0.625161
10	0.516406	25	0.378882	40	0.323908	10	0.760870	25	0.666193	40	0.622904
11	0.499573	26	0.373954	41	0.321253	11	0.750100	26	0.662466	41	0.620710
12	0.484868	27	0.369275	42	0.318682	12	0.740531	27	0.658900	42	0.618576
13	0.471831	28	0.364821	43	0.316192	13	0.731914	28	0.655482	43	0.616500
14	0.460138	29	0.360575	44	0.313778	14	0.724074	29	0.652201	44	0.614478
15	0.449553	30	0.356521	45	0.311436	15	0.716880	30	0.649048	45	0.612508

Table 7: Harmonic mean of the PLBSD for selected values of β when $\alpha = 5, 9$

β	$(\alpha = 5, \beta)$	β	$(\alpha = 5, \beta)$	β	$(\alpha = 5, \beta)$	β	$(\alpha = 9, \beta)$	β	$(\alpha = 9, \beta)$	β	$(\alpha = 9, \beta)$
1	0.229584	16	0.939121	31	0.968482	1	0.112731	16	0.895137	31	0.944740
2	0.549580	17	0.942680	32	0.969464	2	0.379589	17	0.901056	32	0.946430
3	0.688418	18	0.945845	33	0.970386	3	0.536069	18	0.906342	33	0.948020
4	0.762726	19	0.948680	34	0.971255	4	0.631434	19	0.911093	34	0.949518
5	0.808643	20	0.951233	35	0.972074	5	0.694787	20	0.915386	35	0.950932
6	0.839750	21	0.953544	36	0.972848	6	0.739735	21	0.919283	36	0.952269
7	0.862190	22	0.955646	37	0.973580	7	0.773221	22	0.922837	37	0.953535
8	0.879133	23	0.957566	38	0.974273	8	0.799109	23	0.926092	38	0.954736
9	0.892375	24	0.959326	39	0.974931	9	0.819713	24	0.929083	39	0.955876
10	0.903007	25	0.960947	40	0.975556	10	0.836495	25	0.931842	40	0.956960
11	0.911729	26	0.962443	41	0.976151	11	0.850425	26	0.934394	41	0.957992
12	0.919015	27	0.963829	42	0.976718	12	0.862173	27	0.936762	42	0.958976
13	0.925190	28	0.965116	43	0.977258	13	0.872212	28	0.938965	43	0.959915
14	0.930491	29	0.966315	44	0.977774	14	0.880891	29	0.941020	44	0.960812
15	0.935091	30	0.967434	45	0.978267	15	0.888466	30	0.942941	45	0.961669

1. Mean residual life order denoted by $X \underset{MRLO}{\leq} Y$, if $m_X(x) \leq m_Y(x)$ for all x.
2. Hazard rate order denoted by $X \underset{HRO}{\leq} Y$, if $h_X(x) \geq h_Y(x)$ for all x.
3. Stochastic order denoted by $X \underset{SO}{\leq} Y$, if $F_X(x) \geq F_Y(x)$ for all x.
4. Likelihood ratio order denoted by $X \underset{LRO}{\leq} Y$, if $\frac{f_X(x)}{f_Y(x)}$ decreases in x.

It is shown by Shaked and Shanthikumar (1994) that

$$\begin{array}{c} X \underset{LRO}{\leq} Y \Rightarrow X \underset{HRO}{\leq} Y \Rightarrow X \underset{MRLO}{\leq} Y \\ \Downarrow \\ X \underset{SO}{\leq} Y \end{array}$$

.

Theorem 6: Let $X \sim f_X(x; \alpha, \beta)$, $Y \sim f_Y(x; \varphi, \eta)$, and if $\alpha > \varphi$, $\beta > \eta$, then for the PLBSD distribution, we have $X \underset{LRO}{\leq} Y$.

Proof: Let $X \sim f_X(x; \alpha, \beta)$, $Y \sim f_Y(x; \varphi, \eta)$, then

$$\frac{f_X(x; \alpha, \beta)}{f_Y(x; \varphi, \eta)} = \frac{\frac{\alpha^6 \beta}{\alpha^4+120} x^{2\beta-1}\left(x^{4\beta}+1\right) \mathrm{e}^{-\alpha x^{\beta}}}{\frac{\varphi^6 \eta}{\varphi^4+120} x^{2\eta-1}\left(x^{4\eta}+1\right) \mathrm{e}^{-\varphi x^{\eta}}},$$

and

$$\begin{aligned} \log\left(\frac{f_X(x; \alpha, \beta)}{f_Y(x; \varphi, \eta)}\right) &= \log\left[\frac{\frac{\alpha^6 \beta}{\alpha^4+120} x^{2\beta-1}\left(x^{4\beta}+1\right) \mathrm{e}^{-\alpha x^{\beta}}}{\frac{\varphi^6 \eta}{\varphi^4+120} x^{2\eta-1}\left(x^{4\eta}+1\right) \mathrm{e}^{-\varphi x^{\eta}}}\right] \\ &= \log\left[\frac{\alpha^6 \beta\left(\varphi^4+120\right)}{\varphi^6 \eta\left(\alpha^4+120\right)} \frac{x^{2\beta-1}\left(x^{4\beta}+1\right)}{x^{2\eta-1}\left(x^{4\eta}+1\right)} e^{-\left(\alpha x^{\beta}-\varphi x^{\eta}\right)}\right] \\ &= \log\left[\frac{\alpha^6 \beta\left(\varphi^4+120\right)}{\varphi^6 \eta\left(\alpha^4+120\right)}\right] + \log\left[\frac{x^{2\beta-1}\left(x^{4\beta}+1\right)}{x^{2\eta-1}\left(x^{4\eta}+1\right)}\right] - \left(\alpha x^{\beta}-\varphi x^{\eta}\right). \end{aligned}$$

Taking the derivative of the last equation with respect to x yields

$$\begin{aligned} \frac{d}{dx} \log \frac{f_X(x; \alpha, \beta)}{f_Y(x; \varphi, \mu)} &= -\alpha \beta x^{\beta-1} + \eta \varphi x^{\eta-1} + \frac{x^{2\eta-1}+x^{6\eta-1}}{x^{2\beta-1}+x^{6\beta-1}} \times \\ &\left(\begin{array}{c} \frac{(2\beta-1)x^{2\beta-2}+(6\beta-1)x^{6\beta-2}}{x^{2\eta-1}+x^{6\eta-1}} \\ -\frac{\left(x^{2\beta-1}+x^{6\beta-1}\right)\left((2\eta-1)x^{2\eta-2}+(6\eta-1)x^{6\eta-2}\right)}{\left(x^{2\eta-1}+x^{6\eta-1}\right)^2} \end{array}\right) \end{aligned}$$

Hence, if $\alpha > \varphi$, $\beta > \eta$, then $\frac{d}{dx} \log \frac{f_X(x;\alpha,\beta)}{f_Y(x;\varphi,\mu)} < 0$. Therefore, $X \underset{LRO}{\leqslant} Y$, $X \underset{HRO}{\leqslant} Y$, $X \underset{MRLO}{\leqslant} Y$ and $X \underset{SO}{\leqslant} Y$.

8 Order Statistics and Maximum Likelihood Estimation

Let $X_1, X_2, ..., X_m$ be a random sample of size m from the PLBSD. Also, let $X_{(1:m)}, X_{(2:m)}, ...,$ $X_{(m:m)}$ be the corresponding order statistics. The pdf of the ith order statistic is given by

$$f_{(i:m)}(x) = \frac{m!}{(i-1)!(m-i)!}[F(x)]^{i-1}[1-F(x)]^{m-i}f(x) for i = 1, 2, ..., m. \tag{27}$$

Based on Equation (27), the pdf of smallest and largest order statistics $X_{(1:m)}$ and $X_{(m:m)}$, respectively, are given by

$$f_{(1:m)}(x;\beta,\alpha) = \frac{\alpha^6\beta m! x^{2\beta-1}\left(x^{4\beta}+1\right)e^{-\alpha x^\beta}\left[\alpha^4\left(-\Gamma\left(2,x^\beta\alpha\right)\right)+\Gamma\left(6,x^\beta\alpha\right)\right]^{m-1}}{(\alpha^4+120)^m(m-1)!}, \tag{28}$$

$$f_{(m:m)}(x;\alpha,\beta) = \frac{\alpha^6\beta m! x^{2\beta-1}\left(x^{4\beta}+1\right)e^{-\alpha x^\beta}\begin{bmatrix}\alpha^4+\alpha^4\left(-\Gamma\left(2,x^\beta\alpha\right)\right)\\ -\Gamma\left(6,x^\beta\alpha\right)+120\end{bmatrix}^{m-1}}{(\alpha^4+120)^m(m-1)!}. \tag{29}$$

For a random sample of size n, $X_1, X_2, ..., X_n$ from PLBSD with parameters $\alpha > 0$ and $\beta > 0$. The maximum likelihood estimators of PLBSD can be obtained as follows

$$\begin{aligned}
L_{PLBSD}(\alpha,\beta) &= \prod_{i=1}^{n} f_{PLBSD}(x_i;\alpha,\beta)\\
&= \prod_{i=1}^{n} \frac{\alpha^6\beta}{\alpha^4+120}\left(x_i^{6\beta-1}+x_i^{2\beta-1}\right)e^{-\alpha x_i^\beta}\\
&= \left(\frac{\alpha^6\beta}{\alpha^4+120}\right)^n \prod_{i=1}^{n}\left(x_i^{6\beta-1}+x_i^{2\beta-1}\right)\prod_{i=1}^{n} e^{-\alpha x_i^\beta}\\
&= \left(\frac{\alpha^6\beta}{\alpha^4+120}\right)^n \prod_{i=1}^{n}\left(x_i^{6\beta-1}+x_i^{2\beta-1}\right)e^{-\alpha\sum\limits_{i=1}^{n} x_i^\beta}.
\end{aligned}$$

The ln of $L_{PLBSD}(\alpha,\beta)$ is

$$\begin{aligned}
\ln\left(L_{PLBSD}(\alpha,\beta)\right) &= 6n\ln\alpha + \ln\beta - n\ln\left(\alpha^4+120\right) + \ln\prod_{i=1}^{n}\left(x_i^{6\beta-1}+x_i^{2\beta-1}\right) - \alpha\sum_{i=1}^{n} x_i^\beta\\
&= 6n\ln\alpha + \ln\beta - n\ln\left(\alpha^4+120\right) + \sum_{i=1}^{n}\ln\left(x_i^{6\beta-1}+x_i^{2\beta-1}\right) - \alpha\sum_{i=1}^{n} x_i^\beta.
\end{aligned} \tag{30}$$

The derivatives of Equation (30) with respect α and β respectively, are

$$\begin{aligned}
\frac{d\ln\left(L_{PLBSD}(\alpha,\beta)\right)}{d\alpha} &= \frac{6n}{\alpha} - \frac{4n\alpha^3}{\alpha^4+120} - n\sum_{i=1}^{n} x_i^{\beta}, \\
\frac{d\ln\left(L_{PLBSD}(\alpha,\beta)\right)}{d\beta} &= \frac{1}{\beta} + \sum_{i=1}^{n} \frac{2x_i^{2\beta-1}\ln x_i + 6x_i^{6\beta-1}\ln x_i}{x_i^{6\beta-1} + x_i^{2\beta-1}} - n\sum_{i=1}^{n} x_i^{\beta}\ln x_i.
\end{aligned} \tag{31}$$

Since there is no closed from for these equations, then the MLEs, say $\hat{\alpha}$ and $\hat{\beta}$, of α and β, respectively can be solved numerically.

9 Application

To investigate the advantage of the suggested distribution, we considered a real data set provided by Ramos et al. (2013), Al-Omari and Alsmairan (2019), and Murthy et al. (2013). The data represent the failure times for a particular model windshield. The failure times of 84 Aircraft Windshield data are given below

0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779,1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.82,3, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663

We fitted the proposed distribution to the 84 Aircraft Windshield data and compared the results with the Garima (GD), Rama (RD), Ishita (ID), Sushila (SD), Suja (SUD) and length biased Suja (LBSD) distributions with respective densities:

1. Garima distribution, $f(x;\theta) = \frac{\theta}{\theta+2}\left(1+\theta+\theta x\right)e^{-\theta x}; x>0,\ \theta>0.$
2. Rama distribution, $f(x;\theta) = \frac{\theta^4}{\theta^3+6}\left(1+x^3\right)e^{-\theta x}, x>0,\ \theta>0.$
3. Ishita distribution, $f(x;\theta) = \frac{\theta^3}{\theta^3+2}\left(\theta+x^2\right)e^{-\theta x}; x>0,\ \theta>0.$
4. Sushila distribution, $f(x;\theta,\delta) = \frac{\delta^2}{\theta(\delta+1)}\left(1+\frac{x}{\theta}\right)e^{-\frac{\delta}{\theta}x}; x>0, \delta>0,\ \theta>0.$
5. Suja distribution, $f(x;\theta) = \frac{\theta^5}{\theta^4+24}\left(1+x^4\right)e^{-\theta x}; x>0,\ \theta>0.$
6. Length biased Suja distribution, $f(x;\theta) = \frac{\theta^6}{\theta^4+120}\left(x+x^5\right)e^{-\theta x}; x>0, \theta>0.$

The criteria of selecting the best model are MLE's of the models parameters, -2log-likelihood (-2LL), Akaike Information Criterion (AIC), Hannan-Quinn Information Criterion (HQIC), Bayesian Information Criterion (BIC), Consistent Akaike Information Criterion (CAIC) and Kolmogorov-Smirnov ($K.S$), where

Table 8: The MLEs of the model parameter with the corresponding standard errors and 0.95 confidence intervals for the Aircraft data.

Model	MLE	Std. Dev.	0.95%CI
GD	$\hat{\theta} = 0.5586$	0.0516	(0.4575, 0.6597)
RD	$\hat{\theta} = 1.2654$	0.0617	(1.1444, 1.3864)
ID	$\hat{\theta} = 0.0322$	0.0013	(0.0297, 0.0347)
SUD	$\hat{\theta} = 1.6098$	0.0667	(1.5021, 1.7543)
$LBSD$	$\hat{\theta} = 2.1174$	0.0845	(1.9519, 2.2830)
SD	$\hat{\theta} = 0.0277$	0.0209	(-0.0133, 0.0687)
	$\hat{\delta} = 0.0214$	0.0160	(-0.0100, 0.0528)
$PLBSD$	$\hat{\alpha}$ =2.6010	0.2240	(2.1620, 3.0400)
	$\hat{\beta}$ =0.8810	0.0748	(0.7344, 1.0275)

Table 9: The AIC, CAIC, BIC, HQIC, K.S, and -2LL for the Aircraft data

Model	AIC	CAIC	BIC	HQIC	K.S	-2LL
GD	321.649	321.697	324.092	322.632	0.284	59.825
RD	668.077	668.126	670.520	669.060	0.439	333.039
ID	2641.466	2641.49	2644.827	2642.830	0.120	1319.733
SD	293.695	293.842	298.581	295.660	0.183	144.848
SUD	279.634	279.683	282.077	280.617	0.149	138.817
$LBSD$	263.503	263.551	265.945	264.485	0.069	130.751
$PLBSD$	239.256	230.402	244.141	241.221	0.058	117.628

- AIC $= -2MLL + 2\kappa$,
- CAIC $= -2MLL + \frac{2\kappa n}{n-\kappa-1}$,
- HQIC $= 2Log\{Log(n)[\kappa - 2MLL]\}$,
- BIC $= -2MLL + \kappa\, Log(n)$,
- $K.S = Sup_n\,|F_n(x) - F(x)|$, $F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I_{x_i \le x}$,

where κ is the number of parameters and n is the sample size.

Table (9) shows that the PLBSD distribution provides the best fit for the data set as it has lower AIC, CAIC, BIC, HQIC and K.S values than the other competitor models. Hence, the PLBSD can be used fit the 84 Aircraft Windshield data.

10 Conclusions

This paper suggests a new continuous two-parameter lifetime distribution named as the power length biased Suja distribution. The moments of the distribution including the mean, variance, coefficients of skewness, kurtosis and coefficient of variation. The Bonferroni and Lorenz curves and the Gini index of the PLBSD as well as the stochastic ordering are presented. The maximum likelihood estimators of the model parameters are derived as well as distributions of order statistics are provided. The Rényi entropies, stochastic ordering and mean and median deviations about the mean are derived. A real data set is fitted to investigate the usefulness of the PLBSD.

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